

Review for Exam 1

1. **Surfaces.** Describe the following surfaces.

- (a) $x^2 + y^2 = 9$ (b) $x^2 + y^2 + z^2 = 4$ (c) $z = 1$
(d) $2x + 3y + z = 6$ (e) $z = \sqrt{x^2 + y^2}$ (f) $z = x^2 + y^2$

2. **Review of Vectors.**

- (a) Let $\vec{a} = \langle 3, 4, 0 \rangle$ and $\vec{b} = \langle -1, 4, 2 \rangle$. Find $|\vec{a}|$, $2\vec{a} + 3\vec{b}$, $3\vec{a} - 2\vec{b}$. Find the normalization of \vec{a} .
(b) Let $\vec{a} = \langle 1, 0, 1 \rangle$ and $\vec{b} = \langle 1, -1, 0 \rangle$. Find $\vec{a} \cdot \vec{b}$. Find the projection of \vec{b} onto \vec{a} .
(c) Let $\vec{a} = \langle -5, 3, 7 \rangle$ and $\vec{b} = \langle 6, -8, 2 \rangle$. Find $\vec{a} \cdot \vec{b}$ and $\vec{a} \times \vec{b}$. Determine if the vectors are parallel, perpendicular or neither.

3. **Lines and Planes.**

- (a) Find an equation of the line through the point $(-2, 4, 10)$ and parallel to the vector $\langle 3, 1, -8 \rangle$.
(b) Find an equation of the line through the point $(1, 0, 6)$ and perpendicular to the plane $x + 3y + z = 5$.
(c) Find an equation of the line through the points $(3, 1, -1)$ and $(3, 2, -6)$.
(d) Find an equation of the plane through the point $(6, 3, 2)$ and perpendicular to the vector $\langle -2, 1, 5 \rangle$.
(e) Find an equation of the plane through the point $(4, -2, 3)$ and parallel to the plane $3x - 7z = 12$.
(f) Find an equation of the plane through the points $(0, 1, 1)$, $(1, 0, 1)$ and $(1, 1, 0)$.
(g) Find an equation of the line of the intersection of the planes $x + y - z = 0$ and $2x - 5y - z = 1$.

4. **Curves in Space.**

- (a) Consider the curve $x = \cos t$ $y = \sin t$ $z = t$. Find an equation of the tangent line to the curve at the point where $t = 0$. Find the length of the curve from $t = 0$ to $t = 1$.
(b) Let C be the curve of intersection of the cylinder $x^2 + y^2 = 1$ with the plane $y + z = 2$. Find parametric equations of this curve and an equation of the tangent line to the curve at the point where $t = 0$ in parametrization. Using the calculator, estimate the length of the curve from $t = 0$ to $t = \pi/2$.
(c) Consider the curve C which is the intersection of the surfaces $x^2 + y^2 = 9$ and $z = 1 - y^2$.
i) Find the parametric equations that represent the curve C . ii) Find the equation of the tangent line to the curve C at point $(0, 3, -8)$. iii) Find the length of the curve from $(3, 0, 1)$ to $(0, 3, -8)$. You can use the calculator for the integral that you are going to get.

- (b) The number N of bacteria in a culture depends on temperature T and pressure P which depend on time t in minutes. Assume that 3 minutes after the experiment started, the pressure is increasing at a rate of 0.1 kPa/min and the temperature at a rate of 0.5 K/min. The number of bacteria changes at the rates of 3 bacteria per kPa and 5 bacteria per Kelvin. i) Find the rate at which the number of bacteria is increasing 3 minutes after the experiment started.
- ii) Assume that the rates of 3 bacteria per kPa and 5 bacteria per Kelvin are constant. If there is 300 bacteria initially when $T = 305$ K and $P = 102$ kPa, estimate the number of bacteria when $T = 309$ K and $P = 100$ kPa.
- (c) The number of flowers N in a closed environment depends on the amount of sunlight S that the flowers receive and the temperature T of the environment. Assume that the number of flowers changes at the rates $N_S = 2$ and $N_T = 4$. i) If there are 100 flowers when $S = 50$ and $T = 70$, estimate the number of flowers when $S = 52$ and $T = 73$.
- ii) If the temperature depends on time as $T(t) = 85 - \frac{8}{1+t^2}$ and the amount of sunlight decreases on time as $S(t) = \frac{1}{t}$, find the rate of change of the flower population at time $t = 2$ days.
- (d) The temperature at a point (x, y) is $T(x, y)$, measured in degrees Celsius. A bug crawls so that its position after t seconds is given by $x = \sqrt{1+t}$, $y = 2 + \frac{1}{3}t$ where x and y are measured in centimeters. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. Determine how fast the temperature rises on the bug's path after 3 seconds.
- (e) An object moves in the space away from its initial position so that after t hours it is at $x = 2t$ and $y = 2t^2 - 1$ and $z = \sqrt{3t+1}$ miles from its initial position. Find the speed of that object 5 hours after it started moving. (Recall that the speed is the length of the tangent vector at a point.)

Solutions

More detailed solutions of the problems can be found on the class handouts.

1. Surfaces.

- (a) Cylinder, base is a circle $x^2 + y^2 = 9$ in xy -plane. (b) Sphere, center at origin, radius 2.
 (c) Horizontal plane, passes $(0, 0, 1)$. (d) Plane, passes $(3, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 6)$.
 (e) Cone, obtained by rotating a line $z = y$ in zy -plane about z -axis. (f) Paraboloid, obtained by rotating a parabola $z = y^2$ in zy -plane about z -axis.

2. Review of Vectors.

- (a) $|\vec{a}| = 5$, $2\vec{a} + 3\vec{b} = \langle 3, 20, 6 \rangle$, $3\vec{a} - 2\vec{b} = \langle 11, 4, -4 \rangle$. $\frac{\vec{a}}{|\vec{a}|} = \langle \frac{3}{5}, \frac{4}{5}, 0 \rangle$.
 (b) $\vec{a} \cdot \vec{b} = 1$ The projection of \vec{b} onto \vec{a} is $\langle \frac{1}{2}, 0, \frac{1}{2} \rangle$.
 (c) $\vec{a} \cdot \vec{b} = -40 \neq 0$ so the vectors are not perpendicular, $\vec{a} \times \vec{b} = \langle 62, 52, 22 \rangle \neq \langle 0, 0, 0 \rangle$ so the vectors are not parallel.

3. Lines and Planes.

- (a) $x = -2 + 3t$ $y = 4 + t$ $z = 10 - 8t$ (b) $x = 1 + t$ $y = 3t$ $z = 6 + t$
 (c) $x = 3$ $y = 1 + t$ $z = -1 - 5t$ (d) $-2x + y + 5z = 1$
 (e) $3x - 7z = -9$ (f) $x + y + z = 2$ (g) $x = 1 + 6t$ $y = t$ $z = 1 + 7t$

4. Curves in Space.

- (a) Tangent: $x = 1, y = t, z = t$. Length: $\sqrt{2}$ (b) Tangent: $x = 1, y = t, z = 2 - t$. Length: 1.91
 (c) i) $x = 3 \cos t, y = 3 \sin t, z = 1 - y^2 = 1 - 9 \sin^2 t$. ii) $(0, 3, -8)$ corresponds to $t = \pi/2$. Plugging $\pi/2$ in derivative gives you $\langle -3, 0, 0 \rangle$. Tangent line: $x = -3t$ $y = 3$ $z = -8$. iii) $(3, 0, 1)$ corresponds to $t = 0$ and $(0, 3, -8)$ to $t = \pi/2$. So, the bounds of integration are 0 to $\pi/2$. The length is 10.48.
 (d) i) $y = 4 \cos t, z = 4 \sin t, x = 8 - y^2 - z = 8 - 16 \cos^2 t - 4 \sin t$. ii) $(-8, -4, 0)$ corresponds to $t = \pi$. Plugging π in derivative gives you $\langle 4, 0, -4 \rangle$. Tangent line: $x = 4t - 8$ $y = -4$ $z = -4t$. iii) $(4, 0, 4)$ corresponds to $t = \pi/2$ and $(-8, -4, 0)$ to $t = \pi$. So, the bounds of integration are $\pi/2$ to π . The length is 14.515.
 (e) A set of parametric equations for the three curves in the intersection is
 $x = 2 \cos t, y = 2 \sin t, z = 0$ with $0 \leq t \leq \frac{\pi}{2}$,
 $x = t, y = 0, z = 4 - t^2$ with $0 \leq t \leq 2$, and
 $x = 0, y = t, z = 4 - t^2$ with $0 \leq t \leq 2$.

The three derivative vectors and length elements are

$$\begin{aligned} x' = -2 \sin t, y' = 2 \cos t, z' = 0 &\Rightarrow ds = \sqrt{4 \sin^2 t + 4 \cos^2 t} dt = \sqrt{4} dt = 2 dt \\ x' = 1, y' = 0, z' = -2t &\Rightarrow ds = \sqrt{1 + 4t^2} dt, \text{ and} \\ x' = 0, y' = 1, z' = -2t &\Rightarrow ds = \sqrt{1 + 4t^2} dt. \end{aligned}$$

The total length is $\int_0^{\pi/2} 2 dt + \int_0^2 \sqrt{1 + 4t^2} dt + \int_0^2 \sqrt{1 + 4t^2} dt = \pi + 4.65 + 4.65 \approx 12.44$.

5. Partial Derivatives.

- (a) $z_x = 6x + 2y$, $z_y = 2x - 10y$, $z_{xx} = 6$, $z_{xy} = z_{yx} = 2$, $z_{yy} = -10$
- (b) $z_x = e^x \sin y$, $z_y = e^x \cos y$, $z_{xx} = e^x \sin y$, $z_{xy} = z_{yx} = e^x \cos y$, $z_{yy} = -e^x \sin y$
- (c) $z_x = 2axe^{x^2-xy} + ax^2e^{x^2-xy}(2x - y) = a(2x + 2x^3 - x^2y)e^{x^2-xy}$, $z_y = ax^2e^{x^2-xy}(-x) = -ax^3e^{x^2-xy}$. Then $z_{xx} = a(2 + 6x^2 - 2xy)e^{x^2-xy} + a(2x + 2x^3 - x^2y)e^{x^2-xy}(2x - y)$ and $z_{yy} = -ax^3e^{x^2-xy}(-x) = ax^4e^{x^2-xy}$. Differentiating z_x with respect to y get $z_{xy} = a(-x^2)e^{x^2-xy} + a(2x + 2x^3 - x^2y)e^{x^2-xy}(-x) = a(-x^2 - 2x^2 - 2x^4 + x^3y)e^{x^2-xy} = a(-3x^2 - 2x^4 + x^3y)e^{x^2-xy}$. Alternatively, differentiating z_y with respect to x get $z_{xy} = -3ax^2e^{x^2-xy} - ax^3e^{x^2-xy}(2x - y) = a(-3x^2 - 2x^4 + x^3y)e^{x^2-xy}$
- (d) $z_x = \ln(xy^2) + 1$, $z_y = 2x/y$, $z_{xx} = 1/x$, $z_{xy} = z_{yx} = 2/y$, $z_{yy} = -2x/y^2$
- (e) $z_x = -(y^2 + 2xz)/(2yz + x^2)$, $z_y = -(2xy + z^2)/(2yz + x^2)$. At $(1, 1, 1)$, $z_x = -1$, $z_y = -1$.
- (f) $z_x = -(1 + \sin(x + y + z))/(-y + \sin(x + y + z))$ and $z_y = -(-z + \sin(x + y + z))/(-y + \sin(x + y + z))$. At $(0, 1, -1)$, $z_x = 1$ and $z_y = 1$.
- (g) $z'(t) = (6x + 2y)(2t) + (2x - 10y)(-3t^2)$; $z'(0) = 0$
- (h) $z'(t) = (\ln(x + 2y) + x/(x + 2y))(-\sin t) + (2x/(x + 2y))(\cos t)$; $z'(0) = 2$.
- (i) $z_s = (e^x y + y^2)t + (e^x + 2xy)2s$; $z_t = (e^x y + y^2)s + (e^x + 2xy)2t$; $z_s(1, 1) = 4e + 12$ and $z_t(1, 1) = 4e + 12$
- (j) $z_s = (2x + y)(-e^t \sin s) + x(e^t \cos s)$, $z_t = (2x + y)(e^t \cos s) + x(e^t \sin s)$, $z_s(\pi, 0) = 1$ and $z_t(\pi, 0) = 2$

6. Tangent plane.

- (a) $8x + 10y - z = 9$ (b) $e^3y - z = e^3$
- (c) $F_x = 2x, F_y = 4y, F_z = 6z$. At $(4, -1, 1)$ this produces vector $\langle 8, -4, 6 \rangle$. The tangent plane is $4x - 2y + 3z = 21$.
- (d) $F_x = y^2 + 2xz, F_y = 2xy + z^2, F_z = 2yz + x^2$. At $(1, 1, 1)$ this produces vector $\langle 3, 3, 3 \rangle$. The tangent plane is $x + y + z = 3$.
- (e) $F_x = 1 + \sin(x + y + z), F_y = -z + \sin(x + y + z), F_z = -y + \sin(x + y + z)$. At $(0, 1, -1)$ this produces vector $\langle 1, 1, -1 \rangle$. The tangent plane is $x + y - z = 2$.

7. Linear Approximation. (a) $f(2.02, 3.1) \approx 5.38$ (b) $f(.9, 1.99) \approx 2.92$ (c) $f(1.95, 1.08) \approx 2.847$ (d) $f(6.9, 2.06) \approx -0.28$

8. Applications.

- (a) -27 liter per second
- (b) i) Since $P'(3) = 0.1$, $T'(3) = 0.5$, $N_P = 3$, and $N_T = 5$, and $N'(t) = N_T T' + N_P P'$ We have that $N'(3) = 5 \cdot 0.5 + 3 \cdot 0.1 = 2.8$ bacteria/minute. ii) Using linear approximation formula, $N(T, P) \approx N(T_0, P_0) + N_T \cdot (T - T_0) + N_P \cdot (P - P_0) \Rightarrow N(309, 100) \approx N(305, 102) + 5(309 - 305) + 3(100 - 102) = 300 + 5(4) + 3(-2) = 314$ bacteria.
- (c) i) $100 + 2(52 - 50) + 4(73 - 70) = 116$ flowers. ii) $N' = N_S S' + N_T T' = 2 \frac{-1}{4} + 4 \frac{32}{25} = 4.62$ flowers/day.
- (d) 2 degrees Celsius per second (e) Speed = 20.1 miles per hour.