Calculus 3
Lia Vas

Maximum and Minimum Values

Let $z = f(x, y)$ be a function of two variables. To find the local maximum and minimum values, we:

1. Find the first partial derivatives f_x and f_y. Then find all points (a, b) at which the partial derivatives are zero or are not defined. Such points are called **critical points**.

2. Find the second partial derivatives f_{xx}, f_{xy}, f_{yx} and f_{yy}, and find the **determinant** D

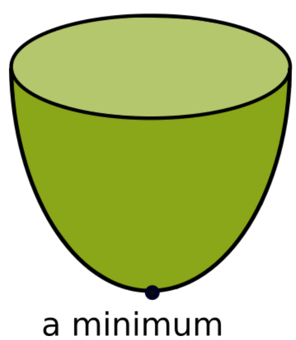
$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

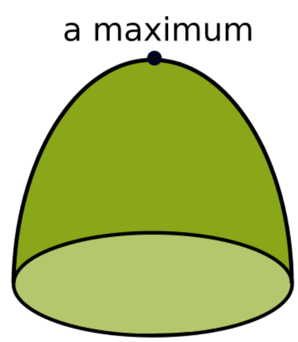
3. Consider the sign of f_{xx} and D at all critical points from part 1.

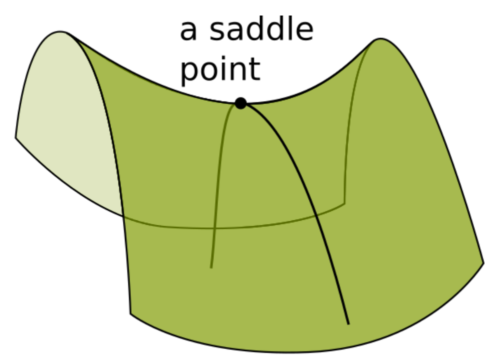
   a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local **minimum**.

   b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local **maximum**.

   c) If $D < 0$, then $f(a, b)$ is **not** a local minimum or maximum. It is a **saddle point**.







Practice Problems. Find the maximum and minimum values of f.

1. $f(x, y) = 9 - 2x + 4y - x^2 - 4y^2$

2. $f(x, y) = x^2 + y^2 + x^2y + 4$

3. $f(x, y) = \frac{x^2y^2 - 8x + y}{xy}$

4. $f(x, y) = xy - 2x - y$

5. $f(x, y) = x^2y + 3xy^2 - 9xy + 2$

Solutions.

1. Find derivatives: $f_x = -2 - 2x$, $f_y = 4 - 8y$. Find critical points: $f_x = -2 - 2x = 0 \Rightarrow -2x = 2 \Rightarrow x = -1$, and $f_y = 4 - 8y = 0 \Rightarrow 4 = 8y \Rightarrow y = \frac{1}{2}$. So, the only critical point is $(-1, \frac{1}{2})$. Find the second derivatives: $f_{xx} = -2$, $f_{xy} = 0$, and $f_{yy} = -8$. Find D : $D = \begin{vmatrix} -2 & 0 \\ 0 & -8 \end{vmatrix} = (-2)(-8) - 0 = 16$. Since $f_{xx} = -2 < 0$ and $D = 16 > 0$, there is a maximum at $(-1, \frac{1}{2})$. Find the maximum value to be $f(-1, 1/2) = 11$.

2. Find derivatives: $f_x = 2x + 2xy$, $f_y = 2y + x^2$. Find critical points: $f_x = 2x + 2xy = 0 \Rightarrow 2x(1 + y) = 0 \Rightarrow 2x = 0$ or $1 + y = 0 \Rightarrow x = 0$ or $y = -1$.

   Plugging $x = 0$ in the second equation gives you $f_y = 2y + 0^2 = 0 \Rightarrow y = 0$. Thus, one critical points is $(0, 0)$.

   Plugging $y = -1$ in the second equation gives you $f_y = -2 + x^2 = 0 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$. So, another two critical points are $(\pm\sqrt{2}, -1)$.

   Find the second derivatives: $f_{xx} = 2+2y$, $f_{xy} = 2x$, and $f_{yy} = 2$. Find $D : D = \begin{vmatrix} 2 + 2y & 2x \\ 2x & 2 \end{vmatrix} = 4 + 4y - 4x^2$.

   Consider the values of the second derivatives in each of the three critical points.

   At $(0, 0)$, $f_{xx} = 2 + 2(0) = 2 > 0$ and $D = 4 + 4(0) - 4(0)^2 = 4 > 0$. So, there is a minimum at $(0, 0)$. Find the minimum value to be $f(0, 0) = 4$.

   At $(\sqrt{2}, -1)$, $f_{xx} = 2 + 2(-1) = 0$ and $D = 4 + 4(-1) - 4(\sqrt{2})^2 = -8 < 0$. Thus, there is a saddle point at $(\sqrt{2}, -1)$ and no extreme value at this point.

   At $(-\sqrt{2}, -1)$, $f_{xx} = 2 + 2(-1) = 0$ and $D = 4 + 4(-1) - 4(-\sqrt{2})^2 = -8 < 0$. Thus, there is a saddle point at $(-\sqrt{2}, -1)$ and no extreme value at this point.

3. To avoid the quotient rule, you can simplify the function as $f = xy - \frac{8}{y} + \frac{1}{x}$. The derivatives $f_x = y - \frac{1}{x^2} = 0 \Rightarrow y = \frac{1}{x^2}$, and $f_y = x + \frac{8}{y^2} = 0 \Rightarrow x + \frac{8}{(\frac{1}{x^2})^2} = x + 8x^4 = 0 \Rightarrow x(1 + 8x^3) = 0 \Rightarrow x = 0$ or $8x^3 = -1 \Rightarrow x = 0$ or $x = \frac{-1}{2}$. Substitute back in $y = \frac{1}{x^2}$. $x = 0$ does not give you a finite y-value. In addition, the second derivatives are also not defined when x or y are 0. So, there is no extreme value at this point. If $x = -\frac{1}{2}$, $y = 4$. So, $(-\frac{1}{2}, 4)$ is the only critical point.

   Find the second derivatives: $f_{xx} = \frac{2}{x^3}$, $f_{xy} = 1$, and $f_{yy} = \frac{-16}{y^3}$. Find $D : D = \begin{vmatrix} \frac{2}{x^3} & 1 \\ 1 & \frac{-16}{y^3} \end{vmatrix} = \frac{-32}{x^3y^3} - 1$. At $(-\frac{1}{2}, 4)$, $f_{xx} = -16 < 0$ and $D = \frac{32}{8} - 1 = 3 > 0$. So, there is a maximum at $(-\frac{1}{2}, 4)$. Find the maximum value to be $f(-\frac{1}{2}, 4) = -6$.

4. $f_x = y - 2 = 0 \Rightarrow y = 2$, $f_y = x - 1 = 0 \Rightarrow x = 1$. One critical point $(1, 2)$. $D = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0$. So there is a saddle point at $(1, 2)$.

5. $f_x = 2xy + 3y^2 - 9y = y(2x + 3y - 9)$ and $f_y = x^2 + 6xy - 9x = x(x + 6y - 9)$. From the first equation, we have two cases (1) $y = 0$ and (2) $2x + 3y - 9 = 0 \Rightarrow y = \frac{-2x+9}{3}$.

   In the first case, the second equation becomes $x(x - 9) = 0 \Rightarrow x = 0$ or $x = 9$. So, (0,0) and (9, 0) are critical points.

In the second case, $y = \frac{-2x+9}{3}$ and the second equation becomes $x(x + 6\frac{-2x+9}{3} - 9) = x(x - 4x + 18 - 9) = x(-3x + 9) = -3x(x-3)0 \Rightarrow x = 0$ or $x = 3$. When $x = 0, y = \frac{-2(0)+9}{3} = \frac{9}{3} = 3$ and when $x = 3, y = \frac{-2(3)+9}{3} = \frac{3}{3} = 1$. So, $(0,3)$ and $(3,1)$ are also critical points. Hence, there are four critical points total.

After finding all the critical points, find the second derivatives.

$f_{xx} = 2y$, $f_{xy} = f_{yx} = 2x+6y-9$, $f_{yy} = 6x$ and the determinant $D = \begin{vmatrix} 2y & 2x+6y-9 \\ 2x+6y-9 & 6x \end{vmatrix}$.

At point (0, 0), $f_{xx} = 0$ and $D = \begin{vmatrix} 0 & -9 \\ -9 & 0 \end{vmatrix} = -81 < 0$. Thus, there is a saddle at (0,0) so no extreme value there.

At point (9, 0), $f_{xx} = 0$ and $D = \begin{vmatrix} 0 & 9 \\ 9 & 54 \end{vmatrix} = -81 < 0$. Thus, there is a saddle at (9,0) so no extreme value there.

At point $(0,3)$, $f_{xx} = 6 > 0$ and $D = \begin{vmatrix} 6 & 9 \\ 9 & 0 \end{vmatrix} = -81 < 0$. Thus, there is a saddle at $(0,3)$ so no extreme value there.

At point $(3,1)$, $f_{xx} = 2 > 0$ and $D = \begin{vmatrix} 2 & 3 \\ 3 & 18 \end{vmatrix} = 36 - 9 = 27 > 0$ and so there is a minimum at $(3,1)$. The minimum value is $f(3,1) = 9 + 9 - 27 + 2 = -7$.