

## Parametric Surfaces. Substitution

Recall that a curve in space is given by parametric equations as a function of *single* parameter  $t$

$$x = x(t) \quad y = y(t) \quad z = z(t).$$

A curve is a one-dimensional object in space so its parametrization is a function of *one* variable.

Analogously, a surface is a two-dimensional object in space and, as such can be described using *two* variables. If a surface is given by an explicit equation  $z = f(x, y)$  the dependence on two independent variables is clearly visible. However, a parametric representation of an implicit surface  $F(x, y, z) = 0$  may often be very useful. In particular, it may be useful to parametrize such surface using two parameters possibly different from  $x$  and  $y$ . In particular, a surface given by the parametric equations

$$x = x(u, v) \quad y = y(u, v) \quad z = z(u, v)$$

is referred to as a **parametric surface** and the two independent variable  $u$  and  $v$  as **parameters**. The equations above are called the **parametric equations** of the surface. Thus, we have the following.

	Dimension	Parameter(s)	Equations
Curve	1	$t$	$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$
Surface	2	$u, v$	$\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$

### Examples.

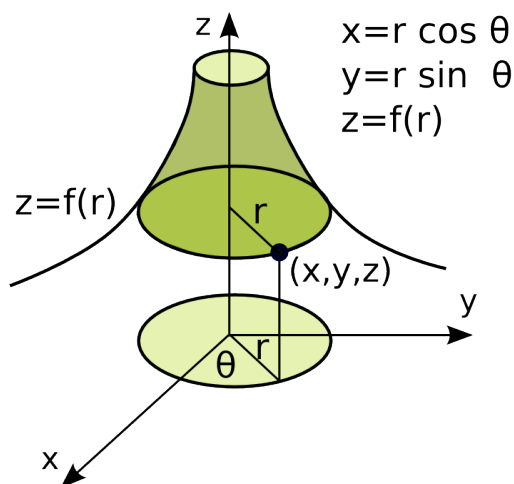
A **surface of revolution**  $z = f(\sqrt{x^2 + y^2})$  can be parametrized using polar coordinates as follows

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = f(r).$$

In particular, the following surfaces of revolution are parametrized in this way.

1. The paraboloid  $z = x^2 + y^2$  has parametric representation by  $x = r \cos \theta$ ,  $y = r \sin \theta$ ,  $z = r^2$ .
2. The cone  $z = \sqrt{x^2 + y^2}$  has parametric representation by  $x = r \cos \theta$ ,  $y = r \sin \theta$ ,  $z = r$ .
3. The upper hemisphere of the sphere  $x^2 + y^2 + z^2 = 9$  has parametric representation by  $x = r \cos \theta$ ,  $y = r \sin \theta$ ,  $z = \sqrt{9 - r^2}$ .

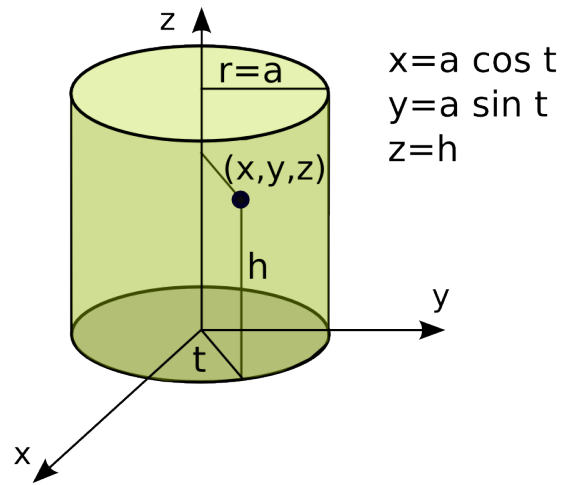
A **cylindrical surface** obtained from a curve in one of the coordinate planes can be parametrized using the curve parametrization and the remaining variable as the second parameter. For example, consider the cylinders below.



4. The cylinder  $x^2 + y^2 = 4$  is based on the circle  $x = 2 \cos t$ , and  $y = 2 \sin t$ . The value of the  $z$ -coordinate of a point on the cylinder represents the height  $h$  of that point. So, the cylinder can be parametrized as

$$x = 2 \cos t, \quad y = 2 \sin t, \quad z = h$$

using the angle  $t$  and the height  $h$  as two parameters.



5. Similarly, the cylinder  $y^2 + z^2 = 4$  is based on the circle  $y = 2 \cos t$ , and  $z = 2 \sin t$  in  $yz$ -plane. The value of the  $x$ -coordinate of a point on the cylinder can also be considered to be the height. Using  $h$  for that value, the cylinder can be parametrized as

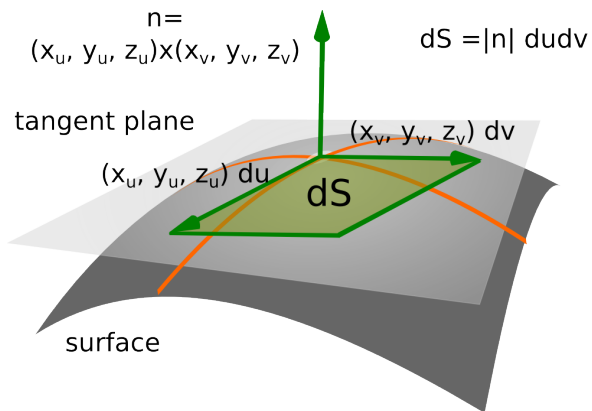
$$x = h, \quad y = 2 \cos t, \quad z = 2 \sin t.$$

### Tangent Plane and Surface Area

The derivative vectors  $\langle x_u, y_u, z_u \rangle$  and  $\langle x_v, y_v, z_v \rangle$  are in the tangent plane of a parametric surface  $x = x(u, v)$ ,  $y = y(u, v)$ ,  $z = z(u, v)$ , (also written shortly as  $\vec{r} = \langle x(u, v), y(u, v), z(u, v) \rangle$ ). Thus, the **tangent plane** of the surface has the perpendicular vector

$$\vec{r}_u \times \vec{r}_v = \langle x_u, y_u, z_u \rangle \times \langle x_v, y_v, z_v \rangle.$$

In case when the surface  $z = z(x, y)$  is given by the parameters  $u = x$  and  $v = y$ , this reduces to the familiar formula  $(-z_x, -z_y, 1)$ .



The surface area  $S$  of this surface over the region  $D$  in  $uv$ -plane can be obtained by integrating surface area elements  $dS$  over sub-rectangles of region  $D$ . The area of each element  $dS$  can be approximated with the area of the parallelogram in the tangent plane just as before when we considered surfaces parametrized using  $x$  and  $y$ .

Recall that the area of a parallelogram formed by two vectors is the length of their cross product. Thus,

$$dS = |\langle x_u, y_u, z_u \rangle \times \langle x_v, y_v, z_v \rangle| du dv \text{ or, written shortly, } dS = |\vec{r}_u \times \vec{r}_v| du dv.$$

The formula for the total surface area  $S$  can be obtained by integrating  $dS$  over entire region  $D$ . Thus

$$S = \iint_D dS = \iint_D |\vec{r}_u \times \vec{r}_v| du dv = \iint_D |\langle x_u, y_u, z_u \rangle \times \langle x_v, y_v, z_v \rangle| du dv.$$

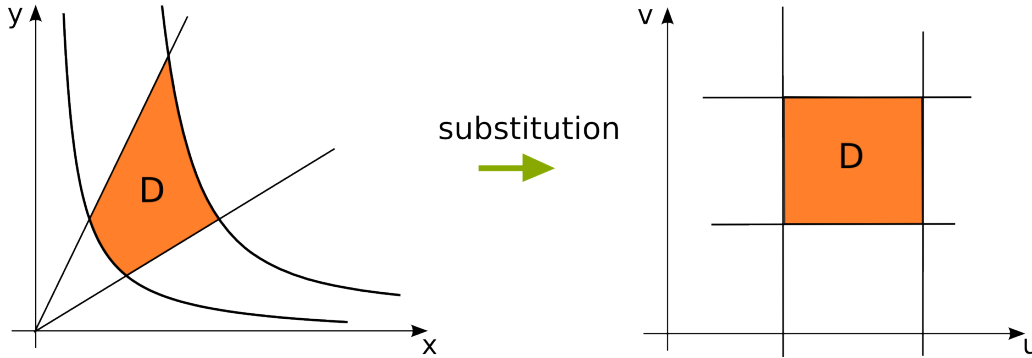
In the case when the surface is parametrized by  $u = x$  and  $v = y$  as  $z = z(x, y)$  this formula becomes the one we have encountered before.

$$S = \iint_D \sqrt{1 + z_x^2 + z_y^2} dx dy$$

## General substitution for double integrals.

In many cases, a region over which a double integral is being taken may have easier representation in another coordinate system, say in  $uv$ -plane, than in  $xy$ -plane. In cases like that, one can transform the region in  $xy$ -plane to a region in  $uv$ -plane by the **substitution**

$$x = g(u, v) \quad y = h(u, v).$$



Thus, a substitution is just a convenient **reparametrization** of a surface  $z = f(x, y)$  when the parameters  $x$  and  $y$  are changed to  $u$  and  $v$ . When evaluating the integral  $\iint_D f(x, y) dx dy$  using substitution, the area element  $dA = dx dy$  becomes  $|J| du dv$  where **the Jacobian determinant**  $J$  is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}.$$

Thus,

$$\iint_D f(x, y) dx dy = \iint_D f(x(u, v), y(u, v)) |J| du dv$$

Note that in one-dimensional case, the Jacobian determinant is simply the derivative of the substitution  $u = u(x)$  solved for  $x$  so that  $x = x(u) \Rightarrow dx = x'(u) du$ .

**Jacobian for polar coordinates.** The polar coordinates  $x = r \cos \theta$  and  $y = r \sin \theta$  can be considered as a substitution in which  $u = r$  and  $v = \theta$ . Thus,  $x_r = \cos \theta$ ,  $x_\theta = -r \sin \theta$  and  $y_r = \sin \theta$ ,  $y_\theta = r \cos \theta$ . The Jacobian is

$$J = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r.$$

This explains the presence of  $r$  in the integrals of the section on Polar Coordinates.

$$\iint_D f(x, y) dx dy = \iint_D f(r \cos \theta, r \sin \theta) r dr d\theta.$$

## Practice problems.

1. Find the equation of the tangent plane to given parametric surface at the specified point.

a)  $x = u + v \quad y = 3u^2 \quad z = u - v; \quad (2, 3, 0)$

b)  $x = uv \quad y = ue^v \quad z = ve^u; \quad (0, 0, 0)$

2. Find the area of the surface.

- The part of the plane  $z = x + 2y$  that lies above the triangle with vertices  $(0,0)$ ,  $(1,1)$  and  $(0,1)$ .
- The part of the surface  $z = y^2 + x^2$  that lies between the cylinders  $x^2 + y^2 = 1$  and  $x^2 + y^2 = 4$ . Write down the parametric equations of the paraboloid and use them to find the surface area.
- The part of the cone  $z = \sqrt{x^2 + y^2}$  that lies between the cylinders  $x^2 + y^2 = 4$  and  $x^2 + y^2 = 9$ . Write down the parametric equations of the cone first. Then find the surface area using the parametric equations.

3. Consider the cylinder  $x^2 + z^2 = 4$ .

- Write down the parametric equations of this cylinder.
- Using the parametric equations, find the tangent plane to the cylinder at the point  $(0, 3, 2)$ .
- Using the parametric equations and formula for the surface area for parametric curves, show that the surface area of the cylinder  $x^2 + z^2 = 4$  for  $0 \leq y \leq 5$  is  $20\pi$ .

4. Use the given substitution to evaluate the integral.

- $\int \int_D (3x + 4y) \, dx \, dy$  where  $D$  is the region bounded by the lines  $y = x$ ,  $y = x - 2$ ,  $y = -2x$ , and  $y = 3 - 2x$ . The substitution  $x = \frac{1}{3}(u + v)$ ,  $y = \frac{1}{3}(v - 2u)$  transforms the region to a rectangle  $0 \leq u \leq 2$  and  $0 \leq v \leq 3$ .
- $\int \int_D xy \, dx \, dy$  where  $D$  is the region in the first quadrant bounded by the curves  $y = x$ ,  $y = 3x$ ,  $y = \frac{1}{x}$ , and  $y = \frac{3}{x}$ . The substitution  $x = \frac{u}{v}$ ,  $y = v$  transforms the region into a region with bounds  $1 \leq u \leq 3$  and  $\sqrt{u} \leq v \leq \sqrt{3u}$ .
- $\int \int_D xy \, dx \, dy$  where  $D$  is the region in the first quadrant bounded by the curves  $y = x$ ,  $y = 3x$ ,  $y = \frac{1}{x}$ , and  $y = \frac{3}{x}$ . The substitution  $x = \sqrt{\frac{u}{v}}$ ,  $y = \sqrt{uv}$  transforms the region into a square  $1 \leq u \leq 3$  and  $1 \leq v \leq 3$ .

### Solutions.

- $\langle x_u, y_u, z_u \rangle = \langle 1, 6u, 1 \rangle$  and  $\langle x_v, y_v, z_v \rangle = \langle 1, 0, -1 \rangle$ . The cross product of these vectors is  $\langle -6u, 2, -6u \rangle$ . Now we need to find  $u$ -value ( $v$ -value is not needed since  $v$  does not appear in this product) that corresponds to the point  $(2, 3, 0)$ . When  $(x, y, z) = (2, 3, 0)$ ,  $x = u + v = 2$  and  $z = u - v = 0$ . From this we have that  $u = v = 1$ . So, the perpendicular vector is  $\langle -6, 2, -6 \rangle$ . The equation of the plane is  $-6(x - 2) + 2(y - 3) - 6(z - 0) = 0 \Rightarrow 3x - y + 3z = 3$
  - $\langle x_u, y_u, z_u \rangle = \langle v, e^v, ve^u \rangle$  and  $\langle x_v, y_v, z_v \rangle = \langle u, ue^v, e^u \rangle$ . The cross product of these vectors is  $\langle e^{uv}(1 - uv), ve^u(u - 1), ue^v(v - 1) \rangle$ . Now we need to find the  $u$  and  $v$  values that correspond to the point  $(0, 0, 0)$ . When  $(x, y, z) = (0, 0, 0)$ ,  $ue^v = 0 \Rightarrow u = 0$  and  $ve^u = 0 \Rightarrow v = 0$ . From this we have that  $u = v = 0$ . So, the perpendicular vector is  $\langle 1, 0, 0 \rangle$ . The equation of the plane is  $1(x - 0) + 0(y - 0) + 0(z - 0) = 0 \Rightarrow x = 0$  ( $yz$ -plane).

2. a) You can use the standard  $xy$ -parametrization here and the formula  $S = \int \int_T \sqrt{1 + z_x^2 + z_y^2} dx dy$  where  $T$  is the given triangle.  $z = x + 2y \Rightarrow z_x = 1$  and  $z_y = 2$ . The bounds for  $x$  are 0 and 1 and  $y$  is bounded by the line connecting (0,0) and (1,1) from below and by the line that connects (0,1) and (1,1) from above. Thus  $x \leq y \leq 1$ . The surface area is  $S = \int_0^1 \int_x^1 \sqrt{1 + 1 + 4} dx dy = \sqrt{6} \int_0^1 (1 - x) dx = \sqrt{6}(1 - \frac{1}{2}) = \frac{\sqrt{6}}{2} = 1.22$ .

b) The paraboloid can be parametrized by  $x = r \cos t$ ,  $y = r \sin t$ ,  $z = x^2 + y^2 = r^2$ .  $\langle x_r, y_r, z_r \rangle = \langle \cos t, \sin t, 2r \rangle$  and  $\langle x_t, y_t, z_t \rangle = \langle -r \sin t, r \cos t, 0 \rangle$ . The cross product of these vectors is  $\langle -2r^2 \cos t, -2r^2 \sin t, r \rangle$ . The length of this product is  $\sqrt{4r^4 \cos^2 t + 4r^4 \sin^2 t + r^2} = \sqrt{4r^4 + r^2} = \sqrt{r^2(4r^2 + 1)} = r\sqrt{4r^2 + 1}$ .

The bounds for the integration are determined by the projection in the  $xy$ -plane which is the region between the circles  $x^2 + y^2 = 1$  and  $x^2 + y^2 = 4$ . Thus  $0 \leq t \leq 2\pi$  and  $1 \leq r \leq 2$ . So, the surface area is  $S = \int_0^{2\pi} dt \int_1^2 r\sqrt{4r^2 + 1} dr = 2\pi 4.91 = 30.85$ .

c) The cone can be parametrized by  $x = r \cos t$ ,  $y = r \sin t$ ,  $z = \sqrt{x^2 + y^2} = r$ .  $\langle x_r, y_r, z_r \rangle = \langle \cos t, \sin t, 1 \rangle$  and  $\langle x_t, y_t, z_t \rangle = \langle -r \sin t, r \cos t, 0 \rangle$ . The cross product of these vectors is  $\langle -r \cos t, -r \sin t, r \rangle$ . The length of this product is  $\sqrt{r^2 \cos^2 t + r^2 \sin^2 t + r^2} = \sqrt{r^2 + r^2} = \sqrt{2r^2} = \sqrt{2}r$ .

The bounds for the integration are determined by the projection in the  $xy$ -plane which is the region between the circles  $x^2 + y^2 = 4$  and  $x^2 + y^2 = 9$ . Thus  $0 \leq t \leq 2\pi$  and  $2 \leq r \leq 3$ . So, the surface area is  $S = \int_0^{2\pi} dt \int_2^3 \sqrt{2}r dr = 2\pi\sqrt{2}(\frac{9}{2} - \frac{4}{2}) = 5\pi\sqrt{2}$ .

3. a) The cylinder can be parametrized as  $x = 2 \cos t$ ,  $y = y$ ,  $z = 2 \sin t$ .

b)  $\langle x_t, y_t, z_t \rangle = \langle -2 \sin t, 0, 2 \cos t \rangle$  and  $\langle x_y, y_y, z_y \rangle = \langle 0, 1, 0 \rangle$ . The cross product is  $\langle -2 \cos t, 0, 2 \sin t \rangle$ . The  $t$ -value that corresponds to (0, 3, 2) can be obtained from  $x = 2 \cos t = 0$  and  $z = 2 \sin t = 2$ . Thus  $t = \frac{\pi}{2}$ . The perpendicular vector is  $\langle 0, 0, 2 \rangle$  and the equation of the plane is  $0(x - 0) + 0(y - 3) + 2(z - 2) = 0 \Rightarrow z = 2$ .

c) We already computed the cross product  $\langle -2 \cos t, 0, 2 \sin t \rangle$ . Its length is  $\sqrt{4 \cos^2 t + 4 \sin^2 t} = \sqrt{4} = 2$ . The  $t$ -bounds are 0 and  $2\pi$  and the  $y$ -bounds are given to be  $0 \leq y \leq 5$ . So, the surface area is  $S = \int_0^{2\pi} \int_0^5 2 dy dt = 2\pi(5)(2) = 20\pi$ .

4. a) Calculate the Jacobian  $J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$ .  $\int \int_D (3x + 4y) dx dy = \int_0^2 \int_0^3 (u + v + \frac{4}{3}(v - 2u)) \frac{1}{3} du dv = \frac{1}{3} \int_0^2 (uv + \frac{v^2}{2} + \frac{4v^2}{6} - \frac{8uv}{3})|_0^3 du = \frac{1}{3} \int_0^2 (3u + \frac{9}{2} + 6 - 8u) du = \frac{1}{3}(6 + 9 + 12 - 16) = \frac{11}{3}$

b) The Jacobian is  $J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & \frac{-u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$ .  $\int \int_D xy dx dy = \int_1^3 \int_{\sqrt{3u}}^{\frac{u}{v}} \frac{u}{v} v \frac{1}{v} du dv = \int_1^3 u du \ln v|_{\sqrt{3u}}^{\frac{u}{v}} = \int_1^3 u du \ln \sqrt{3} = \ln \sqrt{3} \frac{u^2}{2}|_1^3 = 4 \ln \sqrt{3} = 2 \ln 3 = 2.197$

c) The Jacobian is  $J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{2\sqrt{uv}} & \frac{-\sqrt{u}}{2\sqrt{v^3}} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \end{vmatrix} = \frac{1}{4v} + \frac{1}{4v} = \frac{1}{2v}$ .  $\int \int_D xy dx dy = \int_1^3 \int_1^3 \sqrt{\frac{u}{v}} \sqrt{uv} \frac{1}{2v} du dv = \int_1^3 \int_1^3 \frac{u}{2v} du dv = \frac{u^2}{4}|_1^3 \ln v|_1^3 = 2 \ln 3 = 2.197$