

Review for Exam 3

1. Parametric Surfaces. Surface Area.

- (a) Find the equation of the tangent plane to the parametric surface $x = u + v$, $y = 3u^2$, $z = u - v$ at the point $(2, 3, 0)$.
- (b) Find the surface area of the part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$. Write down the parametric equations of the cone first. Then find the surface area using the parametric equations.
- (c) Using the parametric equations and formula for the surface area for parametric curves, show that the surface area of the cylinder $x^2 + z^2 = 4$ for $0 \leq y \leq 5$ is 20π .

2. Triple Integrals and volume.

- (a) $\int \int \int_E x^3 y^2 z \, dx \, dy \, dz$ where $E = \{ (x, y, z) \mid 1 \leq x \leq 2, 0 \leq y \leq x, 0 \leq z \leq y^2 \}$
- (b) $\int \int \int_E xy \, dx \, dy \, dz$ where E is the solid tetrahedron with vertices $(0,0,0)$, $(1, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 3)$.
- (c) $\int \int \int_E \sqrt{x^2 + y^2} \, dx \, dy \, dz$ where E is the region that lies inside the cylinder $x^2 + y^2 = 16$ and between the planes $z = -5$ and $z = 4$.
- (d) $\int \int \int_E 2 \, dx \, dy \, dz$ where E is the solid that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ and between the xy -plane and the plane $z = x + 2$.
- (e) $\int \int \int_E x^2 + y^2 + z^2 \, dx \, dy \, dz$ where E is the unit ball $x^2 + y^2 + z^2 \leq 1$.
- (f) $\int \int \int_E z \, dx \, dy \, dz$ where E is the region between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ in the first octant.
- (g) Find the average value of the function $f(x, y, z) = xyz$ over the cube of side-length 4, in the first octant with one vertex in the origin and edges parallel to the coordinate axes.
- (h) Find the mass and the center of mass of the solid E where E lies under the plane $z = x + y + 1$ and above the region in the xy -plane bounded by the curves $y = \sqrt{x}$, $y = 0$ and $x = 1$, and that has the density function $\rho(x, y, z) = 2$.
- (i) Find the volume of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1$ by using the transformation $x = 2u$, $y = 3v$, $z = 5w$.

3. Line Integrals.

- (a) $\int_C x y^4 \, ds$, C is the right half of the circle $x^2 + y^2 = 16$.
- (b) $\int_C x e^{yz} \, ds$, C is the line segment from $(0,0,0)$ to $(1, 2, 3)$.
- (c) $\int_C (xy + \ln x) \, dy$, C is the parabola $y = x^2$ from $(1,1)$ to $(3,9)$.
- (d) $\int_C xy \, dx + (x - y) \, dy$, C consists of line segments from $(0,0)$ to $(2,0)$ and from $(2,0)$ to $(3,2)$.
- (e) $\int_C z^2 \, dx - z \, dy + 2y \, dz$, C consists of line segments from $(0,0,0)$, $(0,1,1)$, from $(0,1,1)$ to $(1,2,3)$ and from $(1,2,3)$ to $(1, 2,4)$.

- (f) $\int_C z^2 dx + y dy + 2y dz$, where C consists of two parts C1 and C2. C1 is the intersection of the cylinder $x^2 + y^2 = 16$ and the plane $z = 3$ from $(0, 4, 3)$ to $(-4, 0, 3)$. C2 is a line segment from $(-4, 0, 3)$ to $(0, 1, 5)$.
- (g) Find the work done by the force field $\vec{F}(x, y, z) = (x + y^2, y + z^2, z + x^2)$ in moving an object along the curve C which is the intersection of the plane $x + y + z = 1$ and the coordinate planes.
- (h) Find the work done by the force field $\vec{F} = (-y^2, x, z^2)$ in moving an object along the curve C which is the intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$.
- (i) Find the mass and the center of mass of a wire in the shape of the right half of the circle $x^2 + y^2 = 4$ with the density function $\rho(x, y) = k$ where k is a constant.

4. Potential. Independence of Path.

- (a) Check if given vector functions are conservative. If they are, find their potential functions.

i) $\vec{f} = \langle xe^y, ye^x \rangle$

ii) $\vec{f} = \langle x^3y^4, x^4y^3 + 2y \rangle$.

- (b) Check that $\vec{f} = \langle x^3y^4, x^4y^3 + 2y \rangle$ is conservative, find its potential function and use it to evaluate $\int_C \vec{f} d\vec{r}$ where C is $x = \sqrt{t}$, $y = 1 + t^3$, $0 \leq t \leq 1$.
- (c) Check that $\vec{f} = \langle y, x+z, y \rangle$ is conservative, find its potential function and use it to evaluate $\int_C \vec{f} d\vec{r}$ where C is any path from $(2, 1, 4)$ to $(8, 3, -1)$.
- (d) Show that the line integral $\int_C (2xy + z^2)dx + (x^2 + 2yz + 2)dy + (y^2 + 2xz + 3)dz$ where C is any path from $(1, 0, 2)$ to $(0, 1, 4)$, is independent of path and evaluate it.

5. Green's Theorem. Evaluate the following integrals using Green's theorem.

- (a) $\oint_C x^4 dx + xy dy$ where C is the triangle with vertices $(0, 0)$, $(0, 1)$, and $(1, 0)$. Compute the integral also without using Green's Theorem.
- (b) $\oint_C xy^2 dx + x^3 dy$ where C is the rectangle with vertices $(0, 0)$, $(2, 0)$, $(2, 3)$, and $(0, 3)$. Compute the integral also without using Green's Theorem.
- (c) $\oint_C y^2 dx + 3xy dy$ where C is the boundary of the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ above x -axis.
- (d) $\oint_C xy dx + 2x^2 dy$ where C is the line segment from $(-2, 0)$ to $(2, 0)$ and the upper half of the circle $x^2 + y^2 = 4$.
- (e) Use Green's Theorem to find the work done by the force $\vec{f}(x, y) = x(x + y)\vec{i} + xy^2\vec{j}$ in moving a particle along the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$ starting at the origin.

6. Curl and Divergence. Find curl and divergence of the following vector fields.

(a) $\vec{f} = \langle xz, xyz, -y^2 \rangle$

(b) $\vec{f} = \langle e^x \sin y, e^x \cos y, z \rangle$

(c) $\vec{f} = \langle \frac{x}{z}, \frac{y}{z}, \frac{-1}{z} \rangle$

Solutions

More detailed solutions of the problems can be found on the class handouts.

1. Surface Area. Parametric Surfaces.

- (a) Plane $3x - y + 3z = 3$
- (b) Parameterization: $x = r \cos t$, $y = r \sin t$, $z = \sqrt{x^2 + y^2} = \sqrt{r^2} = r$. The length of the cross product is $\sqrt{2}r$. The surface area is $5\pi\sqrt{2}$.
- (c) Parameterization: $x = 2 \cos t$, $y = y$, $z = 2 \sin t$. Bounds: $0 \leq t \leq 2\pi$, $0 \leq y \leq 5$. Length of the cross product is 2. Thus the double integral is $2\pi \cdot 5 \cdot 2 = 20\pi$.

2. Triple Integrals and volume. (a) 13.29 (b) $\frac{1}{10}$ (c) 384π (d) 12π (e) $\frac{4\pi}{5}$ (f) $\frac{15\pi}{16}$ (g) 8 (h) mass = $\frac{79}{30}$, center of mass = $(\frac{358}{553}, \frac{33}{79}, \frac{571}{553})$ (i) 40π

3. Line Integrals.

- (a) 1638.4
- (b) $\sqrt{14}/12(e^6 - 1)$
- (c) 102.68
- (d) $17/3$
- (e) $1/2 + 25/3 + 4 = 77/6$
- (f) $\int_{C_1} = -44$, $\int_{C_2} = 67.83$. So, $\int_C = 67.83 - 44 = 23.83$
- (g) Let C_1 be a line from $(1, 0, 0)$ to $(0, 1, 0)$, C_2 a line from $(0, 1, 0)$ to $(0, 0, 1)$ and C_3 a line from $(0, 0, 1)$ to $(1, 0, 0)$. Find that $\int_{C_1} = \frac{-1}{3}$, $\int_{C_2} = \frac{-1}{3}$, and $\int_{C_3} = \frac{-1}{3}$. Thus $\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} = -1$.
- (h) C has parametrization $x = \cos t$, $y = \sin t$, $z = 2 - y = 2 - \sin t$, $0 \leq t \leq 2\pi$. $\int_C = \int_C -y^2 dx + xdy + z^2 dz = \int_0^{2\pi} \sin^3 t dt + \cos^2 t dt + (2 - \sin t)^2 \cos t dt = \pi$.
- (i) mass = $2k\pi$, center of mass = $(\frac{4}{\pi}, 0)$

4. Potential. Independence of Path. (a) i) Not conservative. ii) Conservative. $F = \frac{1}{4}x^4y^4 + y^2 + c$. (b) $F = \frac{1}{4}x^4y^4 + y^2 + c$, $\int_C = F(1, 2) - F(0, 1) = 7$ (c) $F = xy + yz + c$, $\int_C = F(8, 3, -1) - F(2, 1, 4) = 21 - 6 = 15$ (d). $F = x^2y + z^2x + y^2z + 2y + 3z + c$, $\int_C = F(0, 1, 4) - F(1, 0, 2) = 8$.

5. Green's Theorem. (a) $\frac{1}{6}$ (b) 6 (c) $\frac{14}{3}$ (d) 0 (e) $\frac{-1}{12}$

6. Curl and Divergence. (a) $\text{div} \vec{f} = z + xz$, $\text{curl} \vec{f} = \langle -y(x+2), x, yz \rangle$ (b) $\text{div} \vec{f} = 1$, $\text{curl} \vec{f} = \langle 0, 0, 0 \rangle$ (c) $\text{div} \vec{f} = \frac{2}{z} + \frac{1}{z^2}$, $\text{curl} \vec{f} = \langle \frac{y}{z^2}, \frac{-x}{z^2}, 0 \rangle$.