Recall the parametric equations of a curve in $xy$-plane and compare them with parametric equations of a curve in space.

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<th>Parametric curve in plane</th>
<th>Parametric curve in space</th>
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<td>$x = x(t)$</td>
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<td>$y = y(t)$</td>
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<td>$z = z(t)$</td>
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Given its parametric equations $x = x(t)$, $y = y(t)$, $z = z(t)$, a curve $C$ can be considered to be a vector function, that is a function whose domain is in an interval of real numbers and the range is a set of vectors:

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle.$$  

In this case, the curve $C$ is the graph of the vector function $\vec{r}(t)$. Any value $t = t_0$ from the domain of $\vec{r}(t)$ corresponds to a point $(x_0, y_0, z_0)$ on the curve $C$.

The derivative of a vector function $\vec{r} = \langle x(t), y(t), z(t) \rangle$ is the vector function

$$\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

At point $(x_0, y_0, z_0)$ which corresponds to the value $t_0$ of parameter $t$, the value of the derivative $\vec{r}'(t_0) = \langle x'(t_0), y'(t_0), z'(t_0) \rangle$ represents the velocity vector of the tangent line at $(x_0, y_0, z_0)$.

Note the analogy with the two-dimensional scenario.

### To find a tangent line to the curve $x = x(t), y = y(t)$ at $t = t_0$

- **Point**: $(x(t_0), y(t_0))$
- **Direction vector**: $\langle x'(t_0), y'(t_0) \rangle$

**An equation of the tangent line:**

$$\begin{align*}
x &= x(t_0) + x'(t_0)t \\
y &= y(t_0) + y'(t_0)t
\end{align*}$$

### To find a tangent line to the curve $x = x(t), y = y(t), z = z(t)$ at $t = t_0$

- **Point**: $(x(t_0), y(t_0), z(t_0))$
- **Direction vector**: $\langle x'(t_0), y'(t_0), z'(t_0) \rangle$

**An equation of the tangent line:**

$$\begin{align*}
x &= x(t_0) + x'(t_0)t \\
y &= y(t_0) + y'(t_0)t \\
z &= z(t_0) + z'(t_0)t
\end{align*}$$
Recall that the length $L$ of parametric curve $x = x(t), y = y(t)$ with continuous derivatives on an interval $a \leq t \leq b$ can be obtained by integrating the length element $ds$ from $a$ to $b$.

$$L = \int_{a}^{b} ds.$$ 

The length element $ds$ on a sufficiently small interval can be approximated by the hypotenuse of a triangle with sides $dx$ and $dy$ and so $ds^2 = dx^2 + dy^2 \Rightarrow ds = \sqrt{dx^2 + dy^2} = \sqrt{(x'(t))^2 + (y'(t))^2}dt$.

Analogously, the length element of a space curve satisfies $ds^2 = dx^2 + dy^2 + dz^2$ and so $ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt$.

The expression $\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$ represents the length of the derivative vector $\vec{r}' = \langle x', y', z' \rangle$. Thus the length of a space curve on the interval $a \leq t \leq b$ can be found as

$$L = \int_{a}^{b} ds = \int_{a}^{b} |\vec{r}'(t)|dt = \int_{a}^{b} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt.$$ 

Compare again the two and three dimensional formulas.

| The length of the curve $\vec{r} = \langle x(t), y(t) \rangle$ for $a \leq t \leq b$ is $L = \int_{a}^{b} \sqrt{x'(t)^2 + y'(t)^2}dt = \int_{a}^{b} |\vec{r}'(t)|dt$ | The length of the curve $\vec{r} = \langle x(t), y(t), z(t) \rangle$ for $a \leq t \leq b$ is $L = \int_{a}^{b} \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}dt = \int_{a}^{b} |\vec{r}'(t)|dt$ |
|---|---|

Practice Problems.

1. Describe the following curves. For those without parametric representation, find equations of parametric equations.

(a) The curve given by $x = 1 + t, y = 2 - 2t, z = 1 + 2t$.
(b) The line segment from $(1, 2, -4)$ to $(3, 0, 1)$.
(c) The curve given by $x = \cos t, y = \sin t, z = 2$.
(d) The curve given by $x = \cos t, y = \sin t, z = t$.
(e) The curve in the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $y + z = 2$.
(f) The triangle in the boundary of the part of the plane $3x + 2y + z = 6$ in the first octant.
(g) The boundary of the part of the paraboloid $z = 4 - x^2 - y^2$ in the first octant.
2. For the following curves, find an equation of the tangent line at the point where \( t = 0 \). Find the normalization of a direction vector at \( t = 0 \).

(a) The curve given by \( x = \cos t, y = \sin t, z = t \).

(b) The curve in the intersection of the cylinder \( x^2 + y^2 = 1 \) with the plane \( y + z = 2 \). Use the parametrization of this curve from problem 1 (e).

3. For the following curves, find the length for \( 0 \leq t \leq \frac{\pi}{2} \). Use the calculator to evaluate the integral in part (b).

(a) The curve given by \( x = \cos t, y = \sin t, z = t \).

(b) The curve in the intersection of the cylinder \( x^2 + y^2 = 1 \) with the plane \( y + z = 2 \). Use the parametrization of this curve from problem 1 (e).

4. Consider the curve \( C \) which is the intersection of the surfaces \( x^2 + y^2 = 9 \) and \( z = 1 - y^2 \).

(a) Find the parametric equations that represent the curve \( C \).

(b) Find the equation of the tangent line to the curve \( C \) at point \((0, 3, -8)\).

(c) Find the length of the curve from \((3, 0, 1)\) to \((0, 3, -8)\). You can use the calculator to evaluate the integral that you are going to get.

5. Consider the curve \( C \) which is the intersection of the surfaces \( y^2 + z^2 = 16 \) and \( x = 8 - y^2 - z \).

(a) Find the parametric equations that represent the curve \( C \).

(b) Find the equation of the tangent line to the curve \( C \) at point \((-8, -4, 0)\).

(c) Find the length of the curve from \((4, 0, 4)\) to \((-8, -4, 0)\). Use the calculator to evaluate the integral that you are going to get.

6. Find the length of the boundary of the part of the paraboloid \( z = 4 - x^2 - y^2 \) in the first octant.

**Solutions.**

1. (a) This curve is a line passing the point \((1, 2, 1)\) in the direction of \((1, -2, 2)\).

(b) Any vector colinear with \(\langle 3, 0, 1 \rangle - \langle 1, 2, -4 \rangle = \langle 2, -2, 5 \rangle\) can be used as a direction vector of the line passing two points. You can also use any of \((1,2,-4)\) and \((3, 0, 1)\) for a point on the line. For example, using \((1,2,-4)\) we obtain parametric equations \(x = 1 + 2t, y = 2 - 2tz = -4 + 5t\).

In this case, the initial point corresponds to \( t = 0 \) and the end point to \( t = 1 \). So, the line segment has parametrization \(x = 1 + 2t, y = 2 - 2tz = -4 + 5t\) with \(0 \leq t \leq 1\).

(c) The \(xy\)-equations \(x = \cos t, y = \sin t\), represent a circle of radius 1 in \(xy\)-plane. Thus, the curve is on the cylinder determined by this circle. The \(z\)-equation \(z = 2\) represents the horizontal plane passing 2 on the \(z\)-axis. So, this curve is the intersection of the cylinder with
the horizontal plane: it is a circle of radius 1 centered on the z-axis in the horizontal plane passing z = 2.

(d) The xy-equations \( x = \cos t, y = \sin t \), represent a circle of radius 1 in xy-plane. Thus, the curve is on the cylinder determined by this circle. The equation \( z = t \) has an effect that z-values increase as t-values increase. Thus, this curve is a helix spiraling up the cylinder as t increases. Use Matlab to get a precise graph.

(e) The curve is the intersection of a cylinder with an inclined plane. So, the curve is an ellipse. The xy-equations represent a cylinder based at the circle of radius 1 in xy-plane. Thus, \( x \) and \( y \) can be parametrized as \( x = \cos t, y = \sin t \). To get the z equation, solve the plane equation \( y + z = 2 \) for \( z \), get \( z = 2 - y \) and substitute that \( y = \sin t \). Thus \( z = 2 - \sin t \). This gives us an equation of the ellipse to be \( x = \cos t, y = \sin t, z = 2 - \sin t \). You can use Matlab to get a precise graph.

(f) The triangle in the boundary of the part of the plane \( 3x + 2y + z = 6 \) in the first octant consists of the three line segments, each of which will have a different set of parametric equations. The intersection in xy-plane \( z = 0 \) can be obtained by plugging \( z = 0 \) in \( 3x + 2y + z = 6 \) and using \( x \), for example, as a parameter. Thus, we have part of the line \( 3x + 2y = 6 \Rightarrow y = 3 - \frac{3}{2}x \) between its two intercepts (2,0) and (0,3) and so \( 0 \leq x \leq 2 \)

\[ x = x, y = 3 - \frac{3}{2}x, z = 0 \text{ or } x = t, y = 3 - \frac{3}{2}t, z = 0 \text{ with } 0 \leq t \leq 2. \]

Alternatively, the parametric equations of this line can be obtained as equations of a line passing x and y intercepts of the plane \( 3x + 2y + z = 6 \), (2, 0, 0) and (0,3, 0). Using \((-2,3,0)\) as direction vector and \((0,3,0)\) as a point on the line, we obtain the equations \( x = -2t, y = 3 + 3t, z = 0 \) with \(-1 \leq t \leq 0 \).

Similarly, you can find equations of the remaining two sides of the triangle. The intersection of xz-plane \( y = 0 \) can be obtained by plugging \( y = 0 \) in \( 3x + 2y + z = 6 \) and using \( x \), for example, as a parameter. Thus, we have \( 3x + z = 6 \Rightarrow z = 6 - 3x \) and so \( x = x, y = 0, z = 6 - 3x \) or \( x = t, y = 0, z = 6 - 3t \) with \( 0 \leq t \leq 2 \).

The intersection of yz-plane \( x = 0 \) can be obtained by plugging \( x = 0 \) in \( 3x + 2y + z = 6 \) and using \( y \), for example, as a parameter. Thus, we have \( 2y + z = 6 \Rightarrow z = 6 - 2y \) and so \( x = 0, y = y, z = 6 - 2y \text{ or } x = 0, y = t, z = 6 - 2t \) with \( 0 \leq t \leq 3 \).

(g) The boundary of the part of the paraboloid \( z = 4 - x^2 - y^2 \) in the first octant consists of three curves, each of which will have a different set of parametric equations. The parametrizations
can be obtained by considering intersections with three coordinate planes $x = 0, y = 0$, and $z = 0$ respectively.

The intersection in $xy$-plane $z = 0$ is a circle $0 = 4 - x^2 - y^2 \Rightarrow x^2 + y^2 = 4$ which has parametric equations $x = 2 \cos t, y = 2 \sin t$. Since we are considering just the part with $x \geq 0$ and $y \geq 0$, we have that $0 \leq t \leq \frac{\pi}{2}$. Thus, this curve has parametric equations

$$x = 2 \cos t, y = 2 \sin t, z = 0 \text{ with } 0 \leq t \leq \frac{\pi}{2}.$$ 

The intersection in $xz$-plane $y = 0$ is a parabola $z = 4 - x^2$ with $0 \leq x \leq 2$. Using $x$ as a parameter produces parametric equations $x = x, y = 0, z = 4 - x^2$ or $x = t, y = 0, z = 4 - t^2$ with $0 \leq t \leq 2$.

The intersection in $yz$-plane $x = 0$ is a parabola $z = 4 - y^2$ with $0 \leq y \leq 2$. Using $y$ as a parameter produces parametric equations $x = 0, y = y, z = 4 - y^2$ or $x = 0, y = t, z = 4 - t^2$ with $0 \leq t \leq 2$.

2. (a) To find a point on the tangent, plug $t = 0$ in the parametric equations of the curve. Get $x = 1, y = 0, z = 0$. To get a direction vector, plug $t = 0$ in the derivative $x' = -\sin t, y' = \cos t, z' = 1$. Get $\langle 0, 1, 1 \rangle$. So, the equation of the tangent is $x = 1 + 0t, y = 0 + 1t, z = 0 + 1t \Rightarrow x = 1, y = t, z = t$.

The direction vector $\langle 0, 1, 1 \rangle$ has length $\sqrt{2}$ so its normalization is $\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$.

(b) Use the parametric equations from problem 1 (d) $x = \cos t, y = \sin t, z = 2 - \sin t$. To find a point on the tangent, plug $t = 0$ in the parametric equations. Get $x = 1, y = 0, z = 2$. To get a direction vector, plug $t = 0$ in the derivative $x' = -\sin t, y' = \cos t, z' = -\cos t$. Get $\langle 0, 1, -1 \rangle$. So, the equation of the tangent is $x = 1 + 0t, y = 0 + 1t, z = 2 - 1t \Rightarrow x = 1, y = t, z = 2 - t$.

The direction vector $\langle 0, 1, -1 \rangle$ has length $\sqrt{2}$ so its normalization is $\langle 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$.

3. (a) $L = \int_{0}^{\pi/2} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt = \int_{0}^{\pi/2} \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2} dt = \int_{0}^{\pi/2} \sqrt{1 + 1} dt = \sqrt{2} \int_{0}^{\pi/2} dt = \frac{\sqrt{2}\pi}{2}$.

(b) $L = \int_{0}^{\pi/2} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt = \int_{0}^{\pi/2} \sqrt{(-\sin t)^2 + (\cos t)^2 + (-\cos t)^2} dt$. Using the calculator, $L \approx 1.91$.

4. (a) You can parametrize the cylinder $x^2 + y^2 = 9$ by $x = 3 \cos t$ and $y = 3 \sin t$. From the equation $z = 1 - y^2$, you obtain that $z = 1 - (3 \sin t)^2 = 1 - 9 \sin^2 t$.

(b) To find a direction vector, we need to plug $t$-value that corresponds to the point $(0, 3, -8)$ into the derivative $x' = -3 \sin t, y' = 3 \cos t, z' = -18 \sin t \cos t$. To find this $t$-value, set $x = 0, y = 3$ and $z = -8$ and make sure that you find the $t$-value that satisfies all three equations. From the first equation $x = 3 \cos t = 0 \Rightarrow t = \pm \frac{\pi}{2}$. From the second $y = 3 \sin t = 3 \Rightarrow t = \frac{\pi}{2}$. The value $t = \frac{\pi}{2}$ satisfies the third equation $z = 1 - 9 \sin^2 \frac{\pi}{2} = 1 - 9 = -8$. Thus, $t = \frac{\pi}{2}$.

Plugging this value in the derivatives produces the direction vector $\langle -3, 0, 0 \rangle$. So, the tangent line is $x = 0 - 3t, y = 3 + 0t, z = -8 + 0t \Rightarrow x = -3t, y = 3, z = -8$. 

5
(c) From part (b), we have that $t = \frac{\pi}{2}$ corresponds to the point $(0, 3, -8)$. Thus, $\frac{\pi}{2}$ is the upper bound. To find the lower bound, determine the $t$-value that corresponds to $(3, 0, 1)$. Set $x = 3, y = 0$ and $z = 1$ and make sure that you find the $t$-value that satisfies all three equations. From the first equation $x = 3 \cos t = 3 \Rightarrow \cos t = 1 \Rightarrow t = 0$. From the second $y = 3 \sin t = 0 \Rightarrow \sin t = 0 \Rightarrow t = 0$. The value $t = 0$ satisfies the third equation $z = 1 - 9 \sin^2 t = 1 - 0 = 1$. So, the bounds of integration are $0$ to $\frac{\pi}{2}$.

The length is $L = \int_0^{\pi/2} \sqrt{(-3 \sin t)^2 + (3 \cos t)^2 + (-18 \sin t \cos t)^2} \, dt = 10.48$.

5. (a) You can parametrize the cylinder $y^2 + z^2 = 16$ by $y = 4 \cos t$ and $z = 4 \sin t$. From the equation $x = 8 - y^2 - z$, you obtain that $x = 8 - (4 \cos t)^2 - 4 \sin t = 8 - 16 \cos^2 t - 4 \sin t$.

(b) To find a direction vector, we need to plug $t$-value that corresponds to the point $(-8, -4, 0)$ into the derivative $x' = 32 \cos t \sin t - 4 \cos t, y' = -4 \sin t, z' = 4 \cos t$. To find this $t$-value, set $x = -8, y = -4$ and $z = 0$ and make sure that you find the $t$-value that satisfies all three equations. From the second equation $y = 4 \cos t = -4 \Rightarrow \cos t = -1 \Rightarrow t = \pi$. From the third $z = 4 \sin t = 0 \Rightarrow \sin t = 0 \Rightarrow t = 0$ and $t = \pi$. The value $t = \pi$ agrees with the $t$-value we obtained using the $y$-equation. Plugging this value in the $x$-equation gives you $x = 8 - 16 \cos^2 \pi = -4 \sin \pi = 8 - 16 = -8$ which agrees with the $x$-coordinate of $(-8, -4, 0)$.

Thus, $t = \pi$.

Plugging this value in the derivatives produces the direction vector $(4, 0, -4)$. So, the tangent line is $x = -8 + 4t, y = -4 + 0t, z = 0 - 4t \Rightarrow x = -8 + 4t, y = -4, z = -4t$.

(c) From part (b), we have that $t = \pi$ corresponds to the point $(-8, -4, 0)$. Thus, $\pi$ is the upper bound. To find the lower bound, determine the $t$-value that corresponds to $(4, 0, 4)$. Set $x = 4, y = 0$ and $z = 4$ and make sure that you find the $t$-value that satisfies all three equations. From the second equation $y = 4 \cos t = 0 \Rightarrow \cos t = 0 \Rightarrow t = \pm \frac{\pi}{2}$. From the third $z = 4 \sin t = 4 \Rightarrow \sin t = 1 \Rightarrow t = \frac{\pi}{2}$. So the value $-\frac{\pi}{2}$ obtained from the $y$-equation can be discarded and we obtain that $t = \frac{\pi}{2}$. Plugging this value in the $x$-equation gives you $x = 8 - 16 \cos^2 \frac{\pi}{2} = 8 - 4 = 4$ which agrees with the $x$-coordinate of $(4, 0, 4)$. Thus, the lower bound is $t = \frac{\pi}{2}$.

The length is $L = \int_{\pi/2}^\pi \sqrt{(32 \cos t \sin t - 4 \cos t)^2 + (-4 \sin t)^2 + (4 \cos t)^2} \, dt = 14.515$.

6. Recall that we found parametric equations of the three curves in the intersection to be

\[
\begin{align*}
x &= 2 \cos t, \quad y = 2 \sin t, \quad z = 0 \quad \text{with} \quad 0 \leq t \leq \frac{\pi}{2}, \\
x &= t, \quad y = 0, \quad z = 4 - t^2 \quad \text{with} \quad 0 \leq t \leq 2, \text{ and} \\
x &= 0, \quad y = t, \quad z = 4 - t^2 \quad \text{with} \quad 0 \leq t \leq 2.
\end{align*}
\]

The three derivative vectors and length elements are

\[
\begin{align*}
x' &= -2 \sin t, \quad y' = 2 \cos t, \quad z' = 0 \quad \Rightarrow \quad ds = \sqrt{4 \sin^2 t + 4 \cos^2 t} \, dt = \sqrt{4} \, dt = 2 \, dt \\
x' &= 1, \quad y' = 0, \quad z' = -2t \quad \Rightarrow \quad ds = \sqrt{1 + 4t^2} \, dt, \text{ and} \\
x' &= 0, \quad y' = 1, \quad z' = -2t \quad \Rightarrow \quad ds = \sqrt{1 + 4t^2} \, dt.
\end{align*}
\]

The total length can be calculated as a sum of the three integrals below. Using the calculator for the second two produces

\[
\int_0^{\pi/2} 2 \, dt + \int_0^2 \sqrt{1 + 4t^2} \, dt + \int_0^2 \sqrt{1 + 4t^2} \, dt = \pi + 4.65 + 4.65 \approx 12.44.
\]