Substitution for Triple Integrals. Cylindrical and Spherical Coordinates

General substitution for triple integrals. Just as for double integrals, a region over which a triple integral is being taken may have easier representation in another coordinate system, say in \( uvw \)-space, than in \( xyz \)-space. In cases like that, one can transform the region in \( xyz \)-space to a region in \( uvw \)-space by the substitution

\[
x = x(u, v, w), \quad y = y(u, v, w), \quad \text{and} \quad z = z(u, v, w).
\]

When evaluating the integral \( \int \int \int_E f(x, y, z) \, dx \, dy \, dz \) using substitution, the volume element \( dV = dx \, dy \, dz \) becomes \( |J| \, du \, dv \, dw \) where the Jacobian determinant \( J \) is given by

\[
J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}
\]

Thus,

\[
\int \int \int_E f(x, y, z) \, dx \, dy \, dz = \int \int \int_E f(x(u, v, w), y(u, v, w), z(u, v, w)) \, |J| \, du \, dv \, dw
\]

Two main examples of such substitution are **cylindrical** and **spherical coordinates**.

Cylindrical coordinates.

Recall that the cylinder \( x^2 + y^2 = a^2 \) can be parametrized by \( x = a \cos \theta, \quad y = a \sin \theta \) and \( z = z \). Assuming now that the radius \( a \) is not constant and using the variable \( r \) to denote it just as in polar coordinates, we obtain the cylindrical coordinates

\[
x = r \cos \theta \\
y = r \sin \theta \\
z = z
\]
The Jacobian of cylindrical coordinates is

\[
J = \begin{vmatrix}
    x_r & x_{\theta} & x_z \\
    y_r & y_{\theta} & y_z \\
    z_r & z_{\theta} & z_z
\end{vmatrix} = \begin{vmatrix}
    \cos \theta & -r \sin \theta & 0 \\
    \sin \theta & r \cos \theta & 0 \\
    0 & 0 & 1
\end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r.
\]

Thus, when using cylindrical coordinates to evaluate a triple integral of a function \( f(x, y, z) \) defined over a solid region \( E \) above the surface \( z = g(x, y) \) and below the surface \( z = h(x, y) \) with the projection \( D \) in the \( xy \)-plane. If the projection \( D \) has a representation in the polar coordinates \( D = \{ (r, \theta) \mid \alpha \leq \theta \leq \beta, \ r_1(\theta) \leq r \leq r_2(\theta) \} \), then the triple integral

\[
\int \int \int_E f(x, y, z) \, dx \, dy \, dz = \int_{\alpha}^{\beta} \left( \int_{r_1(\theta)}^{r_2(\theta)} \left( \int_{g(r,\theta)}^{h(r,\theta)} f(r \cos \theta, r \sin \theta, z) \, dz \right) \, r \, dr \right) \, d\theta.
\]

It is also good to keep in mind that in cylindrical coordinates \( x, y \) and \( r \) are related in the same way as in polar coordinates by

\[ x^2 + y^2 = r^2. \]

**Spherical coordinates.**

Besides cylindrical coordinates, another frequently used coordinates for triple integrals are spherical coordinates. Spherical coordinates are mostly used for the integrals over a solid whose definition involves spheres.

If \( P = (x, y, z) \) is a point in space and \( O \) denotes the origin, let

- \( r \) denote the length of the vector \( \overrightarrow{OP} = \langle x, y, z \rangle \), i.e. the distance of the point \( P = (x, y, z) \) from the origin \( O \). Thus,

\[ x^2 + y^2 + z^2 = r^2; \]

- \( \theta \) be the angle between the projection of vector \( \overrightarrow{OP} = \langle x, y, z \rangle \) on the \( xy \)-plane and the vector \( \overrightarrow{i} \) (positive \( x \)-axis); and

- \( \phi \) be the angle between the vector \( \overrightarrow{OP} \) and the vector \( \overrightarrow{k} \) (positive \( z \)-axis).

The conversion equations are

\[ x = r \cos \theta \sin \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \phi. \]

The Jacobian determinant can be computed to be \( J = r^2 \sin \phi. \) Thus,

\[ dx \, dy \, dz = r^2 \sin \phi \, dr \, d\phi \, d\theta. \]

Note that the angle \( \theta \) is the same in cylindrical and spherical coordinates.

It is important to remember that the distance \( r \) is different in cylindrical and in spherical coordinates.

<table>
<thead>
<tr>
<th></th>
<th>Meaning of ( r )</th>
<th>Relation to ( x, y, z )</th>
</tr>
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<tbody>
<tr>
<td>Cylindrical</td>
<td>distance from ((x, y, z)) to (z)-axis</td>
<td>( x^2 + y^2 = r^2 )</td>
</tr>
<tr>
<td>Spherical</td>
<td>distance from ((x, y, z)) to the origin</td>
<td>( x^2 + y^2 + z^2 = r^2 )</td>
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Spherical coordinates parametrization of a sphere. If \( a \) is a positive constant and a point \((x, y, z)\) is on the sphere centered at the origin of radius \( a \), then the coordinates satisfy the equation
\[
x^2 + y^2 + z^2 = a^2.
\]
So, the distance from the origin \( r \) is exactly \( a \) for every such point. In other words, \( r \) is constant and equal to \( a \). Thus, the equation of the sphere in spherical coordinates become simple and short
\[
r = a.
\]

The equations
\[
x = a \cos \theta \sin \phi, \quad y = a \sin \theta \sin \phi, \quad z = a \cos \phi
\]
parametrize the sphere. When these equations are substituted in the expression \( x^2 + y^2 + z^2 \), it simplifies to \( a^2 \) (you should convince yourself of this fact).

Practice problems.

1. Evaluate the triple integral
   a) \( \int \int \int_E \sqrt{x^2 + y^2} \, dx \, dy \, dz \) where \( E \) is the region that lies inside the cylinder \( x^2 + y^2 = 16 \) and between the planes \( z = -3 \) and \( z = 4 \).
   b) \( \int \int \int_E 2 \, dx \, dy \, dz \) where \( E \) is the solid that lies between the cylinders \( x^2 + y^2 = 1 \) \( x^2 + y^2 = 4 \) and between the \( xy \)-plane and the plane \( z = x + 2 \).
   c) \( \int \int \int_E (x^2 + y^2 + z^2) \, dx \, dy \, dz \) where \( E \) is the unit ball \( x^2 + y^2 + z^2 \leq 1 \).
   d) \( \int \int \int_E z \, dx \, dy \, dz \) where \( E \) is the region between the spheres \( x^2 + y^2 + z^2 = 1 \) and \( x^2 + y^2 + z^2 = 4 \) in the first octant.

2. Find the volume of the solid enclosed by the paraboloids \( z = x^2 + y^2 \) and \( z = 36 - 3x^2 - 3y^2 \).

3. Find the volume of the solid enclosed by the paraboloids \( z = x^2 + y^2 \) and \( z = 18 - x^2 - y^2 \).

4. Find the volume of the ellipsoid \( \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1 \) by using the transformation \( x = 2u, \ y = 3v \) \( z = 5w \).

Solutions.

1. a) Use cylindrical coordinates. The interior of the circle \( x^2 + y^2 = 16 \) can be described by \( 0 \leq \theta \leq 2\pi \) and \( 0 \leq r \leq 4 \). The bounds for \( z \) are given by \( z = -5 \) and \( z = 4 \). \( \int \int \int_E \sqrt{x^2 + y^2} \, dx \, dy \, dz = \int_0^{2\pi} \int_0^4 \int_{-5}^4 \sqrt{r^2 + z^2} \, dr \, d\theta \, dz = \int_0^{2\pi} \int_0^4 r \, dr \, d\theta \, \int_{-5}^4 dz = 2\pi \frac{64}{3} (4 + 5) = 384\pi. \)

   b) The region between the circles \( x^2 + y^2 = 1 \) \( x^2 + y^2 = 4 \) has \( 0 \leq \theta \leq 2\pi \) and \( 1 \leq r \leq 2 \). The bounds for \( z \) are \( xy \)-plane \( z = 0 \) and the plane \( z = x + 2 \) which in polar coordinates has the equation \( z = r \cos \theta + 2 \). Thus, using the cylindrical coordinates, \( \int \int \int_E 2 \, dx \, dy \, dz = \int_0^{2\pi} \int_1^2 \int_0^{r \cos \theta + 2} 2r \, dr \, d\theta \, dz = \int_0^{2\pi} 2r \, d\theta \int_1^2 2r \, dr \left( r \cos \theta + 2 \right) = \int_0^{2\pi} \left( \frac{14}{3} \cos \theta + 6 \right) \, d\theta = 12\pi. \)
c) Using spherical coordinates, \( 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi, \) and \( 0 \leq r \leq 1. \) The integral is:

\[
\int \int \int_E x^2 + y^2 + z^2 \, dx \, dy \, dz = \int_0^{2\pi} \int_0^\pi \int_0^1 r^2 \sin \phi \, dr \, d\phi \, d\theta = \int_0^{2\pi} \int_0^\pi r^2 \sin \phi \, d\phi \int_0^1 r^4 \, dr = 2\pi (-\cos \phi)|_0^\pi r^3 |_0^1 = 2\pi (2)^\frac{1}{3} = \frac{4\pi}{3}.
\]

d) Use spherical coordinates. Since the region is in the first octant, \( 0 \leq \theta \leq \frac{\pi}{2} \) and \( 0 \leq \phi \leq \frac{\pi}{2}. \) The bounds for \( r \) are determined by the radii of the spheres, so \( 1 \leq r \leq 2. \) The integral is:

\[
\int \int \int_E \rho \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^\pi \int_1^2 \rho^2 \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^\pi r^2 \cos \phi \, d\phi \, d\theta = \int_0^{2\pi} r^2 \, d\theta \int_0^\pi \rho^2 \cos \phi \, d\phi = \frac{\pi}{2} \left( \frac{r^4}{4} \right) |_1^2 = \frac{15\pi}{16}.
\]

2. Use cylindrical coordinates. The paraboloids have the equations \( z = x^2 + y^2 = r^2 \) and \( z = 36 - 3x^2 - 3y^2 = 36 - 3r^2. \) The first is the lower \( z \)-bound and the second is the upper. The bounds for \( \theta \) are \( 0 \leq \theta \leq 2\pi. \) The paraboloids intersect in a circle. The projection of the circle in the \( xy \)-plane determines the \( r \)-bounds. The integral is:

\[
V = \int \int \int_E \rho^2 \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^\pi \int_0^{\sqrt{36 - 3r^2}} r \cos \phi \, r^2 \sin \phi \, d\phi \, dr \, d\theta = \int_0^{2\pi} \int_0^\pi \rho^3 \, d\rho \int_0^{\sqrt{36 - 3r^2}} r^2 \cos \phi \, d\phi = \frac{\pi}{2} \left( \frac{r^4}{4} \right) |_0^3 = \frac{27\pi}{2}.
\]

3. Very similar to the previous problem. The \( z \)-bounds are \( x^2 + y^2 = r^2 \leq z \leq 18 - x^2 - y^2 = 18 - r^2. \) The integral is:

\[
V = \int \int \int_E \rho^2 \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^\pi \int_0^{\sqrt{18 - r^2}} r \cos \phi \, r^2 \sin \phi \, d\phi \, dr \, d\theta = \int_0^{2\pi} \int_0^\pi \rho^3 \, d\rho \int_0^{\sqrt{18 - r^2}} r \cos \phi \, d\phi = \frac{\pi}{2} \left( \frac{r^4}{4} \right) |_0^3 = \frac{27\pi}{2}.
\]

4. The substitution \( x = 2u, y = 3v \) and \( z = 5w \) converts the ellipsoid into a sphere of radius \( 3. \) The integral is:

1. The Jacobian of the substitution is \( J = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{vmatrix} = 30. \) Thus, the volume is:

\[
V = \int \int \int_E x^2 + y^2 + z^2 \, dx \, dy \, dz = \int \int \int_{E'} x^2 + y^2 + z^2 \, dx \, dy \, dz = \int \int \int_{E'} (2u)^2 + (3v)^2 + (5w)^2 \, dx \, dy \, dz.
\]

Use the spherical coordinates now. The bounds are \( 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi, \) and \( 0 \leq r \leq 1 \) and the Jacobian is \( r^2 \sin \phi. \) Thus, the integral is:

\[
V = \int \int \int_{E'} r^2 \sin \phi \, dr \, d\phi \, d\theta = 30 \int_0^{2\pi} \int_0^\pi r^2 \sin \phi \, d\phi \int_0^1 r^2 \, dr = 30 \int_0^{2\pi} \pi (-\cos \phi)|_0^\pi r^3 |_0^1 = 30 \pi r^3 |_0^1 = 120\pi \frac{1}{3} = 40\pi.
\]