Calculus 3
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Sequences

A **sequence** is a list of numbers indexed by the positive (or nonnegative) integers:

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

For example, $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

A sequence might be given by the formula for its n-th term a_n. For example, $a_n = \frac{1}{n}$. A sequence can also be given **recursively**. For example, $a_1 = 1$, $a_{n+1} = \frac{1}{1+a_n}$.

A sequence with the n-th term a_n is **convergent** if the limit $\lim_{n\to\infty} a_n$ exist (as a real number). Otherwise, it is **divergent.**

The limit $\lim_{n\to\infty} a_n$ exist and is equal to L if and only if $\lim_{x\to\infty} f(x)$ exist and is equal to L where $f(n) = a_n$. This enables us to use the **L'Hopital's Rule** when doing limits of sequences as well.

If f is a continuous function and $a_{n+1} = f(a_n)$ defines a recursive sequence, its limit, if it exists, is its *fixed point* – a number a such that $a = f(a)$. This is because we have that $\lim_{n\to\infty} a_n$ is a in this case, so that $\lim_{n\to\infty} a_{n+1}$ is also a.

Practice Problems.

1. List five first terms of the sequence. Determine whether the above sequences are convergent or divergent. If they are convergent, find their limits.

   a) $a_n = \left(\frac{1}{2}\right)^n$, $n = 1, 2\ldots$ b) $a_n = \frac{n+1}{2n-1}$, $n = 1, 2\ldots$

   c) $a_n = \sin\frac{n\pi}{2}$, $n = 0, 1, 2\ldots$ d) $a_0 = 0$, $a_{n+1} = \sqrt{2 + a_n}$.

   e) $a_1 = 1$, $a_{n+1} = \frac{1}{1+a_n}$.

2. Determine whether the sequences are convergent or divergent. If they are convergent, find their limits.

   a) $a_n = \frac{\ln n}{n}$, $n = 1, 2\ldots$ b) $a_n = \frac{n}{2^n}$, $n = 0, 1, 2\ldots$ c) $a_n = \frac{n+3n^3}{4n^2+35n-7+2n^3}$, $n = 1, 2\ldots$

3. If the n-th term a_n is of the form $\frac{p_m(n)}{q_k(n)}$ where $p_m(x)$ is a polynomial with the leading term ax^m and $q_k(x)$ is a polynomial with the leading term bx^k, discuss the possible values of the limit $\lim_{n\to\infty} a_n$ considering the following three cases: (1) $m < k$, (2) $m = k$, (3) $m > k$.

4. Determine for which values of r is the sequence with the n-th term r^n, $n = 0, 1, 2\ldots$, convergent by discussing the following three cases (1) $-1 < r < 1$, (2) $r > 1$, (3) $r = 1$, (4) $r \leq -1$.

5. When calculating the hydrogen ion concentration $[H^+]$ in a acid-base system, the problem frequently boils down to finding the limit of a recursive sequence. For example, when hydrochloric acid HCl is dissolved in water, we have $[H^+]_1 = C_{HCl}$ and

$$[H^+]_{n+1} = C_{HCl} + \frac{K_w}{[H^+]_n},$$

where C_{HCl} is the analytical concentration of HCl and K_w is the water's autoprotolysis constant that is equal to 10^{-14} at 25 degrees Centigrade. If the analytical concentration of HCL, C_{HCl}, is equal to 10^{-7}, find the hydrogen ion concentration $[H^+]$ and its pH value.

6. **Fibonacci numbers and the golden ratio.** Fibonacci numbers are terms of the following recursive sequence.

$$f_0 = 0, \; f_1 = 1 \quad \text{and} \quad f_{n+2} = f_{n+1} + f_n$$

for $n = 0, 1, \ldots$. Thus one starts with 0 and 1, and then produces the next Fibonacci number by adding the two previous Fibonacci numbers. The following sequence is obtained

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, \ldots$$

This sequence is clearly divergent.

On the other hand, two quantities are said to be in the **golden ratio** if the ratio of the larger to smaller quantity is the same as the ratio of their sum to the larger quantity. So, if a and b are two quantities and $a > b$, then a and b are in the golden ratio if

$$\frac{a}{b} = \frac{a+b}{a}.$$

To find the ratio $x = \frac{a}{b}$, note that the right side is $\frac{a+b}{a} = \frac{a}{a} + \frac{b}{a} = 1 + \frac{1}{x}$. Thus,

$$x = 1 + \frac{1}{x} \;\Rightarrow\; x^2 = x + 1 \;\Rightarrow\; x^2 - x - 1 = 0.$$

The positive solution $\frac{1+\sqrt{5}}{2} \approx 1.618$ of this quadratic equation is prominently used in science as well as in art, architecture and music.

While the Fibonacci sequence is divergent the **quotient of two consecutive terms of the Fibonacci sequence** $\frac{f_{n+1}}{f_n}$ is convergent. Show that its limit is the golden ratio $\frac{1+\sqrt{5}}{2} \approx 1.618$.

Solutions.

1. a) First five terms: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$. $\lim_{n\to\infty} \frac{1}{2^n} = \frac{1}{\infty} = 0$. So, the sequence is convergent.

b) First five terms: 2, 1, $\frac{4}{5}, \frac{5}{7}, \frac{6}{9}$. $\lim_{n\to\infty} \frac{n+1}{2n-1} = \lim_{n\to\infty} \frac{1}{2}$ (you can use L'Hopital's rule for example). So, the limit of the sequence is $\frac{1}{2}$ and so the sequence is convergent.

c) First five terms: 0, 1, 0, -1. The terms of the sequence alternate between 0 and ± 1. Thus, there is not a single number towards which the terms converge. Thus, this sequence is divergent.

d) First five terms: $0, \sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$. Let a stand for the limit of this sequence in case it exists. Note that then $a = \lim_{n\to\infty} a_n$ and $a = \lim_{n\to\infty} a_{n+1}$ as well. To find the value of a let $n \to \infty$ in the equation $a_{n+1} = \sqrt{2+a_n}$. The left side converges to a and the right side to $\sqrt{2+a}$. So, a can be found from the equation $a = \sqrt{2+a} \Rightarrow a^2 = 2 + a \Rightarrow a^2 - a - 2 = 0 \Rightarrow a = 2$ or $a = -1$. Since -1 is an extraneous root (it does not satisfy the equation $a = \sqrt{2+a}$), the limit of the sequence is 2.

e) First five terms: 1, $\frac{1}{1+1}$, $\frac{1}{1+\frac{1}{1+1}}$, $\frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}$, $\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}}$. Let a stand for the limit of this sequence in case it exists. Note that then $a = \lim_{n\to\infty} a_n$ and $a = \lim_{n\to\infty} a_{n+1}$ as well. To find the value of a let $n \to \infty$ in the equation $a_{n+1} = \frac{1}{1+a_n}$. The left side converges to a and the right side to $\frac{1}{1+a}$. So, a can be found from the equation $a = \frac{1}{1+a} \Rightarrow a(1+a) = 1 \Rightarrow a^2 + a - 1 = 0 \Rightarrow a = 0.618$ or $a = -1.618$. Since all the terms of the sequence are positive, the sequence converges towards the positive value $a = 0.618$.

2. a) $\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = \frac{1}{\infty} = 0$.

   b) $\lim_{n\to\infty} \frac{n}{2^n} = \lim_{n\to\infty} \frac{1}{2^n \ln 2} = \frac{1}{\infty} = 0$.

   c) Note that a_n is a quotient of two polynomials. Recall that the leading term of a polynomial determines it behavior at ∞. Thus $\lim_{n\to\infty} \frac{n+3n^3}{4n^2+35n-7+2n^3} = \lim_{n\to\infty} \frac{3n^3}{2n^3} = \frac{3}{2}$.

3. Since the leading term of a polynomial determines its behavior at infinity, the limit $\lim_{n\to\infty} \frac{p_m(n)}{q_k(n)}$ is equal to $\lim_{n\to\infty} \frac{an^m}{bn^k}$. So, if $m < k$ this limit is of the form $\frac{a}{bn^{k-m}} \to \frac{a}{\infty} = 0$.

   If $m = k$ this limit is of the form $\frac{an^m}{bn^m} = \frac{a}{b}$.

   if $m > k$ this limit is of the form $\frac{an^{m-k}}{b} \to \pm\infty$ (∞ if $\frac{a}{b}$ is positive and $-\infty$ if $\frac{a}{b}$ is negative).

4. (1) If $-1 < r < 1$, the powers of r approach 0 as n approaches infinity (consider, for example, what happens if $r = \frac{1}{2}$ or $r = \frac{-1}{2}$). (2) If $r > 1$, the powers of r become larger so the limit of the sequence is ∞. (3) If $r = 1$, the sequence is constant 1 and its limit is 1. (4) If $r \leq -1$, the terms alternate from positive to negative. So, the limit does not exist. In conclusion, the sequence is convergent just for $-1 < r \leq 1$.

5. The hydrogen ion concentration $[H^+]$ can be obtained as a limit of the recursive sequence. With the given values, the recursive equation becomes $[H^+]_1 = 10^{-7}$ and $[H^+]_{n+1} = 10^{-7} + \frac{10^{-14}}{[H^+]_n}$. Let x stand for the limit of this sequence. Then x can be found from the equation $x = 10^{-7} + \frac{10^{-14}}{x} \Rightarrow x^2 - 10^{-7}x - 10^{-14} = 0$. The positive solution $1.618 \cdot 10^{-7}$ gives us the value of $[H^+]$ and its pH value of 6.7910.

6. Let us denote $a_n = \frac{f_{n+1}}{f_n}$. Dividing the equation $f_{n+2} = f_{n+1} + f_n$ by f_{n+1}, we obtain $\frac{f_{n+2}}{f_{n+1}} = 1 + \frac{f_n}{f_{n+1}}$. Note that the term on the left is $a_{n+1} = \frac{f_{n+2}}{f_{n+1}}$ and that the right side is $1 + \frac{1}{a_n}$. Thus, the recursive formula of the quotient sequence a_n is $a_{n+1} = 1 + \frac{1}{a_n}$.

   Denote the limit by x. Thus $x = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$ and so value x satisfies the equation $x = 1 + \frac{1}{x}$. Multiply by x to get $x^2 = x + 1 \Rightarrow x^2 - x - 1 = 0 \Rightarrow x = \frac{1\pm\sqrt{1+4}}{2} = \frac{1\pm\sqrt{5}}{2} \Rightarrow x = \frac{1+\sqrt{5}}{2} \approx 1.618$ or $x = \frac{1-\sqrt{5}}{2} \approx -0.618$. The negative solution is not relevant since all terms of the sequence are positive. Thus, the limit is the golden ratio $\frac{1+\sqrt{5}}{2} \approx 1.618$.