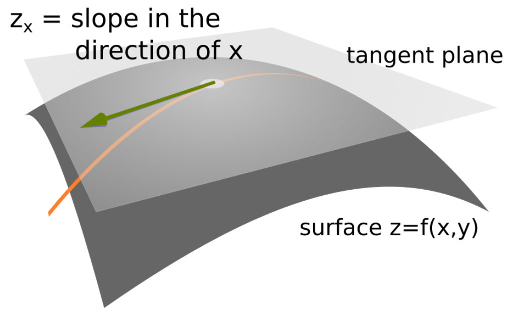
Calculus 3
Lia Vas

Partial Derivatives

Let $z = f(x, y)$ be a function of two variables. The **partial derivative of z with respect to x** is obtained by regarding y as a constant and differentiating z with respect to x. Notation:

$$z_x \quad \text{or} \quad \frac{\partial z}{\partial x}.$$

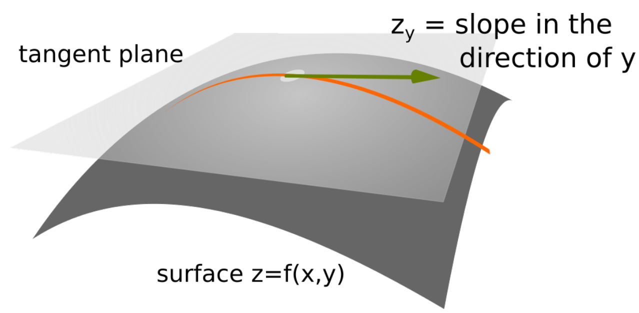


The derivative z_x at a point (x_0, y_0, z_0) on the surface $z = f(x, y)$, represent *the rate of change of function $z = f(x, y_0)$ in the direction of x-axis.* This value is the slope of the line tangent to the curve obtained by intersection of the surface $z = f(x, y)$ and the vertical plane $y = y_0$.

The vector on the above figure has coordinates $\langle 1, 0, z_x \rangle$. The coordinate values signify the fact that if x values change by 1 unit and y-values do not change (or change by 0 units), the value of the function changes by z_x units.

The **partial derivative of z with respect to y** is obtained by regarding x as a constant and differentiating z with respect to y. Notation:

$$z_y \quad \text{or} \quad \frac{\partial z}{\partial y}.$$



This derivative at a point (x_0, y_0, z_0) on the surface $z = f(x, y)$, represent*the rate of change of function $z = f(x_0, y)$ in the direction of x-axis.* This value is the slope of the line tangent to the curve obtained by intersection of the surface $z = f(x, y)$ and the vertical plane $x = x_0$.

The vector on the above figure has coordinates $\langle 0, 1, z_y \rangle$. The coordinate values signify the fact that if y values change by 1 unit and x-values do not change (or change by 0 units), the value of the function changes by z_y units.

The second partial derivatives can be found by differentiating the first partial derivatives. Differentiating z_x with respect to x produces

$$(z_x)_x = z_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$$

and with respect to y produces

$$(z_x)_y = z_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}.$$

Differentiating z_y with respect to x produces

$$(z_y)_x = z_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$$

and with respect to y produces

$$(z_y)_y = z_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}.$$

If z_{xy} and z_{yx} are both continuous, then

$$z_{xy} = z_{yx}.$$

In this case, differentiating z_x with respect to y and z_y with respect to x produces the same result.

Practice problems. Find the first and the second partial derivatives.

1. $z = 3x^2 + 2xy - 5y^2$
2. $z = e^x \sin y$
3. $z = e^{x^2 - xy}$
4. $z = x \ln(xy^2)$
5. $z = ax^2 e^{x^2 - xy}$ where a is a constant ("PChem problem").
6. Assuming that a is also a variable in the previous problem, find $\frac{\partial z}{\partial a}$.

Solutions.

1. $z_x = 6x + 2y$, $z_y = 2x - 10y$, $z_{xx} = 6$, $z_{xy} = z_{yx} = 2$, $z_{yy} = -10$.
2. $z_x = e^x \sin y$, $z_y = e^x \cos y$, $z_{xx} = e^x \sin y$, $z_{xy} = z_{yx} = e^x \cos y$, $z_{yy} = -e^x \sin y$.
3. $z_x = e^{x^2-xy}(2x-y)$, $z_y = e^{x^2-xy}(-x)$, $z_{xx} = e^{x^2-xy}(2x-y)^2 + e^{x^2-xy}(2)$, $z_{xy} = z_{yx} = e^{x^2-xy}(2x-y)(-x) - e^{x^2-xy}$, $z_{yy} = e^{x^2-xy}x^2$.
4. $z_x = \ln(xy^2) + x\frac{y^2}{xy^2} = \ln(xy^2) + 1$, $z_y = x\frac{2xy}{xy^2} = \frac{2x}{y}$, $z_{xx} = \frac{y^2}{xy^2} = \frac{1}{x}$, $z_{xy} = z_{yx} = \frac{2}{y}$, $z_{yy} = \frac{-2x}{y^2}$.
5. $z_x = 2axe^{x^2-xy} + ax^2e^{x^2-xy}(2x-y) = a(2x + 2x^3 - x^2y)e^{x^2-xy}$,
   $z_y = ax^2e^{x^2-xy}(-x) = -ax^3e^{x^2-xy}$.

   Then $z_{xx} = a(2 + 6x^2 - 2xy)e^{x^2-xy} + a(2x + 2x^3 - x^2y)e^{x^2-xy}(2x-y)$ and
   $z_{yy} = -ax^3e^{x^2-xy}(-x) = ax^4e^{x^2-xy}$.

   To find z_{xy}, either differentiate z_x with respect to y and get $z_{xy} = a(-x^2)e^{x^2-xy} + a(2x + 2x^3 - x^2y)e^{x^2-xy}(-x) = a(-x^2 - 2x^2 - 2x^4 + x^3y)e^{x^2-xy} = a(-3x^2 - 2x^4 + x^3y)e^{x^2-xy}$

   or differentiate z_y with respect to x and get $z_{xy} = -3ax^2e^{x^2-xy} - ax^3e^{x^2-xy}(2x-y) = a(-3x^2 - 2x^4 + x^3y)e^{x^2-xy}$
6. $z = ax^2e^{x^2-xy} \Rightarrow \frac{\partial z}{\partial a} = x^2e^{x^2-xy}$