Calculus 3
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Parametric Surfaces

Recall that a curve in space is given by parametric equations as a function of *single* parameter t

$$x = x(t) \quad y = y(t) \quad z = z(t).$$

A curve is a one-dimensional object in space so its parametrization is a function of *one* variable.

Analogously, a surface is a two-dimensional object in space and, as such can be described using *two* variables. If a surface is given by an explicit equation $z = f(x, y)$ the dependence on two independent variables is clearly visible. However, a parametric representation of an implicit surface $F(x, y, z) = 0$ may often be very useful. In particular, it may be useful to parametrize such surface using two parameters possibly different from x and y. In particular, a surface given by the parametric equations

$$x = x(u, v) \quad y = y(u, v) \quad z = z(u, v)$$

is referred to as a **parametric surface** and the two independent variable u and v as **parameters.** The equations above are called the **parametric equations** of the surface. These equations can be written shortly as

$$\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle.$$

To summarize, we have the following.

	Dimension	Parameter(s)	Equations
Curve	1	t	$\vec{r}(t) \quad = \langle x(t), y(t), z(t)\rangle$
Surface	2	u, v	$\vec{r}(u, v) = \langle x(u, v),\ y(u, v),\ z(u, v)\rangle$

Examples.

1. If a surface is given by a formula $z = f(x, y)$, it can be parametrized by taking x and y to be two parameters and considering the parametric equations

   $$x = x, \quad y = y \quad z = f(x, y).$$

   For example, the plane $3x + 2y + z = 6$ can be parametrized as

   $$x = x, \quad y = y \quad z = 6 - 3x - 2y.$$

   Note that the given equation also implies that $2y = 6 - 3x - z$ so that $y = 3 - \frac{3}{2}x - \frac{1}{2}z$. So, another parametrization of the same plane is

   $$x = x, \quad y = 3 - \frac{3}{2}x - \frac{1}{2}z \quad z = z.$$

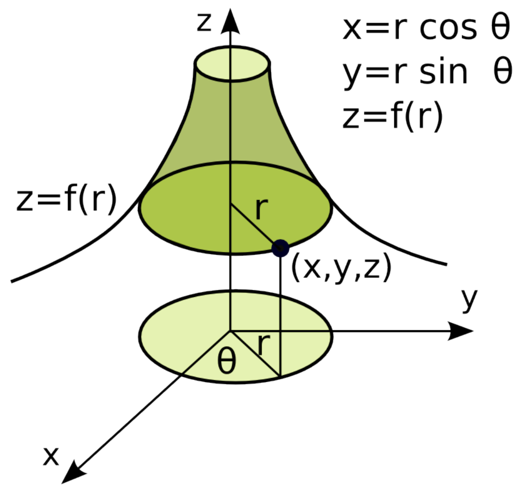
   Thus, just as in the case of parametrizations of a curve, same surface can be parametrized on many different ways. One can choose a suitable parametrization based on specific problem considered or based on some other preference.

2. Every **surface of revolution** $z = f(\sqrt{x^2 + y^2})$ can be parametrized as $x = x, y = y, z = f(\sqrt{x^2 + y^2})$. However, if a projection of a part one may consider on th xy-plane produces a disc or some part of it, polar coordinates are much more suitable. Comparing the graphs of the same cone parametrized using x and y and parametrized using polar coordinates (see page 8 of Matlab notes) illustrates this point.

Since $\sqrt{x^2+y^2}$ is r in polar coordinates, the surface can be parametrized as follows

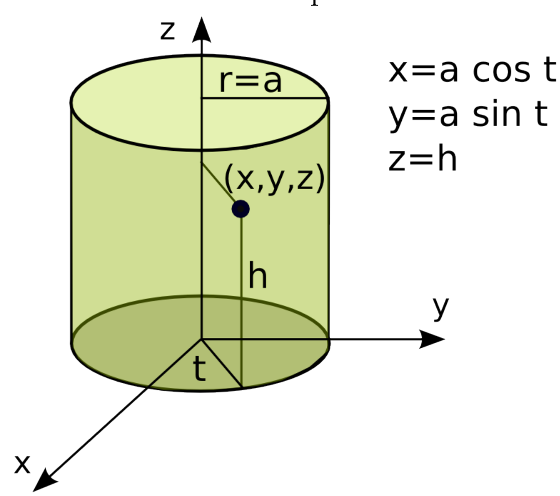
$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = f(r).$$

In particular, the following surfaces of revolution are parametrized in this way.



- The paraboloid $z = x^2 + y^2$ can be parametrized by $x = r\cos\theta$, $y = r\sin\theta$, $z = r^2$ and the paraboloid $z = 3x^2 + 3y^2$, for example, can be parametrized by $x = r\cos\theta$, $y = r\sin\theta$, $z = 3r^2$.
- The cone $z = \sqrt{x^2+y^2}$ has a parametric representation by $x = r\cos\theta$, $y = r\sin\theta$, $z = r$. The cone $z = 5 - 2\sqrt{x^2+y^2}$, for example, has parametric representation by $x = r\cos\theta$, $y = r\sin\theta$, $z = 5 - 2r$.
- The upper hemisphere $z = \sqrt{9 - x^2 - y^2}$ of the sphere $x^2 + y^2 + z^2 = 9$ has parametric representation by $x = r\cos\theta$, $y = r\sin\theta$, $z = \sqrt{9 - r^2}$.

3. A **cylindrical surface** obtained from a curve in one of the coordinate planes can be parametrized using the curve parametrization and the remaining variable as the second parameter. For example, consider the cylinders below.



- The cylinder $x^2 + y^2 = 4$ is based on the circle $x = 2\cos t$, and $y = 2\sin t$. The value of the z-coordinate of a point on the cylinder represents the height h of that point. So, the cylinder can be parametrized as

$$x = 2\cos t, \quad y = 2\sin t, \quad z = h$$

using the angle t and the height h as two parameters.

Note the difference between the x and y equations of the surfaces of revolution in the previous three examples and in this one. In previous examples, just as on the figure with a surface of revolution, the value of r is not constant. In contrast, the value of r on a vertical cylinder is fixed (see the figure above on which the value of r is fixed and equal to a).

- Similarly, the cylinder $y^2 + z^2 = 4$ is based on the circle $y = 2\cos t$, and $z = 2\sin t$ in yz-plane. The value of the x-coordinate of a point on the cylinder can also be considered to be the height. Using h for that value, the cylinder can be parametrized as

$$x = h, \quad y = 2\cos t, \quad z = 2\sin t.$$

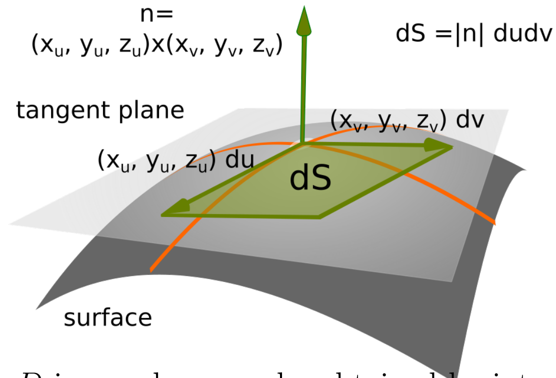
Tangent Plane and Surface Area

A parametric surface $\vec{r} = \langle x(u, v), y(u, v), z(u, v)\rangle$ has derivative vectors $\vec{r_u} = \langle x_u, y_u, z_u\rangle$ and $\vec{r_v}\langle x_v, y_v, z_v\rangle$ in the tangent plane at any point. Thus, the vector

$$\overrightarrow{r_u} \times \overrightarrow{r_v} = \langle x_u, y_u, z_u\rangle \times \langle x_v, y_v, z_v\rangle.$$

is perpendicular to the tangent plane.

In case when the surface $z = z(x, y)$ is given by the parameters $u = x$ and $v = y$, this reduces to the familiar formula $\langle -z_x, -z_y, 1\rangle$.



The surface area S of this surface over the region D in uv-plane can be obtained by integrating surface area elements dS over sub-rectangles of region D. The area of each element dS can be approximated with the area of the parallelogram in the tangent plane just as before when we considered surfaces parametrized using x and y.

Recall that the area of a parallelogram formed by two vectors is the length of their cross product. Thus,

$$dS = |\langle x_u, y_u, z_u\rangle \times \langle x_v, y_v, z_v\rangle| dudv \text{ or, written shortly, } dS = |\overrightarrow{r_u} \times \overrightarrow{r_v}| dudv.$$

The formula for the total surface area S can be obtained by integrating dS over entire region D. Thus

$$S = \int\int_D dS = \int\int_D |\overrightarrow{r_u} \times \overrightarrow{r_v}|\ dudv = \int\int_D |\langle x_u, y_u, z_u\rangle \times \langle x_v, y_v, z_v\rangle|\ dudv.$$

In the case when the surface is parametrized by $u = x$ and $v = y$ as $z = z(x, y)$ this formula becomes the one we have encountered before.

$$S = \int\int_D \sqrt{1 + z_x^2 + z_y^2}\ dxdy$$

Practice problems.

1. Find the equation of the tangent plane to given parametric surface at the specified point.

   a) $x = u + v \quad y = 3u^2 \quad z = u - v; \quad (2, 3, 0)$

   b) $x = uv \quad y = ue^v \quad z = ve^u; \quad (0, 0, 0)$

2. Find the area of the surface.

   a) The part of the plane $z = x + 2y$ that lies above the triangle with vertices (0,0), (1,1) and (0,1).

   b) The part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$. Write down the parametric equations of the cone first. Then find the surface area using the parametric equations.

   c) The part of the surface $z = y^2 + x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. Write down the parametric equations of the paraboloid and use them to find the surface area.

3. Consider the cylinder $x^2 + z^2 = 4$.

a) Write down the parametric equations of this cylinder.

b) Using the parametric equations, find the tangent plane to the cylinder at the point $(0, 3, 2)$.

c) Using the parametric equations and formula for the surface area for parametric curves, show that the surface area of the cylinder $x^2 + z^2 = 4$ for $0 \le y \le 5$ is 20π.

Solutions.

1. a) $\overrightarrow{r} = \langle u+v, 3u^2, u-v\rangle$ so that $\overrightarrow{r_u} = \langle 1, 6u, 1\rangle$ and $\overrightarrow{r_v} = \langle 1, 0, -1\rangle$, and, hence, $\overrightarrow{r_u} \times \overrightarrow{r_v} = \langle -6u, 2, -6u\rangle$. Next, we need to find the u-value (the v-value is not needed since v does not appear in this product) that corresponds to the point $(2, 3, 0)$. When $(x, y, z) = (2, 3, 0)$, $x = u + v = 2$ and $z = u - v = 0$. From this we have that $u = v = 1$ and this agrees with $3 = y = 3u^2$. Plugging $u = v = 1$ in the cross product produces a normal vector $\langle -6, 2, -6\rangle$ so an equation of the tangent plane is $-6(x-2) + 2(y-3) - 6(z-0) = 0 \Rightarrow 3x - y + 3z = 3$.

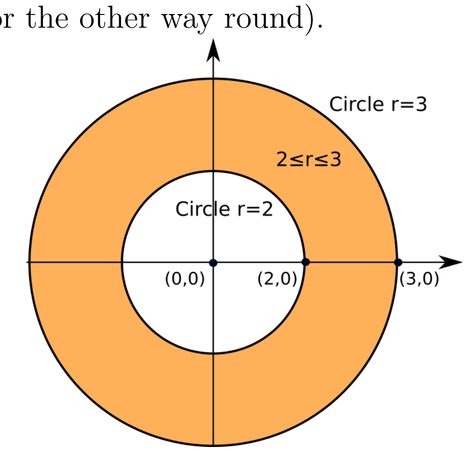
b) $\overrightarrow{r} = \langle uv, ue^v, ve^u\rangle$ so that $\overrightarrow{r_u} = \langle v, e^v, ve^u\rangle$ and $\overrightarrow{r_v} = \langle u, ue^v, e^u\rangle$, and, hence, $\overrightarrow{r_u} \times \overrightarrow{r_v} = \langle e^{uv}(1-uv), ve^u(u-1), ue^v(v-1)\rangle$. The u and v values that correspond to the point $(0, 0, 0)$ can be obtained from $(x, y, z) = (0, 0, 0)$, $ue^v = 0 \Rightarrow u = 0$ and $ve^u = 0 \Rightarrow v = 0$. From this we have that $u = v = 0$. Plugging $u = v = 1$ in the cross product produces a normal vector $\langle 1, 0, 0\rangle$ so an equation of the tangent plane is $1(x-0) + 0(y-0) + 0(z-0) = 0 \Rightarrow x = 0$ (the yz-plane).

2. a) You can use the standard xy-parametrization here so that $\overrightarrow{r} = \langle x, y, x+2y\rangle$ and $\overrightarrow{r_x} = \langle 1, 0, 1\rangle$, $\overrightarrow{r_y} = \langle 0, 1, 2\rangle$, and, hence, $\overrightarrow{r_x} \times \overrightarrow{r_y} = \langle -1, -2, 1\rangle$ so $|\overrightarrow{r_x} \times \overrightarrow{r_y}| = \sqrt{1+4+1} = \sqrt{6}$. Note that the same answer is obtained by the "old" formula for $dS = \sqrt{1 + z_x^2 + z_y^2}\, dxdy$ since $z = x + 2y \Rightarrow z_x = 1$ and $z_y = 2$. So, $S = \int\int_T \sqrt{6}\, dxdy$ where T is the given triangle. The bounds for x are 0 and 1 and y is bounded by the line connecting (0,0) and (1,1) from below and by the line that connects (0,1) and (1,1) from above. Thus $x \le y \le 1$. The surface area is

$$S = \int_0^1 \int_x^1 \sqrt{6}\, dxdy = \sqrt{6} \int_0^1 (1-x)dx = \sqrt{6}(1 - \frac{1}{2}) = \frac{\sqrt{6}}{2} = 1.22.$$

b) The cone can be parametrized by $x = r\cos t$, $y = r\sin t$, $z = \sqrt{x^2+y^2} = r$. Thus, $\overrightarrow{r} = \langle r\cos t, r\sin t, r\rangle$ and "u" is r and "v" is t (or the other way round).

Hence, $\overrightarrow{r_r} = \langle \cos t, \sin t, 1\rangle$ and $\overrightarrow{r_t} = \langle -r\sin t, r\cos t, 0\rangle$ so that $\overrightarrow{r_r} \times \overrightarrow{r_t} = \langle -r\cos t, -r\sin t, r\rangle$ and $|\overrightarrow{r_r} \times \overrightarrow{r_t}| = \sqrt{r^2\cos^2 t + r^2\sin^2 t + r^2} = \sqrt{r^2 + r^2} = \sqrt{2r^2} = \sqrt{2}r$.
The bounds for the integration are determined by the projection in the xy-plane which is the region between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$. Thus $0 \le t \le 2\pi$



and $2 \le r \le 3$. So, the surface area is

$$S = \int_0^{2\pi} dt \int_2^3 \sqrt{2} r dr = 2\pi\sqrt{2}(\frac{9}{2} - \frac{4}{2}) = 5\pi\sqrt{2}.$$

c) The paraboloid can be parametrized by $x = r\cos t$, $y = r\sin t$, $z = x^2 + y^2 = r^2$. Thus, $\overrightarrow{r} = \langle r\cos t, r\sin t, r^2\rangle$ and "u" is r and "v" is t (or the other way round). Hence, $\overrightarrow{r_r} = \langle \cos t, \sin t, 2r\rangle$ and $\overrightarrow{r_t} = \langle -r\sin t, r\cos t, 0\rangle$ so that $\overrightarrow{r_r} \times \overrightarrow{r_t} = \langle -2r^2\cos t,\ -2r^2\sin t, r\rangle$ and $|\overrightarrow{r_r} \times \overrightarrow{r_t}| = \sqrt{4r^4\cos^2 t + 4r^4\sin^2 t + r^2} = \sqrt{4r^4 + r^2} = \sqrt{r^2(4r^2+1)} = r\sqrt{4r^2+1}$.
The bounds for the integration are determined by the projection in the xy-plane which is the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. Thus $0 \le t \le 2\pi$ and $1 \le r \le 2$. So, the surface area is

$$S = \int_0^{2\pi} dt \int_1^2 r\sqrt{4r^2+1} dr \approx 2\pi 4.91 \approx 30.85.$$

3. a) The cylinder can be parametrized as $x = 2\cos t$, $y = h$, $z = 2\sin t$. Thus, $\overrightarrow{r} = \langle 2\cos t, h, 2\sin t\rangle$ and "u" is t and "v" is h (or the other way round).

b) $\overrightarrow{r_t} = \langle -2\sin t, 0, 2\cos t\rangle$ and $\overrightarrow{r_h} = \langle 0, 1, 0\rangle$ so that $\overrightarrow{r_t} \times \overrightarrow{r_h} = \langle -2\cos t, 0, -2\sin t\rangle$.
The t-value that corresponds to $(0, 3, 2)$ can be obtained from $x = 2\cos t = 0$ and $z = 2\sin t = 2$. Thus $t = \frac{\pi}{2}$. Plugging this value in the cross product produces a normal vector $\langle 0, 0, -2\rangle$. So, an equation of the tangent plane is $0(x-0)+0(y-3)-2(z-2) = 0 \Rightarrow z = 2$.

c) We already computed $\overrightarrow{r_t} \times \overrightarrow{r_h} = \langle -2\cos t, 0, -2\sin t\rangle$. The length is $\sqrt{4\cos^2 t + 4\sin^2 t} = \sqrt{4} = 2$. The t-bounds are 0 and 2π and the y-bounds are given to be $0 \le y \le 5$. Since $y = h$, $0 \le h \le 5$. So, the surface area is

$$S = \int_0^{2\pi}\int_0^5 2 dy dt = 2\pi(5)(2) = 20\pi.$$