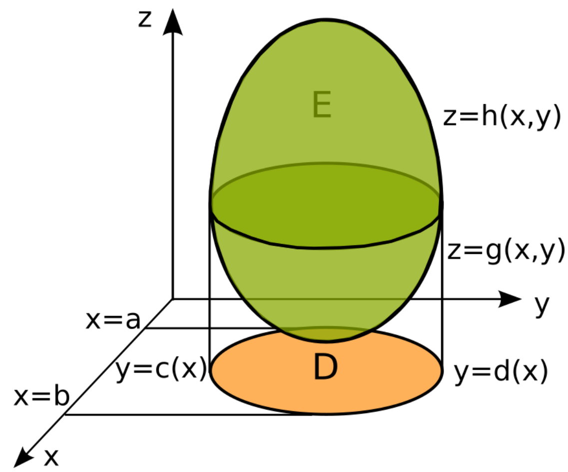
Calculus 3
Lia Vas

Triple Integrals

Let $f(x, y, z)$ be a function of three variables defined on a solid region E consisting of points (x, y, z) such that

$$\begin{aligned} a \leq & \quad x \leq b, \\ c(x) \leq & \quad y \leq d(x), \\ g(x,y) \leq & \quad z \leq h(x,y) \end{aligned}$$



The **triple integral** of f over E is

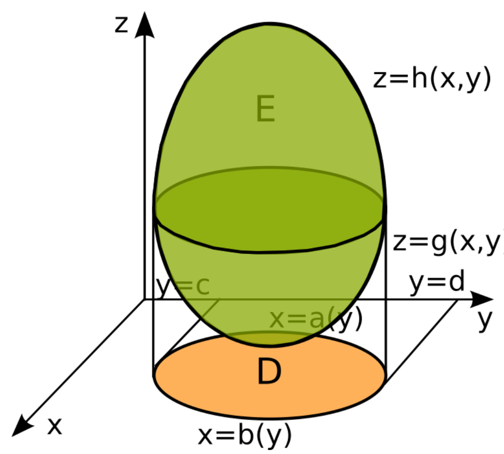
$$\int\int\int_E f(x, y, z)\, dx\, dy\, dz$$

can be computed as follows.

$$\int\int\int_E f(x, y, z)\, dx\, dy\, dz = \int_a^b \left(\int_{c(x)}^{d(x)} \left(\int_{g(x,y)}^{h(x,y)} f(x, y, z)\, dz \right) dy \right) dx$$

In case the region is given with a bit different dependence between the bounds, for example as

$$\begin{aligned} c \leq & \quad y \leq d, \\ a(y) \leq & \quad x \leq b(y), \\ g(x,y) \leq & \quad z \leq h(x,y) \end{aligned}$$



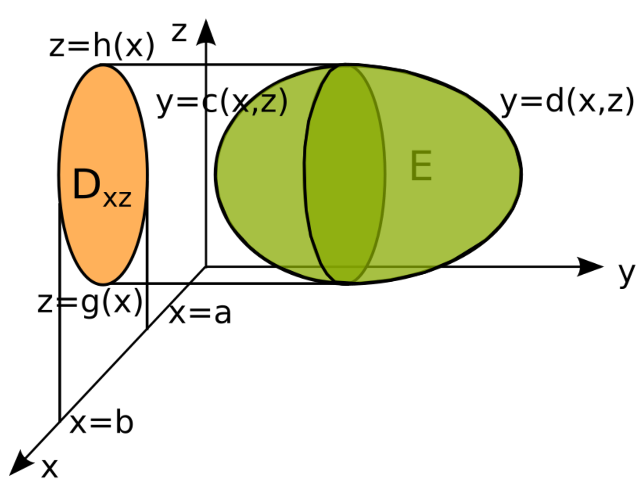
the order of integration has to be different and, in this case, as follows.

$$\int\int\int_E f(x, y, z)\, dx\, dy\, dz =$$

$$\int_c^d \left(\int_{a(y)}^{b(y)} \left(\int_{g(x,y)}^{h(x,y)} f(x, y, z)\, dz \right) dx \right) dy$$

Yet another possible scenario is given in the next figure. In this case, the region is given by the bounds as follows.

$$\begin{aligned} a \leq & \quad x \leq b, \\ g(x) \leq & \quad z \leq h(x), \\ c(x,z) \leq & \quad y \leq d(x,z) \end{aligned}$$



The order of integration in this case is as follows.

$$\int\int\int_E f(x, y, z)\, dx\, dy\, dz =$$

$$\int_a^b \left(\int_{g(x)}^{h(x)} \left(\int_{c(x,z)}^{d(x,z)} f(x, y, z)\, dy \right) dz \right) dx$$

Applications of triple integral

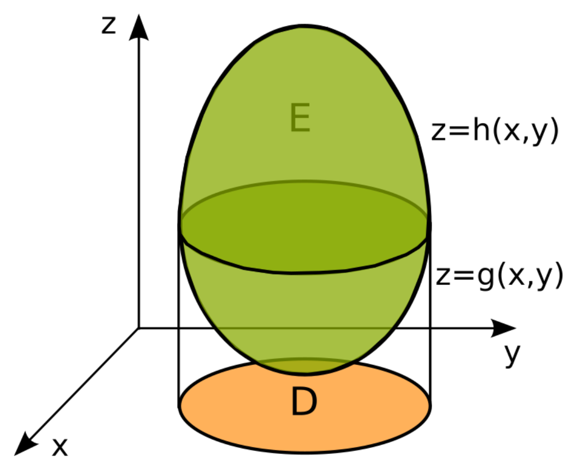
We exhibit the use of triple integrals for computing (1) volumes of solid regions in space (2) the average value of functions $f(x,y,z)$, and (3) the mass and the center of mass of a solid region with density $\rho = \rho(x,y,z)$.

Volume. Recall that the area of a region D in xy-plane can be obtained by integrating the area element $dA = dxdy$ over D,

$$A = \int\int_D dx\, dy.$$

Analogously, the volume of a solid region E in xyz-space can be obtained by integrating the volume element $dV = dxdydz$ over E

$$\boxed{V = \int\int\int dx\; dy\; dz.}$$



In particular, if E is the solid region between surfaces $z = h(x,y)$ and $z = g(x,y)$ with $h(x,y) \geq g(x,y)$ and D is its projection on xy-plane, then the volume of E can be computed as

$$V(E) = \int\int\int_E dx\; dy\; dz =$$

$$\int\int_D z\Big|_{g(x,y)}^{h(x,y)} dx\, dy = \int\int_D (h(x,y) - g(x,y))\, dx\, dy.$$

The average value of function. Recall that

- the average value of $y = f(x)$ over interval $[a,b]$ is found as $\boxed{f\text{ave} = \frac{1}{b-a}\int_a^b f(x)\; dx}$ and
- the average value of $z = f(x,y)$ over region D is found as $\boxed{f\text{ave} = \frac{1}{A(D)}\int\int_D f(x,y)\; dx\, dy.}$

In complete analogy, the average value of $u = f(x,y,z)$ over a region E in xyz-space is defined as the value f_{ave} such that the triple integral of $f(x,y,z)$ is equal to the product of the volume of f_{ave} and the volume $V(E)$ of E. Thus,

$$\boxed{f\text{ave} = \frac{1}{V(E)}\int\int\int_E f(x,y,z)\; dx\; dy\; dz}$$

The mass and the center of mass. If E is a solid in space, its mass m, volume V and density ρ are related by $\rho = \frac{m}{V}$ if the density is a constant function. If the density at a point (x,y,z) is not constant throughout the region E and it is given by a continuous function $\rho(x,y,z)$, then ρ is the quotient of differentials $\frac{dm}{dV}$. Thus, **the total mass** can be found by integration.

$$m = \int\int\int_E dm = \int\int\int_E \rho(x,y,z)\; dV \quad \Rightarrow$$

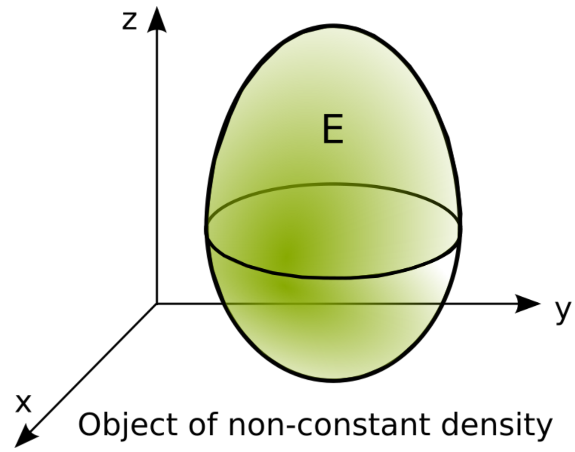
$$m = \int\int\int_E \rho(x,y,z)\; dx\, dy\, dz.$$

The formulas for the coordinates $(\overline{x}, \overline{y}, \overline{z})$ of the **center of mass** of E can be obtained by similar arguments using the formulas for the moments about the three axis and are given by

$$\overline{x} = \frac{1}{m} \int\int\int_E x\, \rho(x,y,z)\, dx\, dy\, dz$$

$$\overline{y} = \frac{1}{m} \int\int\int_E y\, \rho(x,y,z)\, dx\, dy\, dz$$

$$\overline{z} = \frac{1}{m} \int\int\int_E z\, \rho(x,y,z)\, dx\, dy\, dz$$



Practice problems.

1. Evaluate the triple integral

   a)
   $$\int\int\int_E x^3 y^2 z\, dx\, dy\, dz$$
   where $E = \{\ (x,y,z) \mid 1 \le x \le 2,\ \ 0 \le y \le x,\ \ 0 \le z \le y^2\ \}$

   b)
   $$\int\int\int_E 2x\, dx\, dy\, dz$$
   where $E = \{\ (x,y,z) \mid 0 \le y \le 2,\ \ 0 \le x \le \sqrt{4-y^2},\ \ 0 \le z \le y\ \}$

   c)
   $$\int\int\int_E 6xy\, dx\, dy\, dz$$
   where E lies under the plane $z = x + y + 1$ and above the region in the xy-plane bounded by the curves $y = \sqrt{x}$, $y = 0$ and $x = 1$.

   d)
   $$\int\int\int_E xy\, dx\, dy\, dz$$
   where E is the solid tetrahedron with vertices (0,0,0), (1, 0, 0), (0, 2, 0) and (0, 0, 3).

2. Find the volume of the tetrahedron bounded by the coordinate planes and the plane $2x + 3y + 6z = 12$.

3. Find the average value of the function $f(x,y,z) = xyz$ over the cube with side length 4 that lies in the first octant with one vertex in the origin and edges parallel to the coordinate axes.

4. Find the mass and the center of mass of the solid E given in problem 1c) and that has the density function $\rho(x,y,z) = 2$.

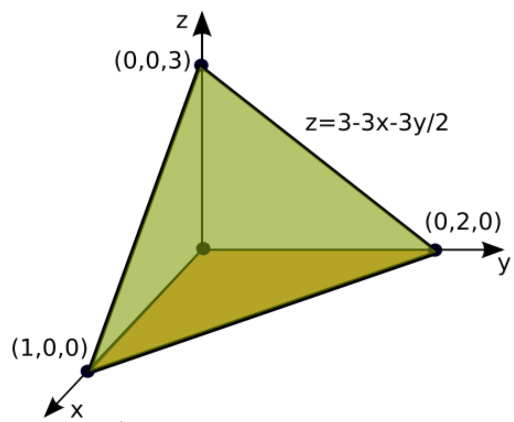
Solutions.

1. a) $\int_1^2 \int_0^x \int_0^{y^2} x^3y^2z\ dx\ dy\ dz = \int_1^2 x^3dx\ \int_0^x y^2dy\ \int_0^{y^2} z\ dz = \int_1^2 x^3dx\ \int_0^x y^2dy\ \ \frac{z^2}{2}\Big|_0^{y^2} = \int_1^2 x^3dx\ \int_0^x \frac{y^6}{2}dy = \int_1^2 x^3dx\ \ \frac{y^7}{14}\Big|_0^x = \int_1^2 \frac{x^{10}}{14}dx = \frac{x^{11}}{14(11)}\Big|_1^2 = 13.29.$

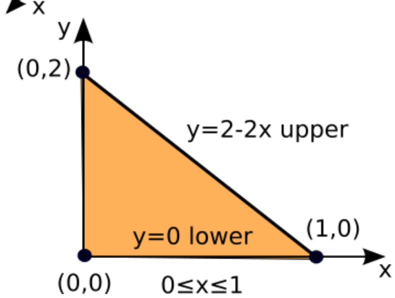
   b) You should evaluate the integral with respect to y last since all the other variables have y in the bounds. $\int_0^2 dy\ \int_0^{\sqrt{4-y^2}} 2x\ dx\ \int_0^y dz = \int_0^2 dy\ \int_0^{\sqrt{4-y^2}} 2x\ dx\ y = \int_0^2 y\,dy\ x^2\Big|_0^{\sqrt{4-y^2}} = \int_0^2 y(4-y^2)\,dy = (2y^2 - \frac{y^4}{4})\Big|_0^2 = 8-4 = 4.$

   c) Sketch the region in xy-plane first. The x-bounds are $0 \leq x \leq 1$. The y-bounds are $0 \leq y \leq \sqrt{x}$. The z-bounds are determined by the plane $z = x+y+1$ and the xy-plane and so $0 \leq z \leq x+y+1$. So $\int_0^1 \int_0^{\sqrt{x}} \int_0^{x+y+1} 6xy\ dx\ dy\ dz = \int_0^1 6xdx\ \int_0^{\sqrt{x}} y\ dy\ \int_0^{x+y+1} dz = \int_0^1 6xdx\ \int_0^{\sqrt{x}} y\ dy(x+y+1) = \int_0^1 6xdx\ \ (x\frac{y^2}{2} + \frac{y^3}{3} + \frac{y^2}{2})\Big|_0^{\sqrt{x}} = \int_0^1 6xdx\ (x\frac{x}{2} + \frac{x^{3/2}}{3} + \frac{x}{2}) = \frac{65}{28}.$

   d) First, find an equation of the plane which contains the points $P = (1,0,0), Q = (0,2,0)$ and $R = (0,0,3)$. Recall that the vectors $\overrightarrow{PQ} = (-1,2,0)$ and $\overrightarrow{PR} = (-1,0,3)$ are in the plane so the cross product $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = (6,3,2)$ is perpendicular to the plane. The equation of plane, using point P for example, is $6(x-1)+3y+2z = 0 \Rightarrow 6x+3y+2z = 6 \Rightarrow z = 3-3x-\frac{3}{2}y.$



   Thus, the upper bound for z is $3-3x-\frac{3}{2}y$. The lower bound for z is 0. In xy-plane we have a triangle with vertices (0,0), (1,0) and (0,2). So, the bounds for x are $0 \leq x \leq 1$. The lower bound for y is 0 and the upper bound is the line passing (0,2) and (1,0). The equation of this line is $y = -2x+2$. Thus,



   $\int\int\int xy\ dx\ dy\ dz = \int_0^1 xdx\ \int_0^{-2x+2} ydy\ \int_0^{3-3x-\frac{3}{2}y}\ dz = \int_0^1 xdx\ \int_0^{-2x+2} ydy\ (3-3x-\frac{3}{2}y) = \int_0^1 xdx\ (\frac{3y^2}{2} - \frac{3xy^2}{2} - \frac{y^3}{2})\Big|_0^{-2x+2} = \int_0^1 xdx(\frac{3(-2x+2)^2}{2} - \frac{3x(-2x+2)^2}{2} - \frac{(-2x+2)^3}{2}) = \text{use calculator} = \frac{1}{10}.$

2. The upper bound for z can be obtained by solving $2x+3y+6z = 12$ for z. So, $0 \leq z \leq 2-\frac{1}{3}x-\frac{1}{2}y$. The projection of the relevant region in xy-plane is a triangle determined by the coordinate axes and by the line $2x+3y+6(0) = 12 \Rightarrow y = 4-\frac{2}{3}x$ (alternatively, find the equation of the line passing (6,0) and (0,4)). The bounds for x are $0 \leq x \leq 6$. The volume is $V = \int_0^6 dx\ \int_0^{4-\frac{2}{3}x} dy\ \int_0^{2-\frac{1}{3}x-\frac{1}{2}y} dz = \int_0^6 dx\ \int_0^{4-\frac{2}{3}x} dy\ (2-\frac{1}{3}x-\frac{1}{2}y) = \int_0^6 dx\ \ (2y-\frac{1}{3}xy-\frac{y^2}{4})\Big|_0^{4-\frac{2}{3}x} = \int_0^6 dx\ (2(4-\frac{2}{3}x) - \frac{1}{3}x(4-\frac{2}{3}x) - \frac{(4-\frac{2}{3}x)^2}{4}) = \text{use calculator} = 8.$

3. When integrating over the cube, the bounds for all three variables are 0 and 4. The volume of the cube of side 4 is $4^3 = 64$ (this agrees with $V = \int_0^4 \int_0^4 \int_0^4 dxdydz = x\big|_0^4 \, y\big|_0^4 \, z\big|_0^4 = 64$).

   The average value is $f_{ave} = \frac{1}{64} \int_0^4 \int_0^4 \int_0^4 xyz \; dxdydz = \frac{1}{64} \left.\frac{x^2}{2}\right|_0^4 \left.\frac{y^2}{2}\right|_0^4 \left.\frac{z^2}{2}\right|_0^4 = \frac{1}{64} 8^3 = 8$.

4. The bounds are the same as in problem 1c. The mass is $m = \int_0^1 \int_0^{\sqrt{x}} \int_0^{x+y+1} 2 \; dx \; dy \; dz = \int_0^1 2 \; dx \; \int_0^{\sqrt{x}} (x + y + 1) dy = \int_0^1 2(x\sqrt{x} + \frac{x}{2} + \sqrt{x}) \; dx = 2.633$.

   The x-coordinate is $\overline{x} = \frac{1}{2.633} \int_0^1 \int_0^{\sqrt{x}} \int_0^{x+y+1} 2x dx \; dy \; dz = \frac{1}{2.633} \int_0^1 2x(x\sqrt{x} + \frac{x}{2} + \sqrt{x}) \; dx = \frac{1.705}{2.633} = 0.647$. Similarly you find that $\overline{y} = 0.418$, and $\overline{z} = 1.032$.