Calculus 3
Lia Vas

Review for Exam 3

1. **Parametric Surfaces. Surface Area.**

   (a) Find the equation of the tangent plane to the parametric surface $x = u + v$, $y = 3u^2$, $z = u - v$ at the point $(2, 3, 0)$.

   (b) Find the surface area of the part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$. Write down the parametric equations of the cone first. Then find the surface area using the parametric equations.

   (c) Using the parametric equations and formula for the surface area for parametric curves, show that the surface area of the cylinder $x^2 + z^2 = 4$ for $0 \leq y \leq 5$ is 20π.

2. **Triple Integrals and volume.**

   (a) $\int\int\int_E xy\ dx\ dy\ dz$ where E is the solid tetrahedron with vertices (0,0,0), (1, 0, 0), (0, 2, 0) and (0, 0, 3).

   (b) $\int\int\int_E \sqrt{x^2 + y^2}\ dx\ dy\ dz$ where E is the region that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ and between the xy-plane and the plane $z = x + 3$.

   (c) $\int\int\int_E x^2 + y^2 + z^2\ dx\ dy\ dz$ where E is the unit ball $x^2 + y^2 + z^2 \leq 1$.

   (d) $\int\int\int_E z\ \ dx\ \ dy\ \ dz$ where E is the region between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ in the first octant.

   (e) Find the average value of the function $f(x, y, z) = xyz$ over the cube of side-length 4, in the first octant with one vertex in the origin and edges parallel to the coordinate axes.

   (f) Find the volume of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1$ by using the transformation $x = 2u$, $y = 3v$ $z = 5w$.

   (g) Determine the bounds (in spherical coordinates) for the following regions between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$.

      i. The region between the two spheres and above the xy-plane.

      ii. The region between the two spheres and to the right of the xz-plane.

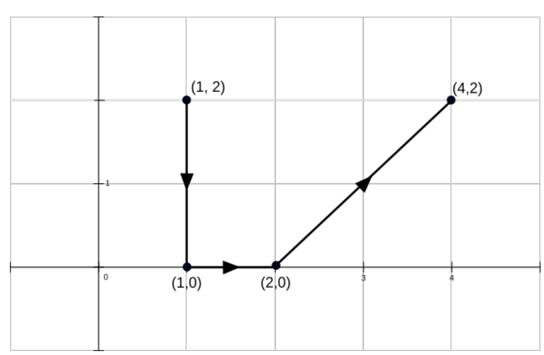
      iii. The region between the two spheres and in front of the yz-plane.

3. **Line Integrals.**

   (a) $\int_C x\ y^4\ ds$, $\quad C$ is the right half of the circle $x^2 + y^2 = 16$.

   (b) $\int_C x\ e^{yz}\ ds$, $\quad C$ is the line segment from (0,0,0) to (1, 2, 3).

   (c) $\int_C xy\ \ dx + (x - y)\ \ dy$, $\quad C$ consists of the three line segments on the figure on the right.



(d) $\int_C (xy + \ln x)\, dy$, C is the parabola $y = x^2$ from (1,1) to (3,9).

(e) $\int_C z^2\, dx + y\, dy + 2y\, dz$, where C consists of two parts C_1 and C_2. C_1 is the intersection of the cylinder $x^2 + y^2 = 16$ and the plane $z = 3$ from $(0, 4, 3)$ to $(-4, 0, 3)$. C_2 is the line segment from $(-4, 0, 3)$ to $(0, 1, 5)$.

(f) Find the work done by the force field $\overrightarrow{F}(x, y, z) = (x+y^2, y+z^2, z+x^2)$ in moving an object along the curve C which is the positively oriented intersection of the plane $x + y + z = 1$ and the coordinate planes.

(g) Find the work done by the force field $\overrightarrow{F} = (-y^2, x, z^2)$ in moving an object along the curve C which is is the positively oriented intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$.

4. **Potential. Independence of Path**.

(a) Check that $\vec{f} = \langle y, x+z, y\rangle$ is conservative, find its potential function and use it to evaluate $\int_C \vec{f} d\vec{r}$ where C is any path from $(2, 1, 4)$ to $(8, 3, -1)$.

(b) Check that $\vec{f} = \langle 2xz + \sin y,\ x\cos y + 3y^2 z,\ x^2 + y^3 + 6z\rangle$ is conservative, find its potential function and use it to evaluate $\int_C \vec{f} d\vec{r}$ where C is the helix $x = \cos t$, $y = \sin t$, $z = t$ for $0 \leq t \leq 2\pi$.

(c) Show that the line integral $\int_C (2xy + z^2)dx + (x^2 + 2yz + 2)dy + (y^2 + 2xz + 3)dz$ where C is any path from (1, 0, 2) to (0, 1, 4), is independent of path and evaluate it.

5. **Green's Theorem.** Evaluate the following integrals using Green's theorem. All the given curves are positively oriented.

(a) $\oint_C x^4 dx + xydy$ where C is the triangle with vertices (0, 0), (0, 1), and (1, 0). Compute the integral also without using Green's Theorem.

(b) $\oint_C y^2 dx + 3xydy$ where C is the boundary of the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ above x-axis.

(c) $\oint_C xydx + 2x^2 dy$ where C is the line segment from $(-2, 0)$ to $(2, 0)$ and the upper half of the circle $x^2 + y^2 = 4$.

(d) Use Green's Theorem to find the work done by the force $\vec{f}(x, y) = x(x + y)\vec{\imath} + xy^2\vec{\jmath}$ in moving a particle along the triangle with vertices (0,0), (1,0) and (0,1) starting at the origin.

6. **Curl and Divergence.** Find curl and divergence of the following vector fields.

(a) $\vec{f} = \langle xz, xyz, -y^2\rangle$ (b) $\vec{f} = \langle e^x \sin y, e^x \cos y, z\rangle$ (c) $\vec{f} = \langle \frac{x}{z}, \frac{y}{z}, \frac{-1}{z}\rangle$

Solutions

<u>More detailed solutions of the problems can be found on the class handouts.</u>

1. Surface Area. Parametric Surfaces.

(a) Plane $3x - y + 3z = 3$

(b) Parameterization: $x = r\cos t$, $y = r\sin t$, $z = \sqrt{x^2+y^2} = \sqrt{r^2} = r$. The length of the cross product is $\sqrt{2}r$. The surface area is $5\pi\sqrt{2}$.

(c) Parameterization: $x = 2\cos t$, $y = y$, $z = 2\sin t$. Bounds: $0 \le t \le 2\pi$, $0 \le y \le 5$. Length of the cross product is 2. Thus the double integral is $2\pi \cdot 5 \cdot 2 = 20\pi$.

2. Triple Integrals and volume. (a) $\frac{1}{10}$ (b) 14π (c) $\frac{4\pi}{5}$ (d) $\frac{15\pi}{16}$ (e) 8 (f) 40π (g) Since the radius of the first sphere is 1 and the radius of the second sphere is 2, the r-bounds are $1 \le r \le 2$ for all three parts.

   i. The values of θ are 0 to 2π because the projection in the xy plane is entire region between two circles. The bounds for ϕ are 0 to $\frac{\pi}{2}$ as the figure in the related handout illustrates.

   ii. The right side of the xz-plane $y = 0$ corresponds to $y > 0$. Hence, the projection in xy-plane is *above* the x-axis. So, the values of θ are 0 to π. The bounds for ϕ are 0 to π as the figure in the related handout illustrates.

   iii. The front of the yz-plane $x = 0$ corresponds to $x > 0$. Hence, the projection in xy-plane is *to the right* of the y-axis. So, the values of θ are $\frac{-\pi}{2}$ to $\frac{\pi}{2}$. The bounds for ϕ are 0 to π as the figure in the related handout illustrates.

3. Line Integrals.

   (a) 1638.4

   (b) 125.48

   (c) $0 + 0 + \frac{32}{3} = \frac{32}{3}$

   (d) 102.68

   (e) $\int_{C_1} = -44$, $\int_{C_2} = 67.83$. So, $\int_C = 67.83 - 44 = 23.83$

   (f) Let C_1 be a line from $(1,0,0)$ to $(0,1,0)$, C_2 a line from $(0,1,0)$ to $(0,0,1)$ and C_3 a line from $(0,0,1)$ to $(1,0,0)$. Find that $\int_{C_1} = \frac{-1}{3}$, $\int_{C_2} = \frac{-1}{3}$, and $\int_{C_3} = \frac{-1}{3}$. Thus $\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} = -1$.

   (g) C has parametrization $x = \cos t$, $y = \sin t$, $z = 2 - y = 2 - \sin t$, $0 \le t \le 2\pi$. $\int_C = \int_C -y^2dx + xdy + z^2dz = \int_0^{2\pi} \sin^3 tdt + \cos^2 tdt + (2-\sin t)^2 \cos tdt = \pi$.

4. Potential. Independence of Path. (a) $F = xy + yz + c$, $\int_C \vec{f}d\vec{r} = F(8,3,-1) - F(2,1,4) = 21 - 6 = 15$ (b) $F = x^2z + x\sin y + y^3z + 3z^2 + c$, $t = 0 \Rightarrow (x,y,z) = (1,0,0)$ and $t = 2\pi \Rightarrow (x,y,z) = (1,0,2\pi)$. So, $\int_C \vec{f}d\vec{r} = F(1,0,2\pi) - F(1,0,0) = 2\pi + 12\pi^2 - 0 \approx 124.72$. (c) $F = x^2y + z^2x + y^2z + 2y + 3z + c$, $\int_C = F(0,1,4) - F(1,0,2) = 8$.

5. Green's Theorem. (a) $\frac{1}{6}$ (b) $\frac{14}{3}$ (c) 0 (d) $\frac{-1}{12}$

6. Curl and Divergence. (a) $\text{div}\vec{f} = z + xz$, $\text{curl}\vec{f} = \langle -y(x+2), x, yz\rangle$ (b) $\text{div}\vec{f} = 1$, $\text{curl}\vec{f} = \langle 0,0,0\rangle$ (c) $\text{div}\vec{f} = \frac{2}{z} + \frac{1}{z^2}$, $\text{curl}\vec{f} = \langle \frac{y}{z^2}, \frac{-x}{z^2}, 0\rangle$.