Recall the parametric equations of a curve in $xy$-plane and compare them with parametric equations of a curve in space.

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<th>Parametric curve in plane</th>
<th>Parametric curve in space</th>
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<td>$x = x(t)$</td>
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<td>$y = y(t)$</td>
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<td>$z = z(t)$</td>
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Given its parametric equations $x = x(t)$, $y = y(t)$, $z = z(t)$, a curve $C$ can be considered to be a vector function, that is a function whose domain is in an interval of real numbers and the range is a set of vectors:

$$\mathbf{r}(t) = (x(t), y(t), z(t)).$$

In this case, the curve $C$ is the graph of the vector function $\mathbf{r}(t)$. Any value $t = t_0$ from the domain of $\mathbf{r}(t)$ corresponds to a point $(x_0, y_0, z_0)$ on the curve $C$.

You can think of $C$ as the trajectory of an object which moves as time passes by and of $\mathbf{r} = \mathbf{r}(t)$ as the position function of that object at time $t$. So, for any $t = t_0$, the point $(x_0, y_0, z_0)$ corresponds to the position at the time $t_0$.

The derivative of a vector function $\mathbf{r} = (x(t), y(t), z(t))$ is the vector function

$$\mathbf{r}'(t) = (x'(t), y'(t), z'(t)).$$

This vector function represents the velocity vector at time $t$.

At point $(x_0, y_0, z_0)$ which corresponds to the value $t_0$ of parameter $t$, the value of the derivative $\mathbf{r}'(t_0) = (x'(t_0), y'(t_0), z'(t_0))$ is parallel to the tangent line so it can be taken as the direction vector of the tangent line. This vector represents the velocity at the time $t = t_0$ and its magnitude is the speed of the object at that time.

Recall the process of finding the tangent line to a parametric curve from Calculus 2. To find an equation of the line tangent to the curve $x = x(t)$, $y = y(t)$ at $t = t_0$, note that this line passes the point $(x(t_0), y(t_0))$ and that the vector $(x'(t_0), y'(t_0))$ can be used as the direction vector. Analogously, to find an equation of the line tangent to the curve $x = x(t)$, $y = y(t)$, $z = z(t)$ at $t = t_0$, note that this line passes the point $(x(t_0), y(t_0), z(t_0))$, and that the vector $(x'(t_0), y'(t_0), z'(t_0))$ can be used as the direction vector. Hence, the tangent line can be described by the relations below.
Recall that the length element $ds$ of a parametric curve $x = x(t), y = y(t)$ with continuous derivatives on an interval $a \leq t \leq b$ can be obtained as the magnitude of the vector $\langle dx, dy \rangle$ so that

$$ds^2 = dx^2 + dy^2 \Rightarrow ds = \sqrt{dx^2 + dy^2} = \sqrt{(x'(t))^2 + (y'(t))^2}dt.$$  

The length of the curve is obtained by integrating the length element $ds$ from $a$ to $b$.

$$L = \int_a^b ds.$$  

Analogously, the length element of a space curve $x = x(t), y = y(t), z = z(t)$ is the magnitude of the vector $d\vec{r} = \langle dx, dy, dz \rangle$ and so

$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$

Note that $d\vec{r} = \vec{r}' dt$ so that

$$ds = |d\vec{r}| = |\vec{r}'|dt = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt$$

Thus the length of a space curve on the interval $a \leq t \leq b$ can be found as

$$L = \int_a^b ds = \int_a^b |\vec{r}'(t)|dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt.$$  

Let us compare again the two and three dimensional formulas.

| The length of the curve $\vec{r} = \langle x(t), y(t) \rangle$ for $a \leq t \leq b$ is $L = \int_a^b |\vec{r}'(t)|dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt$ | The length of the curve $\vec{r} = \langle x(t), y(t), z(t) \rangle$ for $a \leq t \leq b$ is $L = \int_a^b |\vec{r}'(t)|dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt$ |

Note that the quotient $\frac{ds}{dt}$ is equal to $|\vec{r}'|$ which is the speed of the movement.

$$\frac{ds}{dt} = |\vec{r}'| = \text{speed}$$

Because of this the integral above $L = \int_a^b ds$ computes the distance traveled during the time interval $[a, b]$.

**Practice Problems.**
1. Describe the following curves. For those without parametric representation, find equations of parametric equations.

(a) The curve given by \( x = 1 + t, \ y = 2 - 2t, \ z = 1 + 2t. \)
(b) The line segment from (1,2,−4) to (3,0,1).
(c) The curve given by \( x = \cos t, \ y = \sin t, \ z = 2. \)
(d) The curve given by \( x = \cos t, \ y = \sin t, \ z = t. \)
(e) The curve in the intersection of the cylinder \( x^2 + y^2 = 1 \) with the plane \( y + z = 2. \)
(f) The triangle in the boundary of the part of the plane \( 3x + 2y + z = 6 \) in the first octant.
(g) The boundary of the part of the paraboloid \( z = 4 - x^2 - y^2 \) in the first octant.

2. For the following curves, find an equation of the tangent line at the point where \( t = 0. \) Find the normalization of a direction vector at \( t = 0. \)

(a) The curve given by \( x = \cos t, \ y = \sin t, \ z = t. \)
(b) The curve in the intersection of the cylinder \( x^2 + y^2 = 1 \) with the plane \( y + z = 2. \) Use the parametrization of this curve from problem 1 (e).

3. For the following curves, find the length for \( 0 \leq t \leq \frac{\pi}{2}. \) Use the calculator to evaluate the integral in part (b).

(a) The curve given by \( x = \cos t, \ y = \sin t, \ z = t. \)
(b) The curve in the intersection of the cylinder \( x^2 + y^2 = 1 \) with the plane \( y + z = 2. \) Use the parametrization of this curve from problem 1 (e).

4. Find the length of the curve (given below) from (4,0,1) to (0,0,−1).

\[ x = 2 + 2 \cos t, \quad y = 2 \sin t, \quad z = \cos t - \sin^2 t \]

5. Consider the curve \( C \) which is the intersection of the surfaces

\[ x^2 + y^2 = 9 \quad \text{and} \quad z = 1 - y^2. \]

(a) Find the parametric equations that represent the curve \( C. \)
(b) Find the equation of the tangent line to the curve \( C \) at point \( (0,3,−8). \)
(c) Find the length of the curve from \( (3,0,1) \) to \( (0,3,−8) \). You can use the calculator to evaluate the integral that you are going to get.

6. Consider the curve \( C \) which is the intersection of the surfaces

\[ y^2 + z^2 = 16 \quad \text{and} \quad x = 8 - y^2 - z. \]

(a) Find the parametric equations that represent the curve \( C. \)
(b) Find the equation of the tangent line to the curve \( C \) at point \( (−8,−4,0). \)
(c) Find the length of the curve from $(4, 0, 4)$ to $(-8, -4, 0)$. Use the calculator to evaluate the integral that you are going to get.

7. Find the length of the boundary of the part of the paraboloid $z = 4 - x^2 - y^2$ in the first octant.

Solutions.

1. (a) This curve is a line passing the point $(1, 2, 1)$ in the direction of $(1, -2, 2)$.

(b) Any vector colinear with $(3, 0, 1) - (1, 2, -4) = (2, -2, 5)$ can be used as a direction vector of the line passing two points. You can also use any of $(1, 2, -4)$ and $(3, 0, 1)$ for a point on the line. For example, using $(1, 2, -4)$ we obtain parametric equations $x = 1 + 2t, y = 2 - 2tz = -4 + 5t$. Since we are looking only at the segment between the two points and $(1, 2, -4)$ corresponds to $t = 0$ and $(3, 0, 1)$ to $t = 1$, the line segment has parametrization $x = 1 + 2t, y = 2 - 2tz = -4 + 5t$ with $0 \leq t \leq 1$.

(c) The $xy$-equations $x = cos t, y = sin t$, represent a circle of radius 1 in $xy$-plane. Thus, the curve is on the cylinder based at this circle. The $z$-equation $z = 2$ represents the horizontal plane passing 2 on the $z$-axis. So, this curve is the intersection of the cylinder with the horizontal plane: it is a circle of radius 1 centered on the $z$-axis in the horizontal plane passing $z = 2$.

(d) The $xy$-equations $x = cos t, y = sin t$, represent a circle of radius 1 in $xy$-plane. Thus, the curve is on the cylinder based at this circle. The equation $z = t$ has an effect that $z$-values increase as $t$-values increase. Thus, this curve is a helix spiraling up the cylinder as the $t$-values increase. The Matlab notes cover graphing this (and other) curves in Matlab.

(e) The curve is the intersection of a cylinder with an inclined plane. So, the curve is an ellipse. The $xy$-equations represent a cylinder based at the circle of radius 1 in the $xy$-plane. Thus, $x$ and $y$ can be parametrized as $x = cos t, y = sin t$. To get the $z$ equation, solve the plane equation $y + z = 2$ for $z$, get $z = 2 - y$ and substitute that $y = sin t$. Thus $z = 2 - sin t$. This gives us parametric equations of the ellipse $x = cos t, y = sin t, z = 2 - sin t$. 


2. (a) To find a point on the tangent, plug $t = 0$ in the parametric equations of the curve. Get $x = 1, y = 0, z = 0$. To get a direction vector, plug $t = 0$ in the derivative $x' = -\sin t, y' = \cos t, z' = 1$. Get $\langle 0, 1, 1 \rangle$. So, the equation of the tangent is $x = 1 + 0t, y = 0 + 1t, z = 0 + 1t \Rightarrow x = 1, y = t, z = t$.

(f) The three line segments forming the triangle are the intersections of the plane $3x + 2y + z = 6$ with the three coordinate planes. The intersection in the $xy$-plane $z = 0$ can be obtained by plugging $z = 0$ in $3x + 2y + z = 6$ and using $x$, for example, as a parameter. Thus, we have part of the line $3x + 2y = 6 \Rightarrow y = 3 - \frac{3}{2}x$ between its two intercepts (2,0) and (0,3) and so $0 \leq x \leq 2$.

$$x = x, y = 3 - \frac{3}{2}x, z = 0 \text{ or } x = t, y = 3 - \frac{3}{2}t, z = 0 \text{ with } 0 \leq t \leq 2.$$  

Alternatively, the parametric equations of this line can be obtained as equations of a line passing $x$ and $y$ intercepts of the plane $3x + 2y + z = 6$, (2, 0, 0) and (0,3, 0). Using (-2,3,0) as direction vector and (0,3,0) as a point on the line, we obtain the equations $x = -2t, y = 3 + 3t, z = 0$ with $-1 \leq t \leq 0$.

Similarly, you can find equations of the remaining two sides of the triangle. The intersection of $xz$-plane $y = 0$ can be obtained by plugging $y = 0$ in $3x + 2y + z = 6$ and using $x$, for example, as a parameter. Thus, we have $3x + z = 6 \Rightarrow z = 6 - 3x$ and so $x = x, y = 0, z = 6 - 3x$ or $x = t, y = 0, z = 6 - 3t$ with $0 \leq t \leq 2$.

The intersection of $yz$-plane $x = 0$ can be obtained by plugging $x = 0$ in $3x + 2y + z = 6$ and using $y$, for example, as a parameter. Thus, we have $2y + z = 6 \Rightarrow z = 6 - 2y$ and so $x = 0, y = y, z = 6 - 2y$ or $x = 0, y = t, z = 6 - 2t$ with $0 \leq t \leq 3$.

(g) The boundary of the part of the paraboloid $z = 4 - x^2 - y^2$ in the first octant consists of three curves, each of which will have a different set of parametric equations. The parametrizations can be obtained by considering intersections with three coordinate planes $xy(z = 0), yz(x = 0)$, and $xz(y = 0)$.

The intersection in the $xy$-plane $z = 0$ is the circle $0 = 4 - x^2 - y^2 \Rightarrow x^2 + y^2 = 4$ which has parametric equations $x = 2 \cos t, y = 2 \sin t$. Since we are considering just the part in the first quadrant, $0 \leq t \leq \frac{\pi}{2}$. Thus, this curve has parametric equations

$$x = 2 \cos t, y = 2 \sin t, z = 0 \text{ with } 0 \leq t \leq \frac{\pi}{2}.$$

The intersection in the $yz$-plane $x = 0$ is the parabola $z = 4 - y^2$ with $0 \leq y \leq 2$. Using $y$ as a parameter produces parametric equations $x = 0, y = y, z = 4 - y^2$ or $x = 0, y = t, z = 4 - t^2$ with $0 \leq t \leq 2$.

The intersection in the $xz$-plane $y = 0$ is the parabola $z = 4 - x^2$ with $0 \leq x \leq 2$. Using $x$ as a parameter produces parametric equations $x = x, y = 0, z = 4 - x^2$ or $x = t, y = 0, z = 4 - t^2$ with $0 \leq t \leq 2$. 

2. (a) To find a point on the tangent, plug $t = 0$ in the parametric equations of the curve. Get $x = 1, y = 0, z = 0$. To get a direction vector, plug $t = 0$ in the derivative $x' = -\sin t, y' = \cos t, z' = 1$. Get $\langle 0, 1, 1 \rangle$. So, the equation of the tangent is $x = 1 + 0t, y = 0 + 1t, z = 0 + 1t \Rightarrow x = 1, y = t, z = t$. 

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The direction vector \( \langle 0, 1, 1 \rangle \) has length \( \sqrt{2} \) so its normalization is \( \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle \).

(b) Use the parametric equations from problem 1 (d) \( x = \cos t, \ y = \sin t, \ z = 2 - \sin t \). To find a point on the tangent, plug \( t = 0 \) in the parametric equations. Get \( x = 1, \ y = 0, \ z = 2 \). To get a direction vector, plug \( t = 0 \) in the derivative \( x' = -\sin t, \ y' = \cos t, \ z' = -\cos t \). Get \( \langle 0, 1, -1 \rangle \). So, the equation of the tangent is \( x = 1 + 0t, \ y = 0 + 1t, \ z = 2 - 1t \Rightarrow x = 1, \ y = t, \ z = 2 - t \).

The direction vector \( \langle 0, 1, -1 \rangle \) has length \( \sqrt{2} \) so its normalization is \( \langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \rangle \).

3. (a) You can parametrize the cylinder \( x = 2 + 2 \cos t, \ y = 2 + 2 \sin t \). The value \( t = 0 \) agrees with the \( x \)-equation. Check that it agrees with the \( z \)-equation as well: \( 1 = z = \cos(0) - \sin^2(0) = 1 - 0 = 1 \). So, 0 is the lower bound.

When \( (x, y, z) = (0, 0, -1), \) \( x = 2 + 2 \cos t = 0 \Rightarrow \cos t = -1 \Rightarrow t = \pi \) and \( y = 2 \sin t = 0 \Rightarrow t = 0 \) and \( \pi \). The value \( t = \pi \) agrees with the \( x \)-equation. Check that it agrees with the \( z \)-equation as well: \(-1 = z = \cos(\pi) - \sin^2(\pi) = -1 - 0 = -1 \). So, \( \pi \) is the upper bound.

Since \( x' = -2 \sin t, \ y' = 2 \cos t, \) and \( z' = -\sin t - 2 \sin t \cos t \) the length is

\[
L = \int_0^\pi \sqrt{(-2 \sin t)^2 + (2 \cos t)^2 + (-\sin t - 2 \sin t \cos t)^2} dt = \int_0^\pi \sqrt{4 + \sin^2 t (1 + 2 \cos t)^2} dt.
\]

Use the calculator to evaluate this integral and obtain \( L \approx 6.98 \).

4. Find the \( t \)-values corresponding to \( (4, 0, 1) \) to \( (0, 0, -1) \) for the \( t \)-bounds. When \( (x, y, z) = (4, 0, 1), \) \( x = 2 + 2 \cos t = 4 \Rightarrow \cos t = 1 \Rightarrow t = 0 \) and \( y = 2 \sin t = 0 \Rightarrow t = 0 \) and \( \pi \). The value \( t = 0 \) agrees with the \( x \)-equation. Check that it agrees with the \( z \)-equation as well: \( 1 = z = \cos(0) - \sin^2(0) = 1 - 0 = 1 \). So, 0 is the lower bound.

When \( (x, y, z) = (0, 0, -1), \) \( x = 2 + 2 \cos t = 0 \Rightarrow \cos t = -1 \Rightarrow t = \pi \) and \( y = 2 \sin t = 0 \Rightarrow t = 0 \) and \( \pi \). The value \( t = \pi \) agrees with the \( x \)-equation. Check that it agrees with the \( z \)-equation as well: \(-1 = z = \cos(\pi) - \sin^2(\pi) = -1 - 0 = -1 \). So, \( \pi \) is the upper bound.

Since \( x' = -2 \sin t, \ y' = 2 \cos t, \) and \( z' = -\sin t - 2 \sin t \cos t \) the length is

\[
L = \int_0^\pi \sqrt{(-2 \sin t)^2 + (2 \cos t)^2 + (-\sin t - 2 \sin t \cos t)^2} dt = \int_0^\pi \sqrt{4 + \sin^2 t (1 + 2 \cos t)^2} dt.
\]

Use the calculator to evaluate this integral and obtain \( L \approx 6.98 \).

5. (a) You can parametrize the cylinder \( x^2 + y^2 = 9 \) by \( x = 3 \cos t \) and \( y = 3 \sin t \). From the equation \( z = 1 - y^2 \), you obtain that \( z = 1 - (3 \sin t)^2 = 1 - 9 \sin^2 t \).

(b) To find a direction vector, we need to plug \( t \)-value that corresponds to the point \( (0, 3, -8) \) into the derivative \( x' = -3 \sin t, \ y' = 3 \cos t, \ z' = -18 \sin t \cos t \). To find this \( t \)-value, set \( x = 0, y = 3 \) and \( z = -8 \) and make sure that you find the \( t \)-value that satisfies all three equations. From the first equation \( x = 3 \cos t = 0 \Rightarrow t = \pm \frac{\pi}{2} \). From the second \( y = 3 \sin t = 3 \Rightarrow t = \frac{\pi}{2} \). The value \( t = \frac{\pi}{2} \) satisfies the third equation \( z = 1 - 9 \sin^2 \frac{\pi}{2} = 1 - 9 = -8 \). Thus, \( t = \frac{\pi}{2} \).

Plugging this value in the derivatives produces the direction vector \( \langle -3, 0, 0 \rangle \). So, the tangent line is \( x = 0 - 3t, \ y = 3 + 0t, \ z = -8 + 0t \Rightarrow x = -3t, \ y = 3, \ z = -8 \).

(c) From part (b), we have that \( t = \frac{\pi}{2} \) corresponds to the point \( (0, 3, -8) \). Thus, \( \frac{\pi}{2} \) is the upper bound. To find the lower bound, determine the \( t \)-value that corresponds to \( (3, 0, 1) \). Set \( x = 3, y = 0 \) and \( z = 1 \) and make sure that you find the \( t \)-value that satisfies all three equations. From the first equation \( x = 3 \cos t = 3 \Rightarrow \cos t = 1 \Rightarrow t = 0 \). From the second \( y = 3 \sin t = 0 \Rightarrow \sin t = 0 \Rightarrow t = 0 \). The value \( t = 0 \) satisfies the third equation \( z = 1 - 9 \sin^2 0 = 1 - 0 = 1 \). So, the bounds of integration are 0 to \( \frac{\pi}{2} \).

The length is \( L = \int_0^{\frac{\pi}{2}} \sqrt{(-3 \sin t)^2 + (3 \cos t)^2 + (-18 \sin t \cos t)^2} dt = 10.48 \).
6. (a) You can parametrize the cylinder \( y^2 + z^2 = 16 \) by \( y = 4 \cos t \) and \( z = 4 \sin t \). From the equation \( x = 8 - y^2 - z \), you obtain that \( x = 8 - (4 \cos t)^2 - 4 \sin t = 8 - 16 \cos^2 t - 4 \sin t \).

(b) To find a direction vector, we need to plug \( t \)-value that corresponds to the point \((-8, -4, 0)\) into the derivative \( x' = 32 \cos t \sin t - 4 \cos t \), \( y' = -4 \sin t \), \( z' = 4 \cos t \). To find this \( t \)-value, set \( x = -8, y = -4 \) and \( z = 0 \) and make sure that you find the \( t \)-value that satisfies all three equations. From the second equation \( y = 4 \cos t = -4 \) \( \Rightarrow \cos t = -1 \Rightarrow t = \pi \). From the third \( z = 4 \sin t = 0 \Rightarrow \sin t = 0 \Rightarrow t = 0 \) and \( t = \pi \). The value \( t = \pi \) agrees with the \( t \)-value we obtained using the \( y \)-equation. Plugging this value in the \( x \)-equation gives you \( x = 8 - 16 \cos^2 \pi - 4 \sin \pi = 8 - 16 = -8 \) which agrees with the \( x \)-coordinate of \((-8, -4, 0)\). Thus, \( t = \pi \).

Plugging this value in the derivatives produces the direction vector \((4, 0, -4)\). So, the tangent line is \( x = -8 + 4t, y = -4 + 4t, z = 0 - 4t \Rightarrow x = -8 + 4t, y = -4, z = -4t \).

(c) From part (b), we have that \( t = \pi \) corresponds to the point \((-8, -4, 0)\). Thus, \( \pi \) is the upper bound. To find the lower bound, determine the \( t \)-value that corresponds to \((4, 0, 4)\). Set \( x = 4, y = 0 \) and \( z = 4 \) and make sure that you find the \( t \)-value that satisfies all three equations. From the second equation \( y = 4 \cos t = 0 \Rightarrow \cos t = 0 \Rightarrow t = \pm \frac{\pi}{2} \). From the third \( z = 4 \sin t = 4 \Rightarrow \sin t = 1 \Rightarrow t = \frac{\pi}{2} \). So the value \(-\frac{\pi}{2} \) obtained from the \( y \)-equation can be discarded and we obtain that \( t = \frac{\pi}{2} \). Plugging this value in the \( x \)-equation gives you \( x = 8 - 16 \cos^2 \frac{\pi}{2} - 4 \sin \frac{\pi}{2} = 8 - 4 = 4 \) which agrees with the \( x \)-coordinate of \((4, 0, 4)\). Thus, the lower bound is \( t = \frac{\pi}{2} \).

The length is \( L = \int_{\pi/2}^{\pi} \sqrt{(32 \cos t \sin t - 4 \cos t)^2 + (-4 \sin t)^2 + (4 \cos t)^2} dt = 14.515 \).

7. Recall that we found parametric equations of the three curves in the intersection to be

\[
\begin{align*}
x &= 2 \cos t, \quad y = 2 \sin t, \quad z = 0 \quad \text{with } 0 \leq t \leq \frac{\pi}{2}, \\
x &= 0, \quad y = t, \quad z = 4 - t^2 \quad \text{with } 0 \leq t \leq 2, \text{ and} \\
x &= t, \quad y = 0, \quad z = 4 - t^2 \quad \text{with } 0 \leq t \leq 2.
\end{align*}
\]

The three derivative vectors and length elements are

\[
\begin{align*}
x' &= -2 \sin t, \quad y' = 2 \cos t, \quad z' = 0 \quad \Rightarrow \quad ds = \sqrt{4 \sin^2 t + 4 \cos^2 t} dt = 2 dt \\
x' &= 0, \quad y' = 1, \quad z' = -2t \quad \Rightarrow \quad ds = \sqrt{1 + 4t^2} dt, \text{ and} \\
x' &= 1, \quad y' = 0, \quad z' = -2t \quad \Rightarrow \quad ds = \sqrt{1 + 4t^2} dt.
\end{align*}
\]

The total length can be calculated as a sum of the three integrals below. Using the calculator for the second two produces

\[
\int_{0}^{\pi/2} 2 dt + \int_{0}^{2} \sqrt{1 + 4t^2} dt + \int_{0}^{2} \sqrt{1 + 4t^2} dt = \pi + 4.65 + 4.65 \approx 12.44.
\]