**Calculus 3**
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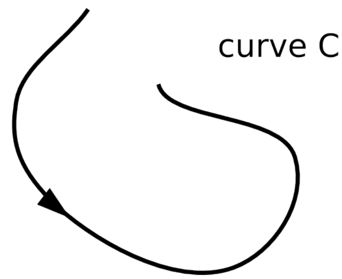
# Space Curves

Recall the parametric equations of a curve in $xy$-plane and compare them with parametric equations of a curve in space.

| Parametric curve in plane | Parametric curve in space |
|---|---|
| $x = x(t)$<br>$y = y(t)$ | $x = x(t)$<br>$y = y(t)$<br>$z = z(t)$ |




Given its parametric equations $x = x(t)$, $y = y(t)$, $z = z(t)$, a curve $C$ can be considered to be a **vector function**, that is a function whose domain is in an interval of real numbers and the range is a set of vectors:

$$\vec{r}(t) = \langle x(t),\ y(t),\ z(t)\rangle.$$

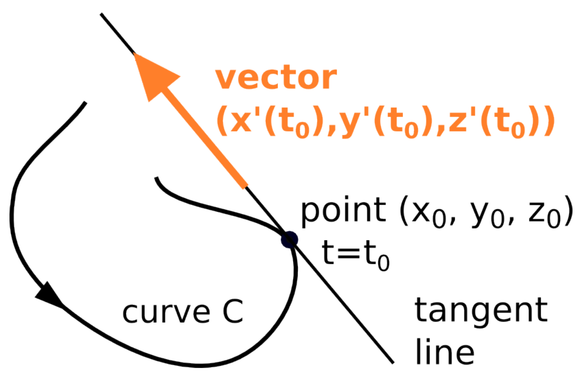
In this case, the curve $C$ is the graph of the vector function $\vec{r}(t)$. Any value $t = t_0$ from the domain of $\vec{r}(t)$ corresponds to a point $(x_0, y_0, z_0)$ on the curve $C$.

You can think of $C$ as the trajectory of an object which moves as time passes by and of $\vec{r} = \vec{r}(t)$ as the **position function** of that object at time $t$. So, for any $t = t_0$, the point $(x_0, y_0, z_0)$ corresponds to the **position at the time** $t_0$.

The **derivative** of a vector function $\vec{r} = \langle x(t),\ y(t),\ z(t)\rangle$ is the vector function

$$\vec{r}\,'(t) = \langle x'(t),\ \ y'(t),\ \ z'(t)\rangle$$

This vector function represents the **velocity vector** at time $t$.

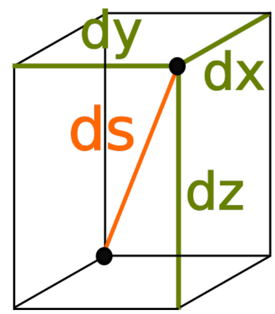



At point $(x_0, y_0, z_0)$ which corresponds to the value $t_0$ of parameter $t$, the value of the derivative $\vec{r}\,'(t_0) = \langle x'(t_0), y'(t_0), z'(t_0)\rangle$ is parallel to the tangent line so it can be taken as the **direction vector** of the tangent line. This vector represents the **velocity** at the time $t = t_0$ and its magnitude is the **speed** of the object at that time.

Recall the process of finding the tangent line to a parametric curve from Calculus 2. To find an equation of the line tangent to the curve $x = x(t)$, $y = y(t)$ at $t = t_0$, note that this line passes the point $(x(t_0), y(t_0))$ and that the vector $\langle x'(t_0), y'(t_0)\rangle$ can be used as the direction vector. Analogously, to find an equation of the line tangent to the curve $x = x(t)$, $y = y(t)$, $z = z(t)$ at $t = t_0$, note that this line passes the point $(x(t_0), y(t_0), z(t_0))$, and that the vector $\langle x'(t_0), y'(t_0), z'(t_0)\rangle$ can be used as the direction vector. Hence, the tangent line can be described by the relations below.

$$x = x(t_0) + x'(t_0)t$$
$$y = y(t_0) + y'(t_0)t$$
$$z = z(t_0) + z'(t_0)t$$

Recall that the length element $ds$ of a parametric curve $x = x(t), y = y(t)$ with continuous derivatives on an interval $a \le t \le b$ can be obtained as the magnitude of the vector $\langle dx, dy \rangle$ so that $ds^2 = dx^2 + dy^2 \Rightarrow ds = \sqrt{dx^2 + dy^2} = \sqrt{(x'(t))^2 + (y'(t))^2}dt$.

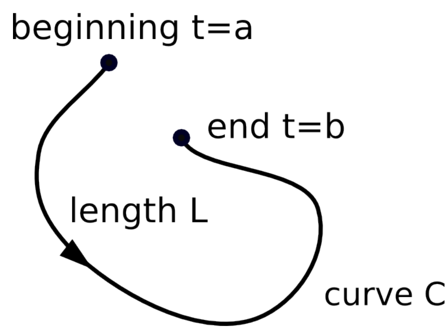



The length of the curve is obtained by integrating the length element $ds$ from $a$ to $b$.

$$L = \int_a^b ds.$$

Analogously, the length element of a space curve $x = x(t), y = y(t), z = z(t)$ is the magnitude of the vector $d\vec{r} = \langle dx, dy, dz \rangle$ and so

$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$




Note that $d\vec{r} = \vec{r}\,'dt$ so that

$$ds = |d\vec{r}| = |\vec{r}\,'|dt =$$
$$\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

Thus the length of a space curve on the interval $a \le t \le b$ can be found as

$$L = \int_a^b ds = \int_a^b |\vec{r}\,'(t)|dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt.$$

Let us compare again the two and three dimensional formulas.

| | |
|---|---|
| The **length** of the curve $\vec{r} = \langle x(t), y(t) \rangle$ for $a \le t \le b$ is $L = \int_a^b |\vec{r}\,'(t)|dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt$ | The **length** of the curve $\vec{r} = \langle x(t), y(t), z(t) \rangle$ for $a \le t \le b$ is $L = \int_a^b |\vec{r}\,'(t)|dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt$ |

Note that the quotient $\frac{ds}{dt}$ is equal to $|\vec{r'}|$ which is the speed of the movement.

$$\frac{ds}{dt} = |\vec{r'}| = \text{ speed}$$

Because of this the integral above $L = \int_a^b ds$ computes the **distance traveled** during the time interval $[a, b]$.

**Practice Problems.**

1. Describe the following curves. For those without parametric representation, find equations of parametric equations.

   (a) The curve given by $x = 1 + t$, $y = 2 - 2t$, $z = 1 + 2t$.

   (b) The line segment from $(1, 2, -4)$ to $(3, 0, 1)$.

   (c) The curve given by $x = \cos t$, $y = \sin t$, $z = 2$.

   (d) The curve given by $x = \cos t$, $y = \sin t$, $z = t$.

   (e) The curve in the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $y + z = 2$.

   (f) The triangle in the boundary of the part of the plane $3x + 2y + z = 6$ in the first octant.

   (g) The boundary of the part of the paraboloid $z = 4 - x^2 - y^2$ in the first octant.

2. For the following curves, find an equation of the tangent line at the point where $t = 0$. Find the normalization of a direction vector at $t = 0$.

   (a) The curve given by $x = \cos t$, $y = \sin t$, $z = t$.

   (b) The curve in the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $y + z = 2$. Use the parametrization of this curve from problem 1 (e).

3. For the following curves, find the length for $0 \leq t \leq \frac{\pi}{2}$. Use the calculator to evaluate the integral in part (b).

   (a) The curve given by $x = \cos t$, $y = \sin t$, $z = t$.

   (b) The curve in the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $y + z = 2$. Use the parametrization of this curve from problem 1 (e).

4. Find the length of the curve (given below) from $(4, 0, 1)$ to $(0, 0, -1)$.

$$x = 2 + 2\cos t, \quad y = 2\sin t, \quad z = \cos t - \sin^2 t$$

5. Consider the curve $C$ which is the intersection of the surfaces

$$x^2 + y^2 = 9 \qquad \text{and} \qquad z = 1 - y^2.$$

   (a) Find the parametric equations that represent the curve $C$.

   (b) Find the equation of the tangent line to the curve $C$ at point $(0, 3, -8)$.

   (c) Find the length of the curve from $(3, 0, 1)$ to $(0, 3, -8)$. You can use the calculator to evaluate the integral that you are going to get.

6. Consider the curve $C$ which is the intersection of the surfaces

$$y^2 + z^2 = 16 \qquad \text{and} \qquad x = 8 - y^2 - z.$$

   (a) Find the parametric equations that represent the curve $C$.

   (b) Find the equation of the tangent line to the curve $C$ at point $(-8, -4, 0)$.

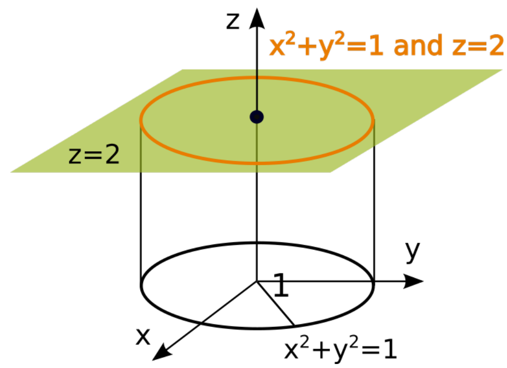
(c) Find the length of the curve from $(4,0,4)$ to $(-8,-4,0)$. Use the calculator to evaluate the integral that you are going to get.

7. Find the length of the boundary of the part of the paraboloid $z = 4 - x^2 - y^2$ in the first octant.

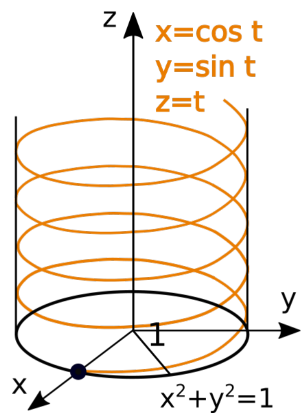
**Solutions.**

1. (a) This curve is a line passing the point $(1,2,1)$ in the direction of $\langle 1,-2,2\rangle$.

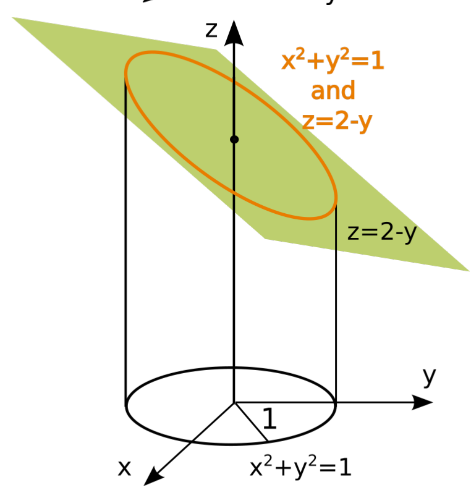
(b) Any vector colinear with $\langle 3,0,1\rangle - \langle 1,2,-4\rangle = \langle 2,-2,5\rangle$ can be used as a direction vector of the line passing two points. You can also use any of $(1,2,-4)$ and $(3, 0, 1)$ for a point on the line. For example, using $(1,2,-4)$ we obtain parametric equations $x = 1+2t, y = 2-2tz = -4+5t$. Since we are looking only at the segment between the two points and $(1,2,-4)$ corresponds to $t = 0$ and $(3, 0, 1)$ to $t = 1$, the line segment has parametrization $x = 1+2t, y = 2-2tz = -4+5t$ with $0 \le t \le 1$.




(c) The $xy$-equations $x = \cos t$, $y = \sin t$, represent a circle of radius 1 in $xy$-plane. Thus, the curve is on the cylinder based at this circle. The $z$-equation $z = 2$ represents the horizontal plane passing 2 on the $z$-axis. So, this curve is the intersection of the cylinder with the horizontal plane: it is a circle of radius 1 centered on the $z$-axis in the horizontal plane passing $z = 2$.




(d) The $xy$-equations $x = \cos t$, $y = \sin t$, represent a circle of radius 1 in $xy$-plane. Thus, the curve is on the cylinder based at this circle. The equation $z = t$ has an effect that $z$-values increase as $t$-values increase. Thus, this curve is a helix spiraling up the cylinder as the $t$-values increase. The Matlab notes cover graphing this (and other) curves in Matlab.




(e) The curve is the intersection of a cylinder with an inclined plane. So, the curve is an ellipse. The $xy$-equations represent a cylinder based at the circle of radius 1 in the $xy$-plane. Thus, $x$ and $y$ can be parametrized as $x = \cos t$, $y = \sin t$. To get the $z$ equation, solve the plane equation $y + z = 2$ for $z$, get $z = 2 - y$ and substitute that $y = \sin t$. Thus $z = 2 - \sin t$. This gives us parametric equations of the ellipse $x = \cos t$, $y = \sin t$, $z = 2 - \sin t$.

(f) The three line segments forming the triangle are the intersections of the plane $3x+2y+z=6$ with the three coordinate planes. The intersection in the $xy$-plane $z=0$ can be obtained by plugging $z=0$ in $3x+2y+z=6$ and using $x$, for example, as a parameter. Thus, we have part of the line $3x+2y=6 \Rightarrow y=3-\frac{3}{2}x$ between its two intercepts (2,0) and (0,3) and so $0 \le x \le 2$

$$x=x, y=3-\frac{3}{2}x, z=0 \text{ or } x=t, y=3-\frac{3}{2}t, z=0 \text{ with } 0 \le t \le 2.$$

Alternatively, the parametric equations of this line can be obtained as equations of a line passing $x$ and $y$ intercepts of the plane $3x+2y+z=6$, (2, 0, 0) and (0,3, 0). Using (-2,3,0) as direction vector and (0,3,0) as a point on the line, we obtain the equations $x=-2t, y=3+3t, z=0$ with $-1 \le t \le 0$.

Similarly, you can find equations of the remaining two sides of the triangle. The intersection of $xz$-plane $y=0$ can be obtained by plugging $y=0$ in $3x+2y+z=6$ and using $x$, for example, as a parameter. Thus, we have $3x+z=6 \Rightarrow z=6-3x$ and so $x=x, y=0, z=6-3x$ or $x=t, y=0, z=6-3t$ with $0 \le t \le 2$.

The intersection of $yz$-plane $x=0$ can be obtained by plugging $x=0$ in $3x+2y+z=6$ and using $y$, for example, as a parameter. Thus, we have $2y+z=6 \Rightarrow z=6-2y$ and so $x=0, y=y, z=6-2y$ or $x=0, y=t, z=6-2t$ with $0 \le t \le 3$.

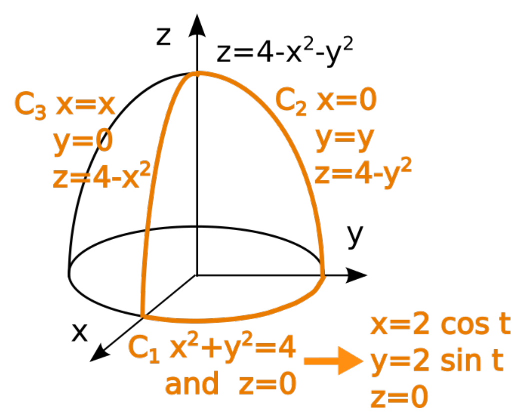
(g) The boundary of the part of the paraboloid $z=4-x^2-y^2$ in the first octant consists of three curves, each of which will have a different set of parametric equations. The parametrizations can be obtained by considering intersections with three coordinate planes $xy(z=0), yz(x=0)$, and $xz(y=0)$.

The intersection in the $xy$-plane $z=0$ is the circle $0=4-x^2-y^2 \Rightarrow x^2+y^2=4$ which has parametric equations $x=2\cos t, y=2\sin t$. Since we are considering just the part in the first quadrant, $0 \le t \le \frac{\pi}{2}$. Thus, this curve has parametric equations

$$x=2\cos t, y=2\sin t, z=0 \text{ with } 0 \le t \le \frac{\pi}{2}.$$

The intersection in the $yz$-plane $x=0$ is the parabola $z=4-y^2$ with $0 \le y \le 2$. Using $y$ as a parameter produces parametric equations $x=0, y=y, z=4-y^2$ or $x=0, y=t, z=4-t^2$ with $0 \le t \le 2$.

The intersection in the $xz$-plane $y=0$ is the parabola $z=4-x^2$ with $0 \le x \le 2$. Using $x$ as a parameter produces parametric equations $x=x, y=0, z=4-x^2$ or $x=t, y=0, z=4-t^2$ with $0 \le t \le 2$.




2. (a) To find a point on the tangent, plug $t=0$ in the parametric equations of the curve. Get $x=1, y=0, z=0$. To get a direction vector, plug $t=0$ in the derivative $x'=-\sin t, y'=\cos t$, $z'=1$. Get $\langle 0,1,1\rangle$. So, the equation of the tangent is $x=1+0t$, $y=0+1t$, $z=0+1t \Rightarrow x=1$, $y=t$, $z=t$.

The direction vector $\langle 0,1,1\rangle$ has length $\sqrt{2}$ so its normalization is $\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$.

(b) Use the parametric equations from problem 1 (d) $x = \cos t$, $y = \sin t$, $z = 2 - \sin t$. To find a point on the tangent, plug $t = 0$ in the parametric equations. Get $x = 1, y = 0, z = 2$. To get a direction vector, plug $t = 0$ in the derivative $x' = -\sin t$, $y' = \cos t$, $z' = -\cos t$. Get $\langle 0, 1, -1\rangle$. So, the equation of the tangent is $x = 1 + 0t$, $y = 0 + 1t$, $z = 2 - 1t \Rightarrow x = 1$, $y = t$, $z = 2 - t$.

The direction vector $\langle 0,1,-1\rangle$ has length $\sqrt{2}$ so its normalization is $\langle 0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\rangle$.

3. (a) $L = \int_0^{\pi/2} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt = \int_0^{\pi/2} \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2} dt =$ $\int_0^{\pi/2} \sqrt{1+1} dt = \sqrt{2} \int_0^{\pi/2} dt = \frac{\sqrt{2}\pi}{2}$.

   (b) $L = \int_0^{\pi/2} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt = \int_0^{\pi/2} \sqrt{(-\sin t)^2 + (\cos t)^2 + (-\cos t)^2} dt$. Using the calculator, $L \approx 1.91$.

4. Find the $t$-values corresponding to $(4, 0, 1)$ to $(0, 0, -1)$ for the $t$-bounds. When $(x, y, z) = (4, 0, 1)$, $x = 2 + 2\cos t = 4 \Rightarrow \cos t = 1 \Rightarrow t = 0$ and $y = 2\sin t = 0 \Rightarrow t = 0$ and $\pi$. The value $t = 0$ agrees with the $x$-equation. Check that it agrees with the $z$-equation as well: $1 = z = \cos(0) - \sin^2(0) = 1 - 0 = 1$. So, 0 is the lower bound.

   When $(x, y, z) = (0, 0, -1)$, $x = 2+2\cos t = 0 \Rightarrow \cos t = -1 \Rightarrow t = \pi$ and $y = 2\sin t = 0 \Rightarrow t = 0$ and $\pi$. The value $t = \pi$ agrees with the $x$-equation. Check that it agrees with the $z$-equation as well: $-1 = z = \cos(\pi) - \sin^2(\pi) = -1 - 0 = -1$. So, $\pi$ is the upper bound.

   Since $x' = -2\sin t$, $y' = 2\cos t$, and $z' = -\sin t - 2\sin t\cos t$ the length is

$$L = \int_0^{\pi} \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (-\sin t - 2\sin t\cos t)^2} dt = \int_0^{\pi} \sqrt{4 + \sin^2 t(1 + 2\cos t)^2} dt.$$

   Use the calculator to evaluate this integral and obtain $L \approx 6.98$.

5. (a) You can parametrize the cylinder $x^2 + y^2 = 9$ by $x = 3\cos t$ and $y = 3\sin t$. From the equation $z = 1 - y^2$, you obtain that $z = 1 - (3\sin t)^2 = 1 - 9\sin^2 t$.

   (b) To find a direction vector, we need to plug $t$-value that corresponds to the point $(0, 3, -8)$ into the derivative $x' = -3\sin t$, $y' = 3\cos t$, $z' = -18\sin t\cos t$. To find this $t$-value, set $x = 0, y = 3$ and $z = -8$ and make sure that you find the $t$-value that satisfies all three equations. From the first equation $x = 3\cos t = 0 \Rightarrow t = \pm\frac{\pi}{2}$. From the second $y = 3\sin t = 3 \Rightarrow t = \frac{\pi}{2}$. The value $t = \frac{\pi}{2}$ satisfies the third equation $z = 1 - 9\sin^2\frac{\pi}{2} = 1 - 9 = -8$. Thus, $t = \frac{\pi}{2}$.

   Plugging this value in the derivatives produces the direction vector $\langle -3, 0, 0\rangle$. So, the tangent line is $x = 0 - 3t$, $y = 3 + 0t$, $z = -8 + 0t \Rightarrow x = -3t$, $y = 3$, $z = -8$.

   (c) From part (b), we have that $t = \frac{\pi}{2}$ corresponds to the point $(0, 3, -8)$. Thus, $\frac{\pi}{2}$ is the upper bound. To find the lower bound, determine the $t$-value that corresponds to $(3, 0, 1)$. Set $x = 3, y = 0$ and $z = 1$ and make sure that you find the $t$-value that satisfies all three equations. From the first equation $x = 3\cos t = 3 \Rightarrow \cos t = 1 \Rightarrow t = 0$. From the second $y = 3\sin t = 0 \Rightarrow \sin t = 0 \Rightarrow t = 0$. The value $t = 0$ satisfies the third equation $z = 1 - 9\sin^2 0 = 1 - 0 = 1$. So, the bounds of integration are 0 to $\frac{\pi}{2}$.

   The length is $L = \int_0^{\pi/2} \sqrt{(-3\sin t)^2 + (3\cos t)^2 + (-18\sin t\cos t)^2} dt = 10.48$.

6. (a) You can parametrize the cylinder $y^2 + z^2 = 16$ by $y = 4\cos t$ and $z = 4\sin t$. From the equation $x = 8 - y^2 - z$, you obtain that $x = 8 - (4\cos t)^2 - 4\sin t = 8 - 16\cos^2 t - 4\sin t$.

   (b) To find a direction vector, we need to plug $t$-value that corresponds to the point $(-8, -4, 0)$ into the derivative $x' = 32\cos t \sin t - 4\cos t$, $y' = -4\sin t$, $z' = 4\cos t$. To find this $t$-value, set $x = -8, y = -4$ and $z = 0$ and make sure that you find the $t$-value that satisfies all three equations. From the second equation $y = 4\cos t = -4 \Rightarrow \cos t = -1 \Rightarrow t = \pi$. From the third $z = 4\sin t = 0 \Rightarrow \sin t = 0 \Rightarrow t = 0$ and $t = \pi$. The value $t = \pi$ agrees with the $t$-value we obtained using the $y$-equation. Plugging this value in the $x$-equation gives you $x = 8 - 16\cos^2 \pi - 4\sin \pi = 8 - 16 = -8$ which agrees with the $x$-coordinate of $(-8, -4, 0)$. Thus, $t = \pi$.

   Plugging this value in the derivatives produces the direction vector $\langle 4, 0, -4 \rangle$. So, the tangent line is $x = -8 + 4t$, $y = -4 + 0t$, $z = 0 - 4t \Rightarrow x = -8 + 4t$, $y = -4$, $z = -4t$.

   (c) From part (b), we have that $t = \pi$ corresponds to the point $(-8, -4, 0)$. Thus, $\pi$ is the upper bound. To find the lower bound, determine the $t$-value that corresponds to $(4, 0, 4)$. Set $x = 4, y = 0$ and $z = 4$ and make sure that you find the $t$-value that satisfies all three equations. From the second equation $y = 4\cos t = 0 \Rightarrow \cos t = 0 \Rightarrow t = \pm\frac{\pi}{2}$. From the third $z = 4\sin t = 4 \Rightarrow \sin t = 1 \Rightarrow t = \frac{\pi}{2}$. So the value $-\frac{\pi}{2}$ obtained from the $y$-equation can be discarded and we obtain that $t = \frac{\pi}{2}$. Plugging this value in the $x$-equation gives you $x = 8 - 16\cos^2 \frac{\pi}{2} - 4\sin \frac{\pi}{2} = 8 - 4 = 4$ which agrees with the $x$-coordinate of $(4, 0, 4)$. Thus, the lower bound is $t = \frac{\pi}{2}$.

   The length is $L = \int_{\pi/2}^{\pi} \sqrt{(32\cos t \sin t - 4\cos t)^2 + (-4\sin t)^2 + (4\cos t)^2 dt} = 14.515$.

7. Recall that we found parametric equations of the three curves in the intersection to be

   $x = 2\cos t, y = 2\sin t, z = 0$ with $0 \le t \le \frac{\pi}{2}$,
   $x = 0, y = t, z = 4 - t^2$ with $0 \le t \le 2$, and
   $x = t, y = 0, z = 4 - t^2$ with $0 \le t \le 2$.

   The three derivative vectors and length elements are

   $x' = -2\sin t, y' = 2\cos t, z' = 0 \Rightarrow$ $ds = \sqrt{4\sin^2 t + 4\cos^2 t dt} = \sqrt{4}dt = 2dt$
   $x' = 0, y' = 1, z' = -2t \Rightarrow$ $ds = \sqrt{1 + 4t^2}dt$, and
   $x' = 1, y' = 0, z' = -2t \Rightarrow$ $ds = \sqrt{1 + 4t^2}dt$.

   The total length can be calculated as a sum of the three integrals below. Using the calculator for the second two produces

$$\int_0^{\pi/2} 2dt + \int_0^2 \sqrt{1 + 4t^2}dt + \int_0^2 \sqrt{1 + 4t^2}dt = \pi + 4.65 + 4.65 \approx 12.44.$$