Calculus 3
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Power Series

A **power series** is a series of the form $\sum_{n=1}^{\infty} a_n(x-a)^n.$

A series of the above form is said to be **a power series centered at** a**.** Note that it is **a function of** x. This function is **defined** for all values of x for which the series **converges** and, for a given power series, exactly one of the following three cases holds:

1. There is a positive number R such that the series converges if $|x-a| < R$ (that is $-R < x-a < R \Rightarrow a-R < x < a+R$) and diverges if $|x-a| > R$. Such number R is called the **radius of convergence**. Note that the convergence at **the endpoints** of the interval $(a-R, a+R)$ has to be checked separately using an appropriate convergence test.

2. The series converges for all values of x. In this case, the interval of convergence is $(-\infty, \infty)$ and the radius R is considered to be ∞.

3. The series converges just when $x = a$. In this case, the the interval of convergence is collapsed to $[a, a]$ and the radius R is considered to be 0.

To find the interval of convergence, in many cases the use of the Ratio or the Root Tests is a good way to start. Keep in mind that those tests require the series to have positive terms, so start by considering $\sum_{n=1}^{\infty} |a_n x^n|$.

Practice Problems. Find the intervals of convergence for the following power series.

1. $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$
2. $\sum_{n=1}^{\infty} \frac{x^n}{n}$
3. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$
4. $\sum_{n=0}^{\infty} n!\, x^n$
5. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n\, 2^n}$
6. $\sum_{n=1}^{\infty} \frac{(x-2)^{n+1}}{n\, 3^n}$
7. $\sum_{n=1}^{\infty} \frac{3^n x^n}{n+1}$
8. $\sum_{n=1}^{\infty} (-1)^n\, n\, 2^n x^n$

Solutions.

1. The series can be considered as a geometric series with $r = \frac{x}{2}$. Thus, it is convergent for $-1 < \frac{x}{2} < 1 \Rightarrow -2 < x < 2$ and divergent for $x \geq 2$ and $x \leq -2$.

   Alternatively, you can use the root test for $\sum_{n=1}^{\infty} |\frac{x}{2}|^n$ and evaluate $\lim_{n\to\infty} \sqrt[n]{|\frac{x}{2}|^n} = |\frac{x}{2}|$. This series is convergent for $|\frac{x}{2}| < 1 \Rightarrow -1 < \frac{x}{2} < 1 \Rightarrow -2 < x < 2$ and divergent for $|\frac{x}{2}| > 1 \Rightarrow x \geq 2$

and $x \leq -2$. Then you have to check the endpoints $|\frac{x}{2}| = 1 \Rightarrow x = \pm 2$ since the root test is inconclusive in this case. When $x = 2$, the series becomes $\sum_{n=1}^{\infty} \frac{2^n}{2^n} = 1 + 1 + \ldots$ which is divergent either by the divergence test or by the geometric series test ($r = 1$). When $x = -2$, the series becomes $\sum_{n=1}^{\infty} \frac{(-2)^n}{2^n} = \sum_{n=1}^{\infty} (-1)^n$ which is divergent either by the divergence test or by the geometric series test ($r = -1$).

2. Use the Ratio Test for $\sum_{n=1}^{\infty} |\frac{x^n}{n}| = \sum_{n=1}^{\infty} \frac{|x|^n}{n}$ to determine the interval of convergence $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)} \frac{n}{|x|^n} = \lim_{n\to\infty} \frac{|x|}{(n+1)} \frac{n}{1} = |x| \lim_{n\to\infty} \frac{n}{n+1} = |x| \frac{1}{1} = |x|$. Thus, the series is convergent when $|x| < 1 \Rightarrow -1 < x < 1$. The series is divergent if $x > 1$ or $x < -1$. Check the convergence on the endpoints of the interval (-1,1).

   When $x = 1$, the series becomes $\sum_{n=1}^{\infty} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$. This is a p-series with $p = 1$. Since $p = 1 \leq 1$, the series is divergent. When $x = -1$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. This is an alternating series with $b_n = \frac{1}{n}$. Since $\frac{1}{n}$ is decreasing and converges to 0, the series is convergent by the Alternating Series Test. Thus, the interval of convergence is $-1 \leq x < 1$.

3. Use the Ratio Test for $\sum_{n=1}^{\infty} |\frac{x^n}{n!}| = \sum_{n=1}^{\infty} \frac{|x|^n}{n!}$ to determine the interval of convergence: $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} \frac{n!}{|x|^n} = \lim_{n\to\infty} \frac{|x|}{1\cdot 2\cdot\ldots\cdot n\cdot(n+1)} \frac{1\cdot 2\cdot\ldots\cdot n}{1} = |x| \lim_{n\to\infty} \frac{1}{n+1} = |x| \frac{1}{\infty} = 0$. Since $0 < 1$ for any value of x, this series is convergent for any value of x (i.e. the interval of convergence is $(-\infty, \infty)$). So, this is an example of a series with infinite radius of convergence. In the next section, we will see that the sum of this series is e^x and the interval of convergence ensures that the equality $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ holds for any x.

4. Use the Ratio Test for $\sum_{n=0}^{\infty} |n!\, x^n| = \sum_{n=0}^{\infty} n!|x|^n$ to determine the interval of convergence: $\lim_{n\to\infty} \frac{(n+1)!|x|^{n+1}}{n!|x|^n} = \lim_{n\to\infty} \frac{1\cdot 2\cdot\ldots\cdot n\cdot(n+1)|x|}{1\cdot 2\cdot\ldots\cdot n} = |x| \lim_{n\to\infty} n+1 = |x|\infty = \infty$ for any $x \neq 0$. Since $\infty > 1$, this series is divergent for any value of $x \neq 0$. Just in the case when $x = 0$, the limit becomes $|x| \lim_{n\to\infty} n + 1 = 0 \lim_{n\to\infty} n + 1 = 0$. Another way to see the convergence at $x = 0$ is to observe that the series is equal to $\sum_{n=0}^{\infty} n!\, 0^n = 1 + 0 + \ldots = 1$ (0^0 is usually defined to be 1). So, the interval of convergence is collapsed to a point $x = 0$ (or the interval $[0, 0]$). The series diverges for all $x \neq 0$ and this is an example of a series with the radius of convergence equal to zero.

5. Use the Ratio Test for $\sum_{n=1}^{\infty} \frac{|x-1|^n}{n2^n}$. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} =$

$$\lim_{n\to\infty} \frac{|x-1|^{n+1}}{(n+1)\, 2^{n+1}} \frac{n\, 2^n}{|x-1|^n} = \lim_{n\to\infty} \frac{|x-1|}{(n+1)2} \frac{n}{1} = \frac{|x-1|}{2} \lim_{n\to\infty} \frac{n}{n+1} = \frac{|x-1|}{2}\, 1 = \frac{|x-1|}{2}$$

   Thus, the series is convergent when $\frac{|x-1|}{2} < 1 \Rightarrow |x-1| < 2 \Rightarrow -2 < x - 1 < 2 \Rightarrow -1 < x < 3$. The series is divergent if $x > 3$ or $x < -1$. Check the convergence on the endpoints of the interval (-1,3).

   When $x = 3$, the series becomes $\sum_{n=1}^{\infty} \frac{2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$. This is a p-series with $p = 1$. Since $p = 1 \leq 1$, the series is divergent. When $x = -1$, the series becomes $\sum_{n=1}^{\infty} \frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. This is an alternating series with $b_n = \frac{1}{n}$. Since $\frac{1}{n}$ is decreasing and converges to 0, the series is convergent by the Alternating Series Test. Thus, the interval of convergence is $-1 \leq x < 3$.

6. Use the Ratio Test for $\sum_{n=1}^{\infty} \frac{|x-2|^{n+1}}{n3^n}$.

$$\lim_{n\to\infty} \frac{|x-2|^{n+2}}{(n+1)\ 3^{n+1}} \frac{n\ 3^n}{|x-2|^{n+1}} = \lim_{n\to\infty} \frac{|x-2|}{(n+1)3}\ \frac{n}{1} = \frac{|x-2|}{3} \lim_{n\to\infty} \frac{n}{n+1} = \frac{|x-2|}{3}$$

Thus, the series is convergent when $\frac{|x-2|}{3} < 1 \Rightarrow |x-2| < 3 \Rightarrow -3 < x-2 < 3 \Rightarrow -1 < x < 5$. The series is divergent if $x > 5$ or $x < -1$. Checking the convergence on the endpoints is similar to the previous problem. For $x = 5$, using the p-test you can obtain that the series is divergent and for $x = -1$, you can use the Alternating Series Test to show convergence. Thus, the interval of convergence is $-1 \le x < 5$.

7. The problem is similar to previous two. Using the Ratio Test obtain that the series is convergent when $3|x| < 1 \Rightarrow |x| < \frac{1}{3} \Rightarrow \frac{-1}{3} < x < \frac{1}{3}$. Test the endpoints then. When $x = \frac{1}{3}$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{n+1}$ which can be tested using the Integral Test or by noting that $\sum_{n=1}^{\infty} \frac{1}{n+1} = \sum_{n=2}^{\infty} \frac{1}{n}$ and using the p-test. In either case, the series is divergent for $x = \frac{1}{3}$. When $x = \frac{-1}{3}$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$. Since $\frac{1}{n+1}$ is decreasing and converges to 0, the series is convergent by the Alternating Series Test. Thus, the interval of convergence is $\frac{-1}{3} \le x < \frac{1}{3}$.

8. Use the Ratio Test for $\sum_{n=1}^{\infty} |-1|^n n 2^n |x|^n = \sum_{n=1}^{\infty} n2^n|x|^n$.

$$\lim_{n\to\infty} \frac{(n+1)2^{n+1}|x|^{n+1}}{n2^n|x|^n} = \lim_{n\to\infty} \frac{(n+1)2|x|}{n} = 2|x| \lim_{n\to\infty} \frac{n+1}{n} = 2|x|$$

Thus, the series is convergent when $2|x| < 1 \Rightarrow |x| < \frac{1}{2} \Rightarrow \frac{-1}{2} < x < \frac{1}{2}$. Test the endpoints then. When $x = \frac{1}{2}$, the series becomes $\sum_{n=1}^{\infty} (-1)^n n$ which can be tested using the Divergence test. Since the limit of the n-th term does not exist, the series is divergent. When $x = \frac{-1}{2}$, the series becomes $\sum_{n=1}^{\infty} (-1)^n n(-1)^n = \sum_{n=1}^{\infty} n$ which can be tested using the Divergence test. Since the limit of the n-th term is infinity, the series is divergent. Thus, the interval of convergence is $\frac{-1}{2} < x < \frac{1}{2}$.