Calculus 3
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Review for Exam 1

1. **Surfaces.** Describe the following surfaces.

   (a) $x^2+y^2=9$ (b) $x^2+y^2+z^2=4$ (c) $z=1$
   (d) $2x+3y+z=6$ (e) $z=\sqrt{x^2+y^2}$ (f) $z=x^2+y^2$

2. **Review of Vectors.**

   (a) Let $\overrightarrow{a}=\langle 3,4,0\rangle$ and $\overrightarrow{b}=\langle -1,4,2\rangle$. Find $|\overrightarrow{a}|$, $2\overrightarrow{a}+3\overrightarrow{b}$, $3\overrightarrow{a}-2\overrightarrow{b}$. Find the normalization of $\overrightarrow{a}$.

   (b) Let $\overrightarrow{a}=\langle 1,0,1\rangle$ and $\overrightarrow{b}=\langle 1,-1,0\rangle$. Find $\overrightarrow{a}\cdot\overrightarrow{b}$. Find the projection of $\overrightarrow{b}$ onto $\overrightarrow{a}$.

   (c) Let $\vec{a}=\langle -5,3,7\rangle$ and $\vec{b}=\langle 6,-8,2\rangle$. Find $\overrightarrow{a}\cdot\overrightarrow{b}$ and $\overrightarrow{a}\times\overrightarrow{b}$. Determine if the vectors are parallel, perpendicular or neither.

3. **Lines and Planes.**

   (a) Find an equation of the line through the point $(-2,4,10)$ and parallel to the vector $\langle 3,1,-8\rangle$.

   (b) Find an equation of the line through the point $(1,0,6)$ and perpendicular to the plane $x+3y+z=5$.

   (c) Find an equation of the line through the points $(3,1,-1)$ and $(3,2,-6)$.

   (d) Find an equation of the plane through the point $(6,3,2)$ and perpendicular to the vector $\langle -2,1,5\rangle$.

   (e) Find an equation of the plane through the point $(4,-2,3)$ and parallel to the plane $3x-7z=12$.

   (f) Find an equation of the plane through the points $(0,1,1)$, $(1,0,1)$ and $(1,1,0)$.

   (g) Find an equation of the line of the intersection of the planes $x+y-z=0$ and $2x-5y-z=1$.

4. **Curves in Space.**

   (a) Consider the curve $x=\cos t \quad y=\sin t \quad z=t$. Find an equation of the tangent line to the curve at the point where $t=0$. Find the length of the curve from $t=0$ to $t=1$.

   (b) Find the length of the curve (given below) from $(4,0,1)$ to $(0,0,-1)$.

   $$x=2+2\cos t,\quad y=2\sin t,\quad z=\cos t-\sin^2 t$$

   (c) Consider the curve C which is the intersection of the surfaces

   $$x^2+y^2=9 \text{ and } z=1-y^2.$$

i) Find the parametric equations that represent the curve C. ii) Find the equation of the tangent line to the curve C at point $(0, 3, -8)$. iii) Find the length of the curve from $(3, 0, 1)$ to $(0, 3, -8)$. You can use the calculator for the integral that you are going to get.

(d) Consider the curve C which is the intersection of the surfaces

$$y^2 + z^2 = 16 \qquad \text{and} \qquad x = 8 - y^2 - z$$

i) Find the parametric equations that represent the curve C. ii) Find the equation of the tangent line to the curve C at point $(-8, -4, 0)$. iii) Find the length of the curve from $(4, 0, 4)$ to $(-8, -4, 0)$. Use the calculator to evaluate the final integral.

(e) Find the length of the boundary of the part of the paraboloid $z = 4 - x^2 - y^2$ in the first octant.

5. **Partial Derivatives.** Find the indicated derivatives.

(a) $z = 3x^2 + 2xy - 5y^2$; z_x, z_y, z_{xx}, z_{xy}, z_{yx} and z_{yy}.

(b) $z = e^x \sin y$; z_x, z_y, z_{xx}, z_{xy}, z_{yx} and z_{yy}.

(c) $z = ax^2 e^{x^2 - xy}$ where a is a constant; z_x, z_y, z_{xx}, z_{xy} and z_{yy}.

(d) $z = x \ln(xy^2)$; z_x, z_y, z_{xx}, z_{xy} and z_{yy}.

(e) $xy^2 + yz^2 + zx^2 = 3$; z_x and z_y at $(1, 1, 1)$.

(f) $x - yz = \cos(x + y + z)$; z_x and z_y at $(0, 1, -1)$.

(g) $z = 3x^2 + 2xy - 5y^2$, $x = 2 + t^2$, $y = 1 - t^3$; $z'(t)$ when $t = 0$.

(h) $z = x \ln(x + 2y)$, $x = \cos t$, $y = \sin t$; $z'(t)$ when $t = 0$.

(i) $z = x^2 + xy$, $x = e^t \cos s$, $y = e^t \sin s$; z_s and z_t at $(\pi, 0)$.

6. **Tangent plane.** Find an equation of the tangent plane to a given surface at a specified point.

(a) $z = y^2 - x^2$, at $(-4, 5, 9)$

(b) $z = e^x \ln y$, at $(3, 1, 0)$

(c) $x^2 + 2y^2 + 3z^2 = 21$, at $(4, -1, 1)$

(d) $xy^2 + yz^2 + zx^2 = 3$; at $(1, 1, 1)$.

(e) $x - yz = \cos(x + y + z)$; at $(0, 1, -1)$.

7. **Linear Approximation.**

(a) If $f(2, 3) = 5$, $f_x(2, 3) = 4$ and $f_y(2, 3) = 3$, approximate $f(2.02, 3.1)$.

(b) If $f(1, 2) = 3$, $f_x(1, 2) = 1$ and $f_y(1, 2) = -2$, approximate $f(.9, 1.99)$.

(c) Find the linear approximation of $z = \sqrt{20 - x^2 - 7y^2}$ at $(2, 1)$ and use it to approximate the value at $(1.95, 1.08)$.

(d) Find the linear approximation of $z = \ln(x - 3y)$ at $(7, 2)$ and use it to approximate the value at $(6.9, 2.06)$.

8. **Applications.**

(a) The number N of bacteria in a culture depends on temperature T and pressure P which depend on time t in minutes. Assume that 3 minutes after the experiment started, the pressure is increasing at a rate of 0.1 kPa/min and the temperature at a rate of 0.5 K/min. The number of bacteria changes at the rates of 3 bacteria per kPa and 5 bacteria per Kelvin.

(i) Find the rate at which the number of bacteria is increasing 3 minutes after the experiment started.

(ii) Assume that the rates of 3 bacteria per kPa and 5 bacteria per Kelvin are constant. If there is 300 bacteria initially when $T = 305$ K and $P = 102$ kPa, estimate the number of bacteria when $T = 309$ K and $P = 100$ kPa.

(b) The pressure of 1 mole of an ideal gas is increasing at a rate of 0.05 kPa/s and the temperature is rising at a rate of 0.15 K/s. The pressure P (in kPa), the volume V (in liters) and the temperature T (in Kelvins) are related by the equation $PV = 8.31T$. Find the rate of change of the volume when the pressure is 20 kPa and temperature 320 K.

(c) The number of flowers N in a closed environment depends on the amount of sunlight S that the flowers receive measured in trillions of photons and the temperature T of the environment measured in Fahrenheit. Assume that the number of flowers changes at the rates of 2 flowers per trillions of photons and 4 flowers per Fahrenheit.

(i) If there are 100 flowers when $S = 50$ and $T = 70$, estimate the number of flowers when $S = 52$ and $T = 73$.

(ii) If the temperature depends on time as $T(t) = 85 - \frac{8}{1+t^2}$ and the amount of sunlight decreases on time as $S(t) = \frac{1}{t}$, find the rate of change of the flower number 2 days after the initial consideration.

(d) The temperature at a point (x, y) is $T(x, y)$, measured in degrees Celsius. A bug crawls so that its position after t seconds is given by $x = \sqrt{1+t}$, $y = 2 + \frac{1}{3}t$ where x and y are measured in centimeters. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. Determine how fast the temperature rises on the bug's path after 3 seconds.

(e) An object moves in the space away from its initial position so that after t hours it is at $x = 2t$ and $y = 2t^2 - 1$ and $z = \sqrt{3t+1}$ miles from its initial position. Find the speed of that object 5 hours after it started moving. (Recall that the speed is the length of the tangent vector at a point.)

Solutions

More detailed solutions of the problems can be found on the class handouts.

1. Surfaces.

(a) Cylinder, base is a circle $x^2 + y^2 = 9$ in xy-plane. (b) Sphere, center at origin, radius 2.

(c) Horizontal plane, passes $(0, 0, 1)$. (d) Plane, passes $(3, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 6)$.

(e) Cone, obtained by rotating a line $z = y$ in zy-plane about z-axis. (f) Paraboloid, obtained by rotating a parabola $z = y^2$ in zy-plane about z-axis.

2. Review of Vectors.

   (a) $|\overrightarrow{a}| = 5$, $2\overrightarrow{a} + 3\overrightarrow{b} = \langle 3, 20, 6\rangle$, $3\overrightarrow{a} - 2\overrightarrow{b} = \langle 11, 4, -4\rangle$. $\frac{\vec{a}}{|\vec{a}|} = \langle \frac{3}{5}, \frac{4}{5}, 0\rangle$.

   (b) $\overrightarrow{a} \cdot \overrightarrow{b} = 1$ The projection of $\overrightarrow{b}$ onto $\overrightarrow{a}$ is $\langle \frac{1}{2}, 0, \frac{1}{2}\rangle$.

   (c) $\overrightarrow{a} \cdot \overrightarrow{b} = -40 \neq 0$ so the vectors are not perpendicular, $\overrightarrow{a} \times \overrightarrow{b} = \langle 62, 52, 22\rangle \neq \langle 0, 0, 0\rangle$ so the vectors are not parallel.

3. Lines and Planes.

   (a) $x = -2 + 3t \quad y = 4 + t \quad z = 10 - 8t$ (b) $x = 1 + t \quad y = 3t \quad z = 6 + t$

   (c) $x = 3 \quad y = 1 + t \quad z = -1 - 5t$ (d) $-2x + y + 5z = 1$

   (e) $3x - 7z = -9$ (f) $x + y + z = 2$ (g) $x = 1 + 6t \quad y = t \quad z = 1 + 7t$

4. Curves in Space.

   (a) Tangent: $x = 1$, $y = t$, $z = t$. Length: $\sqrt{2}$ (b) Obtain the lower bound by solving $(x, y, z) = (4, 0, 1)$ for t and obtain $t = 0$. Similarly, from $(x, y, z) = (0, 0, -1)$ obtain that $t = \pi$ is the upper bound. The length is 6.98.

   (c) i) $x = 3\cos t$, $y = 3\sin t$, $z = 1 - y^2 = 1 - 9\sin^2 t$. ii) $(0, 3, -8)$ corresponds to $t = \pi/2$. Plugging $\pi/2$ in derivative gives you $\langle -3, 0, 0\rangle$. Tangent line: $x = -3t$ $y = 3$ $z = -8$. iii) $(3, 0, 1)$ corresponds to $t = 0$ and $(0, 3, -8)$ to $t = \pi/2$. So, the bounds of integration are 0 to $\pi/2$. The length is 10.48.

   (d) i) $y = 4\cos t$, $z = 4\sin t$, $x = 8 - y^2 - z = 8 - 16\cos^2 t - 4\sin t$. ii) $(-8, -4, 0)$ corresponds to $t = \pi$. Plugging π in derivative gives you $\langle 4, 0, -4\rangle$. Tangent line: $x = 4t - 8$ $y = -4$ $z = -4t$. iii) $(4, 0, 4)$ corresponds to $t = \pi/2$ and $(-8, -4, 0)$ to $t = \pi$. So, the bounds of integration are $\pi/2$ to π. The length is 14.515.

   (e) A set of parametric equations for the three curves in the intersection is
   $x = 2\cos t, y = 2\sin t, z = 0$ with $0 \leq t \leq \frac{\pi}{2}$,
   $x = t, y = 0, z = 4 - t^2$ with $0 \leq t \leq 2$, and
   $x = 0, y = t, z = 4 - t^2$ with $0 \leq t \leq 2$.
   The three derivative vectors and length elements are
   $x' = -2\sin t, y' = 2\cos t, z' = 0 \Rightarrow$ $ds = \sqrt{4\sin^2 t + 4\cos^2 t}dt = \sqrt{4}dt = 2dt$
   $x' = 1, y' = 0, z' = -2t \Rightarrow$ $ds = \sqrt{1 + 4t^2}dt$, and
   $x' = 0, y' = 1, z' = -2t \Rightarrow$ $ds = \sqrt{1 + 4t^2}dt$.
   The total length is $\int_0^{\pi/2} 2dt + \int_0^2 \sqrt{1 + 4t^2}dt + \int_0^2 \sqrt{1 + 4t^2}dt = \pi + 4.65 + 4.65 \approx 12.44$.

5. Partial Derivatives.

   (a) $z_x = 6x + 2y, \quad z_y = 2x - 10y, \quad z_{xx} = 6, \quad z_{xy} = z_{yx} = 2, \quad z_{yy} = -10$

   (b) $z_x = e^x \sin y, \quad z_y = e^x \cos y, \quad z_{xx} = e^x \sin y, \quad z_{xy} = z_{yx} = e^x \cos y, \quad z_{yy} = -e^x \sin y$

(c) $z_x = 2axe^{x^2-xy} + ax^2e^{x^2-xy}(2x-y) = a(2x+2x^3-x^2y)e^{x^2-xy}$, $z_y = ax^2e^{x^2-xy}(-x) = -ax^3e^{x^2-xy}$. Then $z_{xx} = a(2+6x^2-2xy)e^{x^2-xy} + a(2x+2x^3-x^2y)e^{x^2-xy}(2x-y)$ and $z_{yy} = -ax^3e^{x^2-xy}(-x) = ax^4e^{x^2-xy}$. Differentiating z_x with respect to y get $z_{xy} = a(-x^2)e^{x^2-xy} + a(2x+2x^3-x^2y)e^{x^2-xy}(-x) = a(-x^2-2x^2-2x^4+x^3y)e^{x^2-xy} = a(-3x^2-2x^4+x^3y)e^{x^2-xy}$. Alternatively, differentiating z_y with respect to x get $z_{xy} = -3ax^2e^{x^2-xy} - ax^3e^{x^2-xy}(2x-y) = a(-3x^2-2x^4+x^3y)e^{x^2-xy}$

(d) $z_x = \ln(xy^2)+1, \quad z_y = 2x/y, \quad z_{xx} = 1/x, \quad z_{xy} = z_{yx} = 2/y, \quad z_{yy} = -2x/y^2$

(e) $z_x = -(y^2+2xz)/(2yz+x^2)$, $z_y = -(2xy+z^2)/(2yz+x^2)$. At $(1,1,1)$, $z_x = -1$, $z_y = -1$.

(f) $z_x = -(1+\sin(x+y+z))/(-y+\sin(x+y+z))$ and $z_y = -(-z+\sin(x+y+z))/(-y+\sin(x+y+z))$. At $(0,1,-1)$, $z_x = 1$ and $z_y = 1$.

(g) $z'(t) = (6x+2y)(2t) + (2x-10y)(-3t^2); \quad z'(0) = 0$

(h) $z'(t) = (\ln(x+2y) + \frac{x}{x+2y})(-\sin t) + \frac{2x}{x+2y}\cos t;$ When $t = 0$, $x = \cos 0 = 1$ and $y = \sin 0 = 0$. Thus, $z'(0) = (\ln(1+0) + \frac{1}{1+0})(0) + \frac{2}{1+0}(1) = 0+2 = 2$.

(i) $z_s = (2x+y)(-e^t\sin s) + x(e^t\cos s)$, $z_t = (2x+y)(e^t\cos s) + x(e^t\sin s)$, $z_s(\pi,0) = 1$ and $z_t(\pi,0) = 2$

6. Tangent plane.

(a) $8x+10y-z = 9$ (b) $e^3y - z = e^3$

(c) $F_x = 2x, F_y = 4y, F_z = 6z$. At $(4,-1,1)$ this produces vector $\langle 8,-4,6\rangle$. The tangent plane is $4x-2y+3z = 21$.

(d) $F_x = y^2+2xz, F_y = 2xy+z^2, F_z = 2yz+x^2$. At $(1,1,1)$ this produces vector $\langle 3,3,3\rangle$. The tangent plane $x+y+z = 3$.

(e) $F_x = 1+\sin(x+y+z), F_y = -z+\sin(x+y+z), F_z = -y+\sin(x+y+z)$. At $(0,1,-1)$ this produces vector $\langle 1,1,-1\rangle$. The tangent plane is $x+y-z = 2$.

7. Linear Approximation. (a) $f(2.02,3.1) \approx 5.38$ (b) $f(.9,1.99) \approx 2.92$ (c) $f(1.95,1.08) \approx 2.847$ (d) $f(6.9,2.06) \approx -0.28$

8. Applications.

(a) (i) Since $P'(3) = 0.1$, $T'(3) = 0.5$, $N_P = 3$, and $N_T = 5$, and $N'(t) = N_TT' + N_PP'$ We have that $N'(3) = 5\cdot 0.5 + 3\cdot 0.1 = 2.8$ bacteria/minute. (ii) Using linear approximation formula, $N(T,P) \approx N(T_0,P_0) + N_T\cdot(T-T_0) + N_P\cdot(P-P_0) \Rightarrow N(309,100) \approx N(305,102) + 5(309-305) + 3(100-102) = 300 + 5(4) + 3(-2) = 314$ bacteria.

(b) $\frac{dP}{dt} = 0.05$ kPa/s, $\frac{dT}{dt} = 0.15$ K/s at $P_0 = 20$ kPa and $T_0 = 320$ K. We need $\frac{dV}{dt}$ at (P_0,T_0). Solve the given formula for V to get $V = \frac{8.31T}{P}$. Then $\frac{\partial V}{\partial P} = \frac{-8.31T}{P^2}$ and that $\frac{\partial V}{\partial T} = \frac{8.31}{P}$. At $P_0 = 20$ and $T_0 = 320$, we calculate that $\frac{\partial V}{\partial P} = \frac{-8.31(320)}{400} = -6.648$ liters/kPa and that $\frac{\partial V}{\partial T} = \frac{8.31}{20} = 0.4155$ liters/K. Thus, $\frac{dV}{dt} = \frac{\partial V}{\partial P}\frac{dP}{dt} + \frac{\partial V}{\partial T}\frac{dT}{dt} = (-6.648)(0.05) + (0.4155)(0.15) = -.27$ liter per second.

(c) (i) $N \approx N_0 + N_S(S-S_0) + N_T(T-T_0) = 100 + 2(52-50) + 4(73-70) = 116$ flowers. (ii) $S = \frac{1}{t} \Rightarrow S' = \frac{-1}{t^2}$. At $t = 2$, $S' = \frac{-1}{4}$. $T = 85 - \frac{8}{1+t^2} \Rightarrow T' = \frac{16t}{(1+t^2)^2}$. At $t = 2$, $T' = \frac{32}{25}$. Hence, $N' = N_SS' + N_TT' = 2\frac{-1}{4} + 4\frac{32}{25} = 4.62$ flowers per day.

(d) 2 degrees Celsius per second (e) Speed = 20.1 miles per hour.