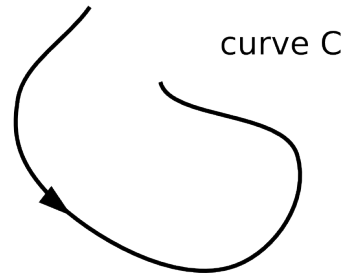Calculus 3
Lia Vas

Space Curves

Recall the parametric equations of a curve in xy-plane and compare them with parametric equations of a curve in space.

Parametric curve in plane	Parametric curve in space
$x = x(t)$ $y = y(t)$	$x = x(t)$ $y = y(t)$ $z = z(t)$

curve C

Given its parametric equations $x = x(t)$, $y = y(t)$, $z = z(t)$, a curve C can be considered to be a **vector function**, that is a function whose domain is in an interval of real numbers and the range is a set of vectors:
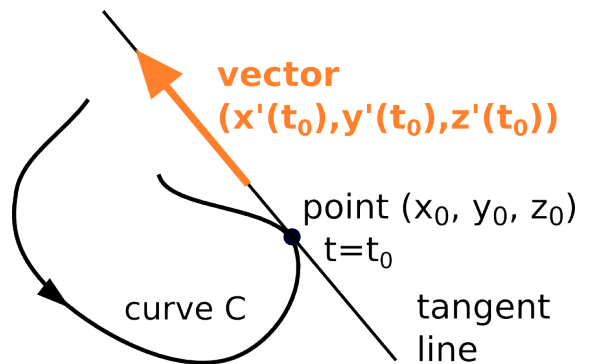$$\vec{r}(t) = \langle x(t), \ y(t), \ z(t)\rangle.$$
In this case, the curve C is the graph of the vector function $\vec{r}(t)$. Any value $t = t_0$ from the domain of $\vec{r}(t)$ corresponds to a point (x_0, y_0, z_0) on the curve C.

The **derivative** of a vector function $\vec{r} = \langle x(t), \ y(t), \ z(t)\rangle$ is the vector function

$$\vec{r}\,'(t) = \langle x'(t), \ y'(t), \ z'(t)\rangle$$

At point (x_0, y_0, z_0) which corresponds to the value t_0 of parameter t, the value of the derivative $\vec{r}\,'(t_0) = \langle x'(t_0), y'(t_0), z'(t_0)\rangle$ represents the **velocity vector of the tangent line** at (x_0, y_0, z_0).

vector $(x'(t_0),y'(t_0),z'(t_0))$

point (x_0, y_0, z_0) $t=t_0$
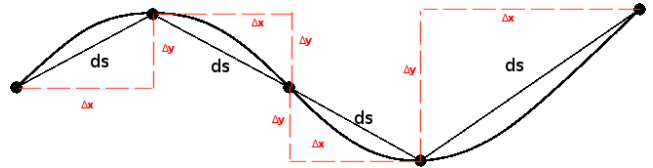
curve C

tangent line

Note the analogy with the two-dimensional scenario.

| To find a tangent line to the curve $x = x(t)$, $y = y(t)$ at $t = t_0$, use
 Point: $(x(t_0), y(t_0))$,
 Direction vector: $\langle x'(t_0), y'(t_0)\rangle$.

 An equation of the **tangent line**:

 $x = x(t_0) + x'(t_0)t$
 $y = y(t_0) + y'(t_0)t$ | To find a tangent line to the curve $x = x(t)$, $y = y(t)$, $z = z(t)$ at $t = t_0$, use
 Point: $(x(t_0), y(t_0), z(t_0))$,
 Direction vector: $\langle x'(t_0), y'(t_0), z'(t_0)\rangle$.

 An equation of the **tangent line**:

 $x = x(t_0) + x'(t_0)t$
 $y = y(t_0) + y'(t_0)t$
 $z = z(t_0) + z'(t_0)t$ |

Recall that the length L of parametric curve $x = x(t), y = y(t)$ with continuous derivatives on an interval $a \leq t \leq b$ can be obtained by integrating the length element ds from a to b.
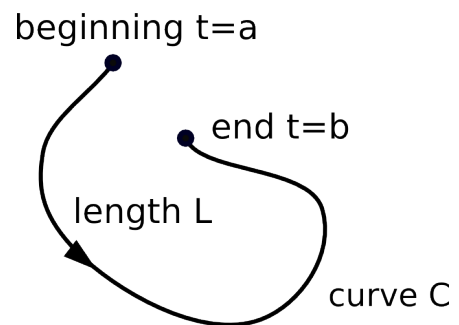
$$L = \int_a^b ds.$$

The length element ds on a sufficiently small interval can be approximated by the hypotenuse of a triangle with sides dx and dy and so $ds^2 = dx^2 + dy^2 \Rightarrow ds = \sqrt{dx^2 + dy^2} = \sqrt{(x'(t))^2 + (y'(t))^2}dt.$

Analogously, the length element of a space curve satisfies $ds^2 = dx^2 + dy^2 + dz^2$ and so $ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt.$

The expression $\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$ represents **the length of the derivative vector** $\vec{r}' = \langle x', y', z' \rangle$. Thus the length of a space curve on the interval $a \leq t \leq b$ can be found as

$$L = \int_a^b ds = \int_a^b |\vec{r}'(t)|dt =$$

$$\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt.$$

Compare again the two and three dimensional formulas.

The **length** of the curve $\vec{r} = \langle x(t), y(t) \rangle$ for $a \leq t \leq b$ is	The **length** of the curve $\vec{r} = \langle x(t), y(t), z(t) \rangle$ for $a \leq t \leq b$ is
$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt = \int_a^b |\vec{r}'(t)|dt$	$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt = \int_a^b |\vec{r}'(t)|dt$

Practice Problems.

1. Describe the following curves. For those without parametric representation, find equations of parametric equations.

   (a) The curve given by $x = 1 + t$, $y = 2 - 2t$, $z = 1 + 2t$.

   (b) The line segment from (1, 2, -4) to (3, 0, 1).

   (c) The curve given by $x = \cos t$, $y = \sin t$, $z = 2$.

   (d) The curve given by $x = \cos t$, $y = \sin t$, $z = t$.

   (e) The curve in the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $y + z = 2$.

   (f) The triangle in the boundary of the part of the plane $3x + 2y + z = 6$ in the first octant.

   (g) The boundary of the part of the paraboloid $z = 4 - x^2 - y^2$ in the first octant.

2. For the following curves, find an equation of the tangent line at the point where $t = 0$. Find the normalization of a direction vector at $t = 0$.

   (a) The curve given by $x = \cos t$, $y = \sin t$, $z = t$.

   (b) The curve in the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $y + z = 2$. Use the parametrization of this curve from problem 1 (e).

3. For the following curves, find the length for $0 \le t \le \frac{\pi}{2}$. Use the calculator to evaluate the integral in part (b).

   (a) The curve given by $x = \cos t$, $y = \sin t$, $z = t$.

   (b) The curve in the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $y + z = 2$. Use the parametrization of this curve from problem 1 (e).

4. Consider the curve C which is the intersection of the surfaces

$$x^2 + y^2 = 9 \quad \text{and} \quad z = 1 - y^2.$$

   (a) Find the parametric equations that represent the curve C.

   (b) Find the equation of the tangent line to the curve C at point $(0, 3, -8)$.

   (c) Find the length of the curve from $(3, 0, 1)$ to $(0, 3, -8)$. You can use the calculator to evaluate the integral that you are going to get.

5. Consider the curve C which is the intersection of the surfaces

$$y^2 + z^2 = 16 \quad \text{and} \quad x = 8 - y^2 - z.$$

   (a) Find the parametric equations that represent the curve C.

   (b) Find the equation of the tangent line to the curve C at point $(-8, -4, 0)$.

   (c) Find the length of the curve from $(4, 0, 4)$ to $(-8, -4, 0)$. Use the calculator to evaluate the integral that you are going to get.

6. Find the length of the boundary of the part of the paraboloid $z = 4 - x^2 - y^2$ in the first octant.

Solutions.

1. (a) This curve is a line passing the point $(1, 2, 1)$ in the direction of $\langle 1, -2, 2 \rangle$.

   (b) Any vector colinear with $\langle 3, 0, 1 \rangle - \langle 1, 2, -4 \rangle = \langle 2, -2, 5 \rangle$ can be used as a direction vector of the line passing two points. You can also use any of $(1,2,-4)$ and $(3, 0, 1)$ for a point on the line. For example, using $(1,2,-4)$ we obtain parametric equations $x = 1 + 2t, y = 2 - 2t z = -4 + 5t$.

   In this case, the initial point corresponds to $t = 0$ and the end point to $t = 1$. So, the line segment has parametrization $x = 1 + 2t, y = 2 - 2t z = -4 + 5t$ with $0 \le t \le 1$.

   (c) The xy-equations $x = \cos t$, $y = \sin t$, represent a circle of radius 1 in xy-plane. Thus, the curve is on the cylinder determined by this circle. The z-equation $z = 2$ represents the horizontal plane passing 2 on the z-axis. So, this curve is the intersection of the cylinder with
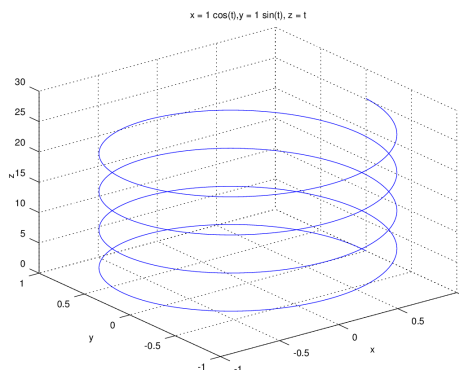
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the horizontal plane: it is a circle of radius 1 centered on the z-axis in the horizontal plane passing $z = 2$.

(d) The xy-equations $x = \cos t$, $y = \sin t$, represent a circle of radius 1 in xy-plane. Thus, the curve is on the cylinder determined by this circle. The equation $z = t$ has an effect that z-values increase as t-values increase. Thus, this curve is a helix spiraling up the cylinder as t increases. Use Matlab to get a precise graph.
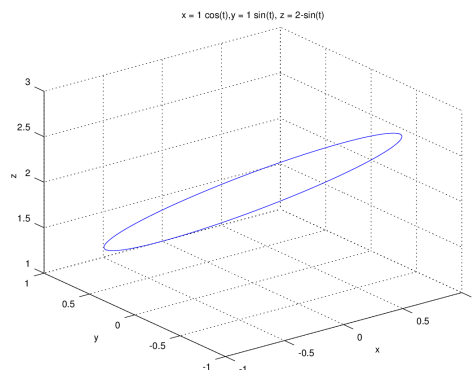
(e) The curve is the intersection of a cylinder with an inclined plane. So, the curve is an ellipse. The xy-equations represent a cylinder based at the circle of radius 1 in xy-plane. Thus, x and y can be parametrized as $x = \cos t$, $y = \sin t$. To get the z equation, solve the plane equation $y + z = 2$ for z, get $z = 2 - y$ and substitute that $y = \sin t$. Thus $z = 2 - \sin t$. This gives us an equation of the ellipse to be $x = \cos t$, $y = \sin t$, $z = 2 - \sin t$. You can use Matlab to get a
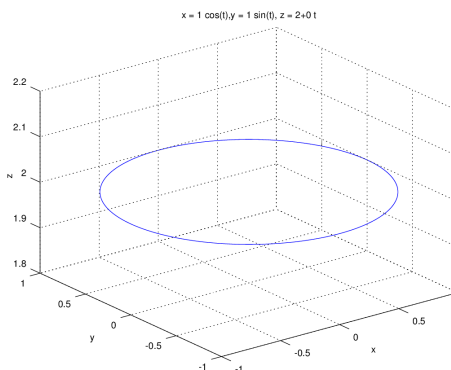
1 (c) Circle in a horizontal plane   1 (d) Helix   1 (e) Ellipse

(f) The triangle in the boundary of the part of the plane $3x + 2y + z = 6$ in the first octant consists of the three line segments, each of which will have a different set of parametric equations. The intersection in xy-plane $z = 0$ can be obtained by plugging $z = 0$ in $3x + 2y + z = 6$ and using x, for example, as a parameter. Thus, we have part of the line $3x + 2y = 6 \Rightarrow y = 3 - \frac{3}{2}x$ between its two intercepts (2,0) and (0,3) and so $0 \le x \le 2$

$$x = x, y = 3 - \frac{3}{2}x, z = 0 \text{ or } x = t, y = 3 - \frac{3}{2}t, z = 0 \text{ with } 0 \le t \le 2.$$

Alternatively, the parametric equations of this line can be obtained as equations of a line passing x and y intercepts of the plane $3x + 2y + z = 6$, (2, 0, 0) and (0,3, 0). Using (-2,3,0) as direction vector and (0,3,0) as a point on the line, we obtain the equations $x = -2t, y = 3 + 3t, z = 0$ with $-1 \le t \le 0$.

Similarly, you can find equations of the remaining two sides of the triangle. The intersection of xz-plane $y = 0$ can be obtained by plugging $y = 0$ in $3x + 2y + z = 6$ and using x, for example, as a parameter. Thus, we have $3x + z = 6 \Rightarrow z = 6 - 3x$ and so $x = x, y = 0, z = 6 - 3x$ or $x = t, y = 0, z = 6 - 3t$ with $0 \le t \le 2$.

The intersection of yz-plane $x = 0$ can be obtained by plugging $x = 0$ in $3x + 2y + z = 6$ and using y, for example, as a parameter. Thus, we have $2y + z = 6 \Rightarrow z = 6 - 2y$ and so $x = 0, y = y, z = 6 - 2y$ or $x = 0, y = t, z = 6 - 2t$ with $0 \le t \le 3$.

(g) The boundary of the part of the paraboloid $z = 4 - x^2 - y^2$ in the first octant consists of three curves, each of which will have a different set of parametric equations. The parametrizations

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can be obtained by considering intersections with three coordinate planes $x = 0, y = 0$, and $z = 0$ respectively.

The intersection in xy-plane $z = 0$ is a circle $0 = 4 - x^2 - y^2 \Rightarrow x^2 + y^2 = 4$ which has parametric equations $x = 2\cos t, y = 2\sin t$. Since we are considering just the part with $x \geq 0$ and $y \geq 0$, we have that $0 \leq t \leq \frac{\pi}{2}$. Thus, this curve has parametric equations

$$x = 2\cos t, y = 2\sin t, z = 0 \text{ with } 0 \leq t \leq \frac{\pi}{2}.$$

The intersection in xz-plane $y = 0$ is a parabola $z = 4 - x^2$ with $0 \leq x \leq 2$. Using x as a parameter produces parametric equations $x = x, y = 0, z = 4 - x^2$ or $x = t, y = 0, z = 4 - t^2$ with $0 \leq t \leq 2$.

The intersection in yz-plane $x = 0$ is a parabola $z = 4 - y^2$ with $0 \leq y \leq 2$. Using y as a parameter produces parametric equations $x = 0, y = y, z = 4 - y^2$ or $x = 0, y = t, z = 4 - t^2$ with $0 \leq t \leq 2$.

2. (a) To find a point on the tangent, plug $t = 0$ in the parametric equations of the curve. Get $x = 1, y = 0, z = 0$. To get a direction vector, plug $t = 0$ in the derivative $x' = -\sin t, y' = \cos t, z' = 1$. Get $\langle 0, 1, 1 \rangle$. So, the equation of the tangent is $x = 1 + 0t, y = 0 + 1t, z = 0 + 1t \Rightarrow x = 1, y = t, z = t$.

The direction vector $\langle 0, 1, 1 \rangle$ has length $\sqrt{2}$ so its normalization is $\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$.

(b) Use the parametric equations from problem 1 (d) $x = \cos t, y = \sin t, z = 2 - \sin t$. To find a point on the tangent, plug $t = 0$ in the parametric equations. Get $x = 1, y = 0, z = 2$. To get a direction vector, plug $t = 0$ in the derivative $x' = -\sin t, y' = \cos t, z' = -\cos t$. Get $\langle 0, 1, -1 \rangle$. So, the equation of the tangent is $x = 1 + 0t, y = 0 + 1t, z = 2 - 1t \Rightarrow x = 1, y = t, z = 2 - t$.

The direction vector $\langle 0, 1, -1 \rangle$ has length $\sqrt{2}$ so its normalization is $\langle 0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \rangle$.

3. (a) $L = \int_0^{\pi/2} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt = \int_0^{\pi/2} \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2} dt = \int_0^{\pi/2} \sqrt{1 + 1} dt = \sqrt{2} \int_0^{\pi/2} dt = \frac{\sqrt{2}\pi}{2}$.

(b) $L = \int_0^{\pi/2} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt = \int_0^{\pi/2} \sqrt{(-\sin t)^2 + (\cos t)^2 + (-\cos t)^2} dt$. Using the calculator, $L \approx 1.91$.

4. (a) You can parametrize the cylinder $x^2 + y^2 = 9$ by $x = 3\cos t$ and $y = 3\sin t$. From the equation $z = 1 - y^2$, you obtain that $z = 1 - (3\sin t)^2 = 1 - 9\sin^2 t$.

(b) To find a direction vector, we need to plug t-value that corresponds to the point $(0, 3, -8)$ into the derivative $x' = -3\sin t, y' = 3\cos t, z' = -18\sin t\cos t$. To find this t-value, set $x = 0, y = 3$ and $z = -8$ and make sure that you find the t-value that satisfies all three equations. From the first equation $x = 3\cos t = 0 \Rightarrow t = \pm\frac{\pi}{2}$. From the second $y = 3\sin t = 3 \Rightarrow t = \frac{\pi}{2}$. The value $t = \frac{\pi}{2}$ satisfies the third equation $z = 1 - 9\sin^2\frac{\pi}{2} = 1 - 9 = -8$. Thus, $t = \frac{\pi}{2}$.

Plugging this value in the derivatives produces the direction vector $\langle -3, 0, 0 \rangle$. So, the tangent line is $x = 0 - 3t, y = 3 + 0t, z = -8 + 0t \Rightarrow x = -3t, y = 3, z = -8$.

(c) From part (b), we have that $t = \frac{\pi}{2}$ corresponds to the point $(0, 3, -8)$. Thus, $\frac{\pi}{2}$ is the upper bound. To find the lower bound, determine the t-value that corresponds to $(3, 0, 1)$. Set $x = 3, y = 0$ and $z = 1$ and make sure that you find the t-value that satisfies all three equations. From the first equation $x = 3\cos t = 3 \Rightarrow \cos t = 1 \Rightarrow t = 0$. From the second $y = 3\sin t = 0 \Rightarrow \sin t = 0 \Rightarrow t = 0$. The value $t = 0$ satisfies the third equation $z = 1 - 9\sin^2 0 = 1 - 0 = 1$. So, the bounds of integration are 0 to $\frac{\pi}{2}$.

The length is $L = \int_0^{\pi/2} \sqrt{(-3\sin t)^2 + (3\cos t)^2 + (-18\sin t \cos t)^2} dt = 10.48$.

5. (a) You can parametrize the cylinder $y^2 + z^2 = 16$ by $y = 4\cos t$ and $z = 4\sin t$. From the equation $x = 8 - y^2 - z$, you obtain that $x = 8 - (4\cos t)^2 - 4\sin t = 8 - 16\cos^2 t - 4\sin t$.

(b) To find a direction vector, we need to plug t-value that corresponds to the point $(-8, -4, 0)$ into the derivative $x' = 32\cos t \sin t - 4\cos t$, $y' = -4\sin t$, $z' = 4\cos t$. To find this t-value, set $x = -8, y = -4$ and $z = 0$ and make sure that you find the t-value that satisfies all three equations. From the second equation $y = 4\cos t = -4 \Rightarrow \cos t = -1 \Rightarrow t = \pi$. From the third $z = 4\sin t = 0 \Rightarrow \sin t = 0 \Rightarrow t = 0$ and $t = \pi$. The value $t = \pi$ agrees with the t-value we obtained using the y-equation. Plugging this value in the x-equation gives you $x = 8 - 16\cos^2 \pi - 4\sin \pi = 8 - 16 = -8$ which agrees with the x-coordinate of $(-8, -4, 0)$. Thus, $t = \pi$.

Plugging this value in the derivatives produces the direction vector $\langle 4, 0, -4 \rangle$. So, the tangent line is $x = -8 + 4t$, $y = -4 + 0t$, $z = 0 - 4t \Rightarrow x = -8 + 4t$, $y = -4$, $z = -4t$.

(c) From part (b), we have that $t = \pi$ corresponds to the point $(-8, -4, 0)$. Thus, π is the upper bound. To find the lower bound, determine the t-value that corresponds to $(4, 0, 4)$. Set $x = 4, y = 0$ and $z = 4$ and make sure that you find the t-value that satisfies all three equations. From the second equation $y = 4\cos t = 0 \Rightarrow \cos t = 0 \Rightarrow t = \pm\frac{\pi}{2}$. From the third $z = 4\sin t = 4 \Rightarrow \sin t = 1 \Rightarrow t = \frac{\pi}{2}$. So the value $-\frac{\pi}{2}$ obtained from the y-equation can be discarded and we obtain that $t = \frac{\pi}{2}$. Plugging this value in the x-equation gives you $x = 8 - 16\cos^2 \frac{\pi}{2} - 4\sin \frac{\pi}{2} = 8 - 4 = 4$ which agrees with the x-coordinate of $(4, 0, 4)$. Thus, the lower bound is $t = \frac{\pi}{2}$.

The length is $L = \int_{\pi/2}^{\pi} \sqrt{(32\cos t \sin t - 4\cos t)^2 + (-4\sin t)^2 + (4\cos t)^2} dt = 14.515$.

6. Recall that we found parametric equations of the three curves in the intersection to be

$x = 2\cos t, y = 2\sin t, z = 0$ with $0 \le t \le \frac{\pi}{2}$,
$x = t, y = 0, z = 4 - t^2$ with $0 \le t \le 2$, and
$x = 0, y = t, z = 4 - t^2$ with $0 \le t \le 2$.

The three derivative vectors and length elements are

$x' = -2\sin t, y' = 2\cos t, z' = 0 \Rightarrow$ $ds = \sqrt{4\sin^2 t + 4\cos^2 t} dt = \sqrt{4} dt = 2dt$
$x' = 1, y' = 0, z' = -2t \Rightarrow$ $ds = \sqrt{1 + 4t^2} dt$, and
$x' = 0, y' = 1, z' = -2t \Rightarrow$ $ds = \sqrt{1 + 4t^2} dt$.

The total length can be calculated as a sum of the three integrals below. Using the calculator for the second two produces

$$\int_0^{\pi/2} 2dt + \int_0^2 \sqrt{1 + 4t^2} dt + \int_0^2 \sqrt{1 + 4t^2} dt = \pi + 4.65 + 4.65 \approx 12.44.$$