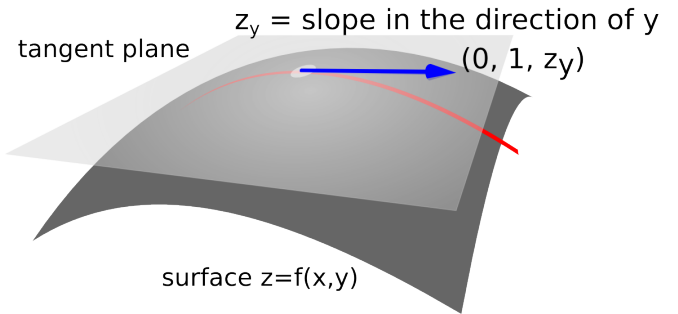
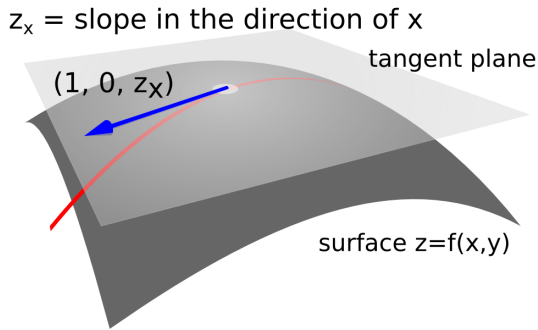


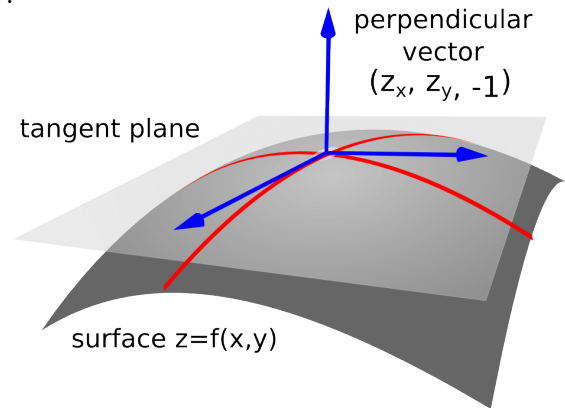
## Tangent Plane. Linear Approximation. The Gradient

**The tangent plane.** Let  $z = f(x, y)$  be a function of two variables with continuous partial derivatives. Recall that the vectors  $\langle 1, 0, z_x \rangle$  and  $\langle 0, 1, z_y \rangle$  are vectors in the tangent plane at any point on the surface.



Thus, the cross product of the vectors  $\langle 1, 0, z_x \rangle$  and  $\langle 0, 1, z_y \rangle$  is perpendicular to the tangent plane. Compute the cross product to be  $\langle -z_x, -z_y, 1 \rangle$ .

So, vector  $\langle -z_x, -z_y, 1 \rangle$ , its opposite  $\langle z_x, z_y, -1 \rangle$  or any of their multiples can be used as vectors for the tangent plane. In particular, an equation of **the tangent plane** of  $z = f(x, y)$  at the point  $(x_0, y_0, z_0)$  can be obtained using  $(x_0, y_0, z_0)$  as point and  $\langle z_x(x_0, y_0), z_y(x_0, y_0), -1 \rangle$  as vector in the plane equation. This produces the equation



$$z_x(x_0, y_0)(x - x_0) + z_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

**The linear approximation.** Solving above equation for  $z$  produces the **linear approximation** of  $z = f(x, y)$  with the tangent plane to point  $(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$ .

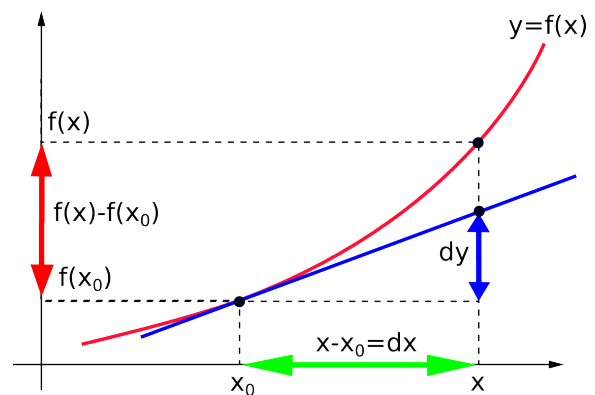
$$z = f(x, y) \approx z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Compare this formula with the linear approximation formula from Calculus 1: if  $y = f(x)$ , the linear approximation of  $y$  with the tangent line to point  $(x_0, f(x_0))$  is given by:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

Also recall that you can think of this formula as follows.

$f(x)$	$\approx$	$f(x_0)$	+	$f'(x_0)$	(	$x - x_0$ )
future value		present value		change rate		time elapsed



In case when  $f$  is a function of two variables, the linear approximation formula can be interpreted as follows.

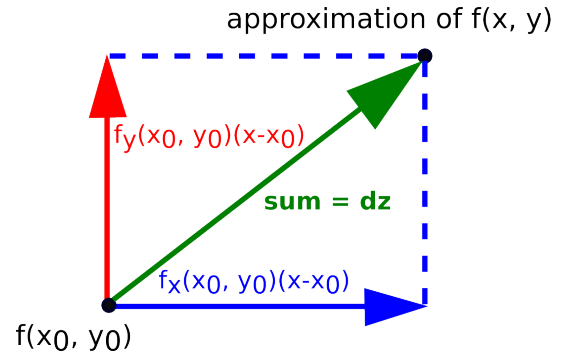
$f(x, y)$	$\approx$	$f(x_0, y_0)$	$+$	$f_x(x_0, y_0)$	$(x - x_0)$	$+$	$f_y(x_0, y_0)$	$(y - y_0)$
future value		present value		rate of change with respect to $x$	increment of $x$		rate of change with respect to $y$	increment of $y$

**The Differential.** The linear approximation formula can be also considered in the form

$$f(x, y) - f(x_0, y_0) \approx f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The expression on the right measures the change in height between the surface and its tangent plane when  $(x_0, y_0)$  changes to  $(x, y)$ . This quantity is called the **differential  $dz$** . Thus,

$$dz = z_x dx + z_y dy$$



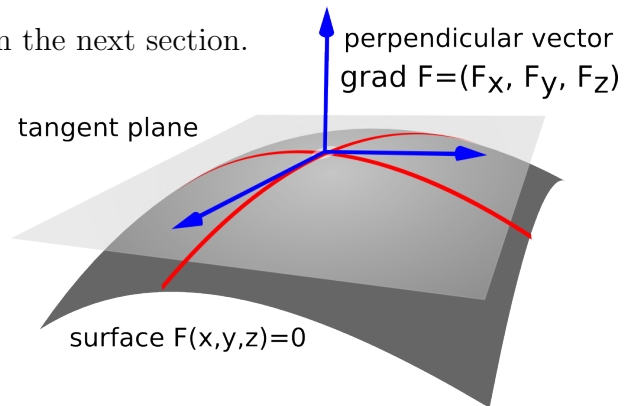
**Implicit functions.** In many situations, a surface can be given by an equation which cannot be solved for  $z$ . Spheres and cylinders are examples of this situation. In such cases, a surface is given by an **implicit function**  $F(x, y, z) = 0$  and the derivatives  $z_x$  and  $z_y$  cannot be found by direct differentiation but are given by the formulas

$$z_x = -\frac{F_x}{F_z} \quad \text{and} \quad z_y = -\frac{F_y}{F_z}.$$

The validity of these formulas will be demonstrated in the next section.

The vector  $\langle z_x, z_y, -1 \rangle$  used as a vector perpendicular to the tangent plane in this case becomes  $\langle -\frac{F_x}{F_z}, -\frac{F_y}{F_z}, -1 \rangle$ . Scaling this vector by  $-F_z$  (multiplying each coordinate by  $-F_z$ ) produces the vector

$$\langle F_x, F_y, F_z \rangle$$



which is still perpendicular to the tangent plane.

Thus, the equation of the **tangent plane** of surface  $F(x, y, z) = 0$  at  $(x_0, y_0, z_0)$  is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

The vector  $\langle F_x, F_y, F_z \rangle$  is called **the gradient** of function  $F$ . It has its two dimensional version as well.

<p>Let <math>f(x, y)</math> be a function of two variables. The <b>gradient</b> of <math>f</math> is the vector <math>\nabla f = \langle f_x, f_y \rangle</math> sometimes also denoted by <math>\text{grad}f</math>.</p> <p>The <b>gradient operator</b> is defined as</p> $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle$ <p>Thus, <math>\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle f_x, f_y \rangle</math></p>	<p>Let <math>F(x, y, z)</math> be a function of three variables. The <b>gradient</b> of <math>F</math> is the vector <math>\nabla F = \langle F_x, F_y, F_z \rangle</math> sometimes also denoted by <math>\text{grad}F</math>.</p> <p>The <b>gradient operator</b> is defined as</p> $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ <p>Thus, <math>\nabla F = \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle = \langle F_x, F_y, F_z \rangle</math></p>
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Using the gradient, a vector equation of the tangent plane of an implicit function  $F$  can be written as  $\nabla F(\vec{r}_0) \cdot (\vec{r} - \vec{r}_0) = 0$ .

### Practice Problems.

- Find an equation of the tangent plane to a given surface at the specified point.
  - $z = y^2 - x^2$ ,  $(-4, 5, 9)$
  - $z = e^x \ln y$ ,  $(3, 1, 0)$
- If  $f(2, 3) = 5$ ,  $f_x(2, 3) = 4$  and  $f_y(2, 3) = 3$ , approximate  $f(2.02, 3.1)$ .
  - If  $f(1, 2) = 3$ ,  $f_x(1, 2) = 1$  and  $f_y(1, 2) = -2$ , approximate  $f(.9, 1.99)$ .
- Find the linear approximation of the given function at the specified point.
  - $z = \sqrt{20 - x^2 - 7y^2}$  at  $(2, 1)$ . Using the linear approximation, approximate the value at  $(1.95, 1.08)$ .
  - $z = \ln(x - 3y)$  at  $(7, 2)$ . Using the linear approximation, approximate value at  $(6.9, 2.06)$ .
- The number  $N$  of bacteria in a culture depends on temperature  $T$  and pressure  $P$  and it changes at the rates of 3 bacteria per kPa and 5 bacteria per Kelvin. If there is 300 bacteria initially when  $T = 305$  K and  $P = 102$  kPa, estimate the number of bacteria when  $T = 309$  K and  $P = 100$  kPa.
- The number of flowers  $N$  in a closed environment depends on the amount of sunlight  $S$  that the flowers receive and the temperature  $T$  of the environment. Assume that the number of flowers changes at the rates  $N_S = 2$  and  $N_T = 4$ . If there are 100 flowers when  $S = 50$  and  $T = 70$ , estimate the number of flowers when  $S = 52$  and  $T = 73$ .
- Find the derivatives  $z_x$  and  $z_y$  of the following surfaces at the indicated points.
  - $x^2 + z^2 = 25$ ,  $(-4, 5, 3)$
  - $y^2 \sin z = xz^2 + 3ze^y$ ,  $(-3, 0, 1)$
- Find an equation of the tangent plane to a given surface at the specified point.
  - $x^2 + 2y^2 + 3z^2 = 21$ ,  $(4, -1, 1)$
  - $xe^{yz} = z$ ,  $(5, 0, 5)$
  - $xy^2 + yz^2 + zx^2 = 3$ ; at  $(1, 1, 1)$ .
  - $x - yz = \cos(x + y + z)$ ; at  $(0, 1, -1)$ .

8. Find the gradient vector field of  $f$  for (a)  $f(x, y) = \ln(x + 2y)$  (b)  $f(x, y, z) = x^2 + y^2 + z^2$ .

**Solutions.**

- (a)  $z_x = -2x$ ,  $z_y = 2y$ . When  $x = -4$  and  $y = 5$ ,  $z_x = 8$  and  $z_y = 10$ . Thus  $\langle z_x, z_y, -1 \rangle = \langle 8, 10, -1 \rangle$ . With point  $(-4, 5, 9)$ , and vector  $\langle 8, 10, -1 \rangle$  obtain the plane  $8(x + 4) + 10(y - 5) - 1(z - 9) = 0 \Rightarrow 8x + 10y - z = 9$ .

(b)  $z_x = e^x \ln y$ ,  $z_y = \frac{e^x}{y}$ . When  $x = 3$  and  $y = 1$ ,  $z_x = 0$  and  $z_y = e^3$ . Using vector  $\langle 0, e^3, -1 \rangle$  and point  $(3, 1, 0)$ , obtain the tangent plane as  $0(x - 3) + e^3(y - 1) - 1(z - 0) = 0 \Rightarrow e^3y - z = e^3$ .
- (a)  $f(2.02, 3.1) \approx f(2, 3) + f_x(2, 3)(2.02 - 2) + f_y(2, 3)(3.1 - 3) = 5 + 4(0.02) + 3(0.1) = 5 + 0.08 + 0.3 = 5.38$

(b)  $f(.9, 1.99) \approx f(1, 2) + f_x(1, 2)(0.9 - 1) + f_y(1, 2)(1.99 - 2) = 3 + 1(-0.1) - 2(-0.01) = 3 - 0.1 + 0.02 = 2.92$
- (a) Let  $z = f(x, y)$ . Find that  $f(2, 1) = \sqrt{20 - 4 - 7} = \sqrt{9} = 3$ . Then find  $z_x = \frac{1}{2}(20 - x^2 - 7y^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{20 - x^2 - 7y^2}}$ ,  $z_y = \frac{1}{2}(20 - x^2 - 7y^2)^{-1/2}(-14y) = \frac{-7y}{\sqrt{20 - x^2 - 7y^2}}$  and plug that  $x = 2, y = 1$ . Obtain that  $f_x(2, 1) = \frac{-2}{3}$  and  $f_y(2, 1) = \frac{-7}{3}$ . Thus,  $f(1.95, 1.08) \approx f(2, 1) + f_x(2, 1)(1.95 - 2) + f_y(2, 1)(1.08 - 1) = 3 - \frac{2}{3}(-0.05) - \frac{7}{3}(0.08) = 2.847$ .

(b) Let  $z = f(x, y)$ . Find that  $f(7, 2) = \ln(x - 3y) = \ln(7 - 6) = \ln 1 = 0$ . Then find  $z_x = \frac{1}{x - 3y}$ ,  $z_y = \frac{-3}{x - 3y}$  and plug that  $x = 7, y = 2$ . Obtain that  $f_x(7, 2) = \frac{1}{1} = 1$  and  $f_y(7, 2) = \frac{-3}{1} = -3$ . Thus,  $f(6.9, 2.06) \approx f(7, 2) + f_x(7, 2)(6.9 - 7) + f_y(7, 2)(2.06 - 2) = 0 + 1(-0.1) - 3(0.06) = -0.1 - 0.18 = -0.28$ .
- Let us denote the initial conditions of 305 K and 102 kPa by  $T_0$  and  $P_0$  so that  $N(305, 102) = 300$ . Let us denote the two given rates by  $N_P$  and  $N_T$  so that  $N_P = 3$ , and  $N_T = 5$ . Using the linear approximation formula,  $N(T, P) \approx N(T_0, P_0) + N_T \cdot (T - T_0) + N_P \cdot (P - P_0)$  with  $T = 309$  and  $P = 100$ , we have that  $N(309, 100) \approx N(305, 102) + 5(309 - 305) + 3(100 - 102) = 300 + 5(4) + 3(-2) = 314$  bacteria.
- Let us use the notation  $S_0$  and  $T_0$  for the initial conditions of 50 for  $S$  and 70 for  $T$ . Thus  $N(50, 70) = 100$ . Using the linear approximation formula and the fact that  $N_S = 2$  and  $N_T = 4$ , we have that  $N(S, T) \approx N(S_0, T_0) + N_S \cdot (S - S_0) + N_T \cdot (T - T_0) \Rightarrow N(52, 73) \approx N(50, 70) + 2(52 - 50) + 4(73 - 70) = 100 + 2(2) + 4(3) = 116$  flowers.
- (a) Consider  $F(x, y, z) = x^2 + z^2 - 25$ . Then  $F_x = 2x$ ,  $F_y = 0$  and  $F_z = 2z$ . At  $(-4, 5, 3)$ ,  $F_x = -8$ ,  $F_y = 0$  and  $F_z = 6$ . So,  $z_x = -\frac{F_x}{F_z} = -\frac{-8}{6} = \frac{4}{3}$  and  $z_y = -\frac{F_y}{F_z} = -\frac{0}{6} = 0$ .

(b) Consider  $F = y^2 \sin z - xz^2 - 3ze^y$ . Then  $F_x = -z^2$ ,  $F_y = 2y \sin z - 3ze^y$  and  $F_z = y^2 \cos z - 2xz - 3e^y$ . At  $(-3, 0, 1)$ ,  $F_x = -1$ ,  $F_y = -3$  and  $F_z = 6 - 3 = 3$ . So,  $z_x = -\frac{F_x}{F_z} = -\frac{-1}{3} = \frac{1}{3}$  and  $z_y = -\frac{F_y}{F_z} = -\frac{-3}{3} = 1$ .
- (a) Consider  $F = x^2 + 2y^2 + 3z^2 - 21 = 0$ . Find  $F_x = 2x$ ,  $F_y = 4y$ , and  $F_z = 6z$ . At  $(4, -1, 1)$ ,  $F_x = 8$ ,  $F_y = -4$ , and  $F_z = 6$ . Using vector  $\langle 8, -4, 6 \rangle$  and point  $(4, -1, 1)$ , obtain the equation of the plane  $8(x - 4) - 4(y + 1) + 6(z - 1) = 0 \Rightarrow 8x - 4y + 6z = 42 \Rightarrow 4x - 2y + 3z = 21$ .

(b) Consider  $F = xe^{yz} - z = 0$ . Then  $F_x = e^{yz}$ ,  $F_y = xze^{yz}$ , and  $F_z = xye^{yz} - 1$ . At  $(5, 0, 5)$ ,  $F_x = 1$ ,  $F_y = 25$ , and  $F_z = -1$ . Using vector  $\langle 1, 25, -1 \rangle$  and point  $(5, 0, 5)$ , obtain the tangent plane  $1(x - 5) + 25(y - 0) - 1(z - 5) = 0 \Rightarrow x + 25y - z = 0$ .

(c)  $F_x = y^2 + 2xz, F_y = 2xy + z^2, F_z = 2yz + x^2$ . At  $(1, 1, 1)$ ,  $F_x = 1 + 2 = 3, F_y = 2 + 1 = 3, F_z = 2 + 1 = 3$  so vector  $\langle 3, 3, 3 \rangle$  is perpendicular to the tangent plane. An equation of the tangent plane is  $x + y + z = 3$ .

(d)  $F_x = 1 + \sin(x + y + z), F_y = -z + \sin(x + y + z), F_z = -y + \sin(x + y + z)$ . At  $(0, 1, -1)$ ,  $F_x = 1 + \sin(0) = 1, F_y = 1 + \sin(0) = 1, F_z = -1 + \sin(0) = -1$  so vector  $\langle 1, 1, -1 \rangle$  is perpendicular to the tangent plane. An equation of the tangent plane is  $x + y - z = 2$ .

8. (a)  $\langle f_x, f_y \rangle = \langle \frac{1}{x+2y}, \frac{2}{x+2y} \rangle$       (b)  $\langle F_x, F_y, F_z \rangle = \langle 2x, 2y, 2z \rangle$ .