**Calculus 3**
**Lia Vas**

# Taylor Series and Polynomials

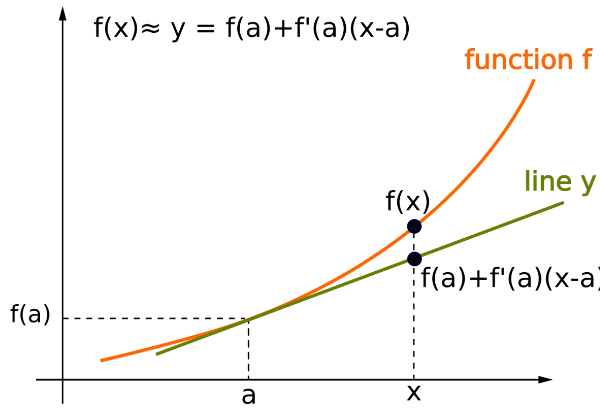
**Taylor polynomial**. Recall that the line which approximates a function $f(x)$ at a point $(a, f(a))$ has the slope $f'(a)$. By the point-slope equation, the equation of this line is

$y - f(a) = f'(a)(x-a) \Rightarrow y = f(a) + f'(a)(x-a)$.

The expression $f(a)+f'(a)(x-a)$ is the **linear approximation** of $f(x)$ at $x = a$.

$$\boxed{f(x) \approx f(a) + f'(a)(x-a)}$$

Note that the **function value and the value of the first derivative** is the same for a function and its linear approximation.




In applications, you can think of the value $f(a)$ as of the **present value**, the value $f(x)$ then represents the **future value**, $(x-a)$ the **time lapsed** and $f'(a)$ the **change rate**. Thus

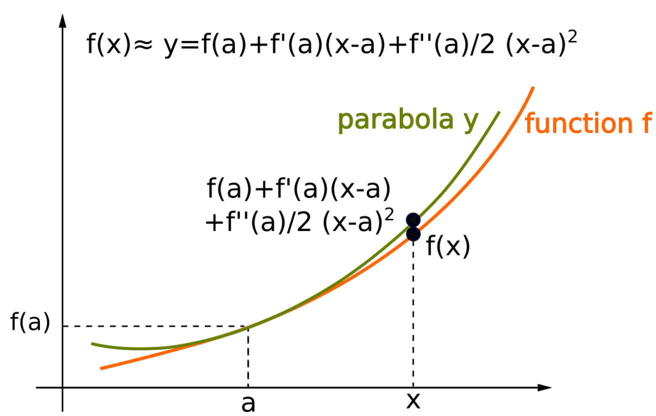
| | | | | | | |
|---|---|---|---|---|---|---|
| $f(x)$ | $\approx$ | $f(a)$ | $+$ | $f'(a)$ | | $(x-a)$ |
| future value | $\approx$ | present value | $+$ | change rate | | time elapsed |

Assume now that we want to increase the accuracy of the approximation by approximating the function with a parabola $y = a_2(x-a)^2 + a_1(x-a) + a_0$ in such a way that the **function value and the value of the first and the second derivatives** are the same for $f(x)$ and parabola $y$ when $x = a$. The condition $f(a) = y(a)$ implies that $a_0 = f(a)$. As $y' = 2a_2(x-a) + a_1$, the condition that $f'(a) = y'(a)$ implies that $f'(a) = 2a_2(a-a) + a_1 = 0 + a_1 = a_1$. So, $a_1 = f'(a)$.

As $y'' = 2a_2$. the condition $f''(a) = y''(a)$ implies that $f''(a) = 2a_2 \Rightarrow a_2 = \frac{f''(a)}{2}$.

This produces the following formula for the polynomial of the second degree which approximates the function $f(x)$ at $x = a$.

$$\boxed{f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2}$$



Similarly, if we are to approximate the formula of a polynomial of third degree $y = a_3(x-a)^3 + a_2(x-a)^2 + a_1(x-a) + a_0$ which approximates function $f(x)$ at $x = a$, we obtain that $a_0 = f(a)$, $a_1 = f'(a)$, $a_2 = \frac{f''(a)}{2}$, and, equating $f'''(a)$ with $y'''(a) \Rightarrow f'''(a) = 2 \cdot 3a_3 = 6a_3$ so that $a_3 = \frac{f'''(a)}{6}$.

Note that the values in the denominators of the expressions $a_0 = f(a) = \frac{f(a)}{1}$, $a_1 = \frac{f'(a)}{1}$, $a_2 = \frac{f''(a)}{2}$, and $a_3 = \frac{f'''(a)}{6}$, match the values of the *factoriel* function of $0, 1, 2$ and 3. Recall that the product $1 \cdot 2 \cdot \ldots n$ is written shortly as $n!$ and is called the **factoriel** of $n$. Thus $1! = 1$, $2! = 1 \cdot 2 = 2$, $3! = 1 \cdot 2 \cdot 3 = 6$, $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$, $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$ and so on. $0!$ is defined to be 1.

Continuing in this way, we obtain the formula for approximating $f(x)$ with a polynomial of $n$-th degree to be

$$f(x) \ \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

The polynomial on the right is called the **Taylor polynomial of $f(x)$ at $x = a$ of order $n$.** The Taylor polynomial centered at 0 is sometimes called **Maclaurin polynomial**.

Approximating a function using its Taylor polynomial is particularly useful when certain phenomena is modeled by a function which is either

- too complex to be manipulated or
- such that its exact formulas is not known but its value and the value of its derivatives are known at a point.

Calculators and software applications (including Matlab for example) manipulate many functions using their Taylor polynomials.

**Taylor series.** If a function $f(x)$ can be expressed as a power series $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ centered at $a$, continuing the process of equating value of the derivatives of function and the power series at $x = a$ as above, we obtain that $a_n = \frac{f^{(n)}(a)}{n!}$. Thus,

$$f(x) \ = \ \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \ (x-a)^n$$

This series is called the **Taylor series for $f(x)$ centered at $a$.** The sum of the first $n+1$ terms of the Taylor series is the Taylor polynomial of $n$-th degree at $x = a$.

**Example 1.** Find the power series expansion centered at 0 for $e^x$. Use the Taylor polynomial of degree 4 to approximate the value of $e$ with a fraction. Compare with the calculator answer.

**Solution.** If $f(x) = e^x$, then $f'(x) = f''(x) = f'''(x) = \ldots = e^x$. Evaluating the function and its derivatives at $x = 0$ produces $f(0) = f'(0) = f''(0) = f'''(0) = \ldots = e^0 = 1$. Thus,

$$e^x \ = \ \sum_{n=0}^{\infty} \frac{1}{n!} x^n \ = \ 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots.$$

The Taylor polynomial of degree 4 is $p_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$. The $x$-value 1 corresponds to the $y$-value $e$ of $e^x$. So, $e = e^1$ is can be approximated by $p_4(1) = 1 + 1 + \frac{1^2}{2} + \frac{1^3}{6} + \frac{1^4}{24} = \frac{48+12+4+1}{24} = \frac{65}{24} \approx 2.708$.

Comparing with the calculator value $e = 2.718\ldots$, you can see that just the first five terms of the polynomial produce the value of $e$ correctly up to the first two digits.

In the above example, we obtained the power series expansion for $e^x$. In the next example, we do the same for the functions $\frac{1}{1-x}$, $\sin x$, and $\cos x$. The expansions of these four functions are considered as *basic expansions* of elementary functions. This expansions can be used for other functions obtained as products, quotients, or composites of these basic four.

**Example 2.** Find the series expansion centered at 0 for the following functions.

(a) $\frac{1}{1-x}$ (b) $\sin x$ (c) $\cos x$

**Solutions.** (a) Let $f(x) = \frac{1}{1-x}$. Then $f'(x) = \frac{1}{(1-x)^2}$, $f''(x) = \frac{2}{(1-x)^3}$, $f'''(x) = \frac{2\cdot 3}{(1-x)^4}$, ... $f^{(n)}(x) = \frac{1\cdot 2\cdot 3\cdot \ldots \cdot n}{(1-x)^{n+1}} = \frac{n!}{(1-x)^{n+1}}$. Evaluating the function and their derivatives at $x = 0$, we have that $f(0) = 1 = 0!$, $f'(0) = 1 = 1!$, $f''(0) = 2 = 2!$, $f'''(0) = 2 \cdot 3 = 3!$, $\ldots$, $f^{(n)}(0) = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n = n!$. Thus,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{n!}{n!}\, x^n = \sum_{n=0}^{\infty} x^n.$$

Alternatively, note that the formula for the sum of geometric series $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ gives the same answer if you put $x$ for $r$.

(b) If $f(x) = \sin x$, then $f'(x) = \cos x \Rightarrow f''(x) = -\sin x \Rightarrow f'''(x) = -\cos x \Rightarrow f^{(4)}(x) = \sin x \Rightarrow f^{(5)}(x) = \cos x$ and the cycle continues. Evaluating the function and its derivatives at 0 produces $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1$, $f^{(4)}(0) = 0$, $f^{(5)}(0) = 1\ldots$ So, the terms with even power of $x$ have zero coefficient and the terms with odd powers of $x$ have 1 or $-1$ alternating as coefficients. Thus,

$$\sin x = 0 + 1x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \ldots = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

(c) Finding expansion for $\cos x$ is similar to finding expansion for $\sin x$. In this case, the terms with odd powers of $x$ have zero coefficients and the terms with the even powers of $x$ are alternating between 1 and $-1$. Thus,

$$\cos x = 1 + 0x - \frac{1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \frac{0}{5!}x^5 - \frac{1}{6!}x^6 + \ldots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Using the four expansions (for $e^x, \frac{1}{1-x}, \sin x, \cos x$) we can obtain the expansions for some related functions as the following example illustrates.

**Example 3.** Find the power series expansions of the functions below centered at $x = 0$.

(a) $e^{2x}$ (b) $\frac{1}{1-x^2}$ (c) $\sin 3x$

**Solutions.** (a) Use that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ to get the expansion for $e^{2x}$ : substitute $x$ by $2x$ in the expansion of $e^x$. Thus

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}.$$

(b) To get the expansion for $\frac{1}{1-x^2}$, substitute $x^2$ for $x$ in the expansion for $\frac{1}{1-x}$. So,

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n}.$$

(c) To get the expansion for $\sin 3x$, substitute $3x$ for $x$ in the expansion of $\sin x$. Hence

$$\sin 3x = \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!} \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+1}}{(2n+1)!}.$$

**Differentiation and integration of power series.** For the $x$-values in the interval of convergence of the power series $\sum_{n=1}^{\infty} a_n (x-a)^n$, you can differentiate and integrate the series by differentiating and integrating each term.

$$\frac{d}{dx}\left(\sum_{n=1}^{\infty} a_n (x-a)^n\right) = \sum_{n=1}^{\infty} a_n n (x-a)^{n-1}$$

$$\int_a^x \left(\sum_{n=1}^{\infty} a_n (x-a)^n\right) dx = \sum_{n=1}^{\infty} a_n \frac{(x-a)^{n+1}}{n+1}$$

Using this fact, the power series can also be used for evaluating integrals which cannot be expressed in terms of elementary functions (i.e. functions whose antiderivatives are not elementary functions). Examples of such integrals include $\int e^{x^2} dx$, $\int \frac{\sin x}{x} dx$, $\int \frac{e^x}{x} dx$ etc.

**Example 4.** The function $e^{-x^2}$, relevant for the Gaussian distribution in statistics, does not have an elementary antiderivative. Evaluate the integral $\int e^{-x^2} dx$ as a series.

**Solution.** Find the power series expansion of $e^{-x^2}$ using the expansion of $e^x$ by replacing $x$ by $-x^2$. Thus,

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}.$$

Then integrate the power series by integrating each term to get

$$\int e^{-x^2} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)}.$$

**Finding the sum of an convergent power series.** So far, we have been expanding given function in a power series. Next, we are reversing this process: for a given power series $\sum_{n=0}^{\infty} a_n (x-a)^n$, we are finding an elementary function $f(x)$ equal to its sum. In this case, the function $f(x)$ is said to be the **closed form** of the series. This process is relevant, among other applications, when finding the solutions of a differential equation in the form of power series.

The equality $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ is valid when the series is convergent and the domain of the function $f(x)$ is equal to the convergence interval of the series.

**Example 5.** Find the sum of the following series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n+2}$ (c) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}$

**Solutions.** Have the four "basic" expansions ($e^x$, $\frac{1}{1-x}$, $\sin x$, and $\cos x$) in front when you are finding the sum. In this problem, all three given series match the expansion for $e^x$ pretty closely. For the first series, only $(-1)^n$ term is "off" but you can group it with $x^n$ to have $(-x)^n$. Thus, the first series is $\sum_{n=0}^{\infty} \frac{1}{n!}(-x)^n$. Note that this series is exactly the series obtained by replacing $x$ by $-x$ in the formula for $e^x$. Hence,

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!}(-x)^n = e^{-x}.$$

For the second series, write $x^{n+2}$ as $x^n\ x^2$ so that $x^2$ can be factored in front of the series. The remaining part matches the previous sum. Hence,

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n+2} = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = x^2 \sum_{n=0}^{\infty} \frac{1}{n!}(-x)^n = x^2 e^{-x}.$$

Since $x^{2n} = (x^2)^n$ in the third series, this series is exactly the series obtained by replacing $x$ by $x^2$ in the expansion of $e^x$. Hence,

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^{2n} = \sum_{n=0}^{\infty} \frac{1}{n!}(x^2)^n = e^{x^2}.$$

Note that each of the three series converges for every $x$ and so their three closed forms are equal to the initial series for every value of $x$.

**Practice Problems.**

1. Find the power series expansion of the following functions centered at given point.

   (a) $x^2 - 2x + 1; \quad x = 0$ (b) $x^2 - 2x + 1; \quad x = 1$ (c) $e^x; \quad x = 1$

   (d) $xe^{2x}; \quad x = 0$ (e) $\frac{1}{1+x}; \quad x = 0$ (f) $\frac{x}{1+x^2}; \quad x = 0$

2. Find the power series expansion of the following functions centered at given point.

   (a) $\frac{1}{(1-x)^2}; \quad x = 0$ (b) $\ln(1-x); \quad x = 0$

3. Evaluate the integral $\int \frac{\sin x}{x}\, dx$ as an infinite series.

4. Find the sum of the following series.

   (a) $\sum_{n=0}^{\infty} 2^n x^n$ (b) $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ (c) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n$

5. Find the Taylor polynomial of the third degree centered at 0 of $f(x) = e^x \sin x$. Use the polynomial to approximate $e^{1/2} \sin \frac{1}{2}$ with a rational number.

6. (a) If $f(2) = 5$, $f'(2) = 3$ and $f''(2) = 1$, approximate $f(2.1)$.

   (b) If $f(2) = 5$, $f'(2) = 3$, $f''(2) = 1$, and $f'''(x) = \frac{1}{2}$ for all $x$, approximate $f(1.9)$.

(c) If $f(1) = f'(1) = -1$, $f''(1) = f'''(1) = 0$ and $f^{iv}(1) = 2$, approximate $f(1.01)$.

7. ("PChem problem") Approximate the function $e^{\frac{hv}{kT}} - 1$ by its Taylor polynomial of the second degree in terms of $v$.

8. (" Physics problem") The magnitude of the electric field $E$ of a single charge $q$ can be described by $E = \frac{kq}{r^2}$ where $r$ is the distance between the field and the charge and $k$ is a proportionality constant. If two opposite charges are at distance $d$ from each other, the formula for the electric field changes to
$$E = \frac{kq}{(r-d)^2} - \frac{kq}{(r+d)^2} = \frac{kq}{r^2(1-\frac{d}{r})^2} - \frac{kq}{r^2(1+\frac{d}{r})^2}$$
Use the Taylor polynomial of the second degree of the function $f(x) = \frac{1}{(1-x)^2}$ (see problem 2 (a)) to show that the magnitude of the electric field $E$ can be approximated as
$$E \approx \frac{4kqd}{r^3}$$
This approximation is accurate if $r$ is much larger than $d$ so that the quotient $\frac{d}{r}$ is small.

**Solutions.**

1. (a) Let $f(x) = x^2 - 2x + 1$. Then $f(0) = 1$, $f'(0) = -2$ and $f''(0) = 2$. All the other derivatives are 0. So, the Taylor series is $1 - 2x + \frac{2}{2}x^2 + 0 + 0 + \ldots = 1 - 2x + x^2$. Note that this is the same polynomial as $f(x)$.

   This answer should not be surprising. In fact, any polynomial is equal to its Taylor series expansion centered at 0.

   (b) Let $f(x) = x^2 - 2x + 1$. Then $f(1) = 0 = f'(1)$ and $f''(1) = 2$. So, the Taylor series expansion centered at 1 is $0 + 0 + \frac{2}{2}(x-1)^2 + 0 + 0 + \ldots = (x-1)^2$. Note that $(x-1)^2$ foils as $x^2 - 2x + 1$.

   (c) Let $f(x) = e^x$. Then $f^{(n)}(x) = e^x$ for any $n$ and $f^{(n)}(1) = e$. So, $e^x = \sum_{n=0}^{\infty} \frac{e}{n!}(x-1)^n = e\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$.

   (d) To get the expansion for $xe^{2x}$, multiply the expansion for $e^{2x}$ (see part (a) of Example 3) by $x$. So, $xe^{2x} = x\sum_{n=0}^{\infty} \frac{2^n x^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^{n+1}}{n!}$.

   (e) To get the expansion for $\frac{1}{1+x}$, substitute $-x$ for $x$ in the expansion for $\frac{1}{1-x}$. So, $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$.

   (f) To get the expansion for $\frac{x}{1+x^2}$, substitute $x^2$ for $x$ in the expansion for $\frac{1}{1+x}$ and multiply it by $x$ So, $\frac{x}{1+x^2} = x\sum_{n=0}^{\infty}(-1)^n(x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n+1}$.

2. (a) Let $f(x) = \frac{1}{(1-x)^2}$. Note that $f(x)$ is the derivative of function $g(x) = \frac{1}{1-x}$ (since $g'(x) = (-1)(1-x)^{-2}(-1) = \frac{1}{(1-x)^2} = f(x)$) for which we already have the power series expansion. So, instead of finding the expansion of $f$ using the formula for Taylor series, it is more efficient to differentiate the series for $g(x) = \frac{1}{1-x}$. So, $g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \Rightarrow f(x) = g'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty} nx^{n-1}$.

   (b) Let $f(x) = \ln(1-x)$. Then $f'(x) = \frac{-1}{1-x} = -\sum_{n=0}^{\infty} x^n \Rightarrow f(x) = \int_0^x \left(-\sum_{n=0}^{\infty} x^n\right) dx = -\sum_{n=0}^{\infty} \int_0^x x^n dx = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$.

3. Divide the expansion for $\sin x$ by $x$ to obtain the expansion for

$$\frac{\sin x}{x} = x^{-1} \sin x = x^{-1} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}.$$

Then integrate the function by integrating the series term by term. So, $\int \frac{\sin x}{x}\, dx =$

$$\int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int x^{2n} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!(2n+1)}.$$

4. (a) Since $\sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} (2x)^n$, you can use the formula for $\frac{1}{1-x}$ but replace $x$ by $2x$. Hence,

$$\sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} (2x)^n = \frac{1}{1-2x}.$$

Note that the interval of convergence corresponds to $|2x| < 1 \Rightarrow -1 < x < 1 \Rightarrow \frac{-1}{2} < x < \frac{1}{2}$.
(b) Since $(-1)^n x^{2n} = (-1)^n (x^2)^n = (-x^2)^n$,

$$\sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-x^2)^n = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}.$$

Note that the interval of convergence corresponds to $|-x^2| < 1 \Rightarrow |x|^2 < 1 \Rightarrow -1 < x < 1$.

(c) The given series matches the expansion for $\cos x$ up to the power of $x$ which is even for the expansion of cosine. Hence, write $x^n$ as $(\sqrt{x}^2)^n = (\sqrt{x})^{2n}$ and replace $x$ by $\sqrt{x}$ in the expansion for cosine to obtain that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\sqrt{x})^{2n} = \cos \sqrt{x}.$$

The series converges for every $x$ so the interval of convergence is $(-\infty, \infty)$.

5. Let $f(x) = e^x \sin x$. While we know the series expansion for each term $e^x$ and $\sin x$, foiling the terms can be quite challenging, so it is easier to find the three derivatives $f', f''$, and $f'''$ and use the formula for the Taylor polynomial. The first three derivatives are $f'(x) = e^x \sin x + e^x \cos x \Rightarrow f''(x) = e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x = 2e^x \cos x \Rightarrow f'''(x) = 2e^x \cos x - 2e^x \sin x$. Evaluating the functon and the derivatives at 0 produces $f(0) = 0$, $f'(0) = 1$, $f''(0) = 2$ and $f'''(0) = 2$. Thus, $e^x \sin x \approx 0 + 1x + \frac{2}{2}x^2 + \frac{2}{6}x^3 = x + x^2 + \frac{x^3}{3}$.

   To approximate $e^{1/2} \sin \frac{1}{2}$ substitute $\frac{1}{2}$ for $x$ in the Taylor polynomial. Obtain $\frac{1}{2} + \frac{1}{4} + \frac{1}{24} = \frac{19}{24} \approx .7917$. Compare with the calculator answer $e^{1/2} \sin \frac{1}{2} \approx .7904$.

6. (a) Using the given derivative values at $x = 2$, the Taylor polynomial of degree 2 is $f(x) \approx f(2) + f'(2)(x-2) + \frac{f''(2)}{2}(x-2)^2 = 5 + 3(x-2) + \frac{1}{2}(x-2)^2$. Evaluating this polynomial at $x = 2.1$ produces an approximation for $f(2.1)$.

$$f(2.1) \approx 5 + 3(2.1-2) + \frac{1}{2}(2.1-2)^2 = 5 + 3(.1) + \frac{1}{2}(.1)^2 = 5.305.$$

(b) Using the given derivative values at $x = 2$, the Taylor polynomial of degree 3 is $f(x) \approx f(2) + f'(2)(x-2) + \frac{f''(2)}{2}(x-2)^2 + \frac{f'''(2)}{6}(x-2)^3 = 5 + 3(x-2) + \frac{1}{2}(x-2)^2 + \frac{1}{12}(x-2)^3$. Evaluating this polynomial at $x = 1.9$ produces an approximation for $f(1.9)$.

$$f(1.9) \approx 5+3(1.9-2)+\frac{1}{2}(1.9-2)^2+\frac{1}{12}(1.9-2)^3 = 5+3(-.1)+\frac{1}{2}(-.1)^2+\frac{1}{12}(-.1)^3 = 4.705.$$

(c) Using the given derivative values at $x = 1$, the Taylor polynomial of degree 4 is $f(x) \approx f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{6}(x-1)^3 + \frac{f^{(4)}(1)}{24}(x-1)^4 = -1 + -1(x-1) + \frac{0}{2}(x-1)^2 + \frac{0}{6}(x-1)^3 + \frac{2}{24}(x-1)^4 = -1 + -1(x-1) + \frac{1}{12}(x-1)^4$. Evaluating this polynomial at $x = 1.01$ produces an approximation for $f(1.01)$.

$$f(1.01) \approx -1 - 1(1.01-1) + \frac{1}{12}(1.01-1)^4 = -1.00999 \approx -1.01.$$

7. The problem is asking you to find the second order Taylor polynomial centered at $v = 0$. Let $f(v) = e^{\frac{hv}{kT}} - 1$. Then $f'(v) = \frac{h}{kT}e^{\frac{hv}{kT}}$ and $f''(v) = \frac{h^2}{k^2T^2}e^{\frac{hv}{kT}}$. Thus $f(0) = 1 - 1 = 0$, $f'(0) = \frac{h}{kT}$, and $f''(0) = \frac{h^2}{k^2T^2}$. So $f(v) \approx \frac{hv}{kT} + \frac{h^2v^2}{2k^2T^2} = \frac{hv(2kT+hv)}{2k^2T^2}$.

8. Recall that in problem 2 (a) we determined that $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1}$ so the second degree approximation is $\frac{1}{(1-x)^2} \approx 1 + 2x + 3x^2$. Substituting $x$ with $-x$ we obtain that $\frac{1}{(1+x)^2} = \frac{1}{(1-(-x))^2} \approx 1 + 2(-x) + 3(-x)^2 = 1 - 2x + 3x^2$. By taking $\frac{d}{r}$ to be $x$, we obtain that $E = \frac{kq}{r^2(1-\frac{d}{r})^2} - \frac{kq}{r^2(1+\frac{d}{r})^2} \approx \frac{kq}{r^2}\left(1 + 2\frac{d}{r} + 3\frac{d^2}{r^2} - \left(1 - 2\frac{d}{r} + 3\frac{d^2}{r^2}\right)\right) = \frac{kq}{r^2}\left(1 + 2\frac{d}{r} + 3\frac{d^2}{r^2} - 1 + 2\frac{d}{r} - 3\frac{d^2}{r^2}\right) = \frac{kq}{r^2}\left(4\frac{d}{r}\right) = \frac{4kqd}{r^3}$.