Calculus 3
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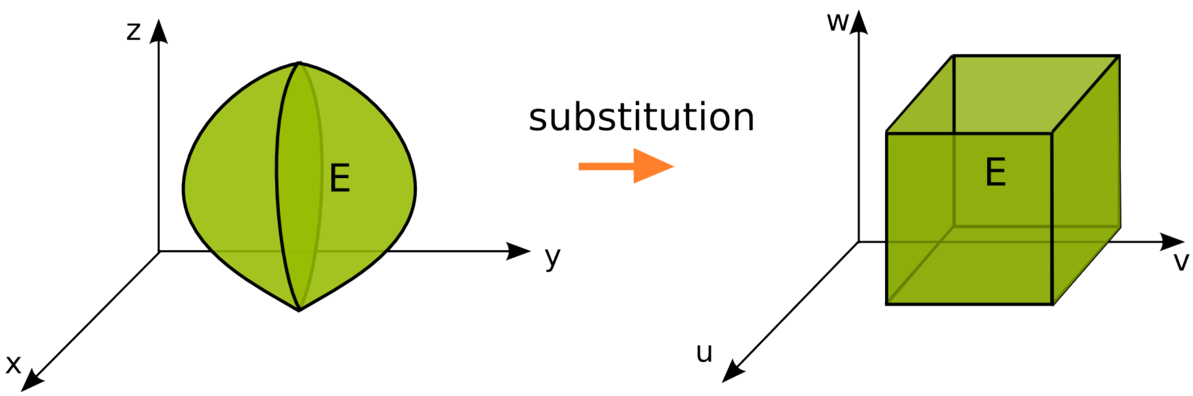
Substitution for Triple Intrgrals. Cylindrical and Spherical Coordinates

General substitution for triple integrals. Just as for double integrals, a region over which a triple integral is being taken may have easier representation in another coordinate system, say in uvw-space, than in xyz-space. In cases like that, one can transform the region in xyz-space to a region in uvw-space by the **substitution**

$$x = x(u, v, w), \quad y = y(u, v, w), \text{ and } z = z(u, v, w).$$



When evaluating the integral $\int\int\int_E f(x, y, z)dxdydz$ using substitution, the volume element $dV = dxdydz$ becomes $|J|dudvdw$ where **the Jacobian determinant** J is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

Thus, $$\int\int\int_E f(x, y, z)\, dx\, dy\, dz = \int\int\int_E f(x(u, v, w), y(u, v, w), z(u, v, w))\, |J| du\, dv\, dw$$

Two main examples of such substitution are **cylindrical and spherical coordinates**.

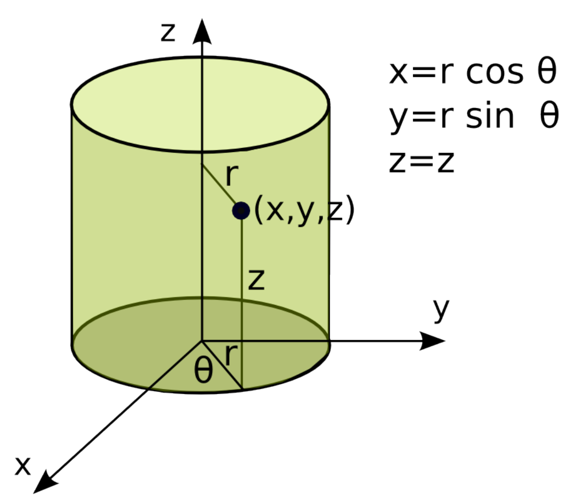
Cylindrical coordinates.

Recall that the cylinder $x^2 + y^2 = a^2$ can be parametrized by $x = a\cos\theta$, $y = a\sin\theta$ and $z = z$. Assuming now that the radius a is not constant and using the variable r to denote it just as in polar coordinates, we obtain the cylindrical coordinates

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$



The Jacobian of cylindrical coordinates is

$$J = \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

Thus, when using cylindrical coordinates to evaluate a triple integral of a function $f(x,y,z)$ defined over a solid region E above the surface $z = g(x,y)$ and below the surface $z = h(x,y)$ with the projection D in the xy-plane. If the projection D has a representation in the polar coordinates $D = \{\ (r,\theta) \mid \alpha \leq \theta \leq \beta,\ \ r_1(\theta) \leq r \leq r_2(\theta)\ \}$, then the triple integral

$$\int\int\int_E f(x,y,z)\ dx\ dy\ dz = \int_\alpha^\beta \left(\int_{r_1(\theta)}^{r_2(\theta)} \left(\int_{g(r,\theta)}^{h(r,\theta)} f(r\cos\theta, r\sin\theta, z)\ dz \right)\ r\ \ dr \right)\ d\theta$$

It is also good to keep in mind that in cylindrical coordinates x, y and r are related in the same way as in polar coordinates by

$$x^2 + y^2 = r^2.$$

Spherical coordinates.

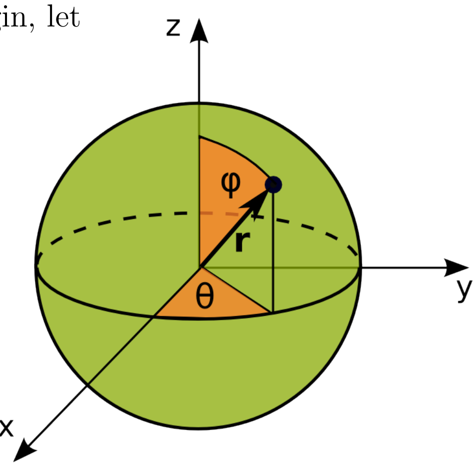
Besides cylindrical coordinates, another frequently used coordinates for triple integrals are **spherical coordinates.** Spherical coordinates are mostly used for the integrals over a solid whose definition involves spheres.

If $P = (x,y,z)$ is a point in space and O denotes the origin, let

- r denote the length of the vector $\overrightarrow{OP} = \langle x,y,z\rangle$, i.e. the distance of the point $P = (x,y,z)$ from the origin O. Thus,

$$x^2 + y^2 + z^2 = r^2;$$

- θ be the angle between the projection of vector $\overrightarrow{OP} = \langle x,y,z\rangle$ on the xy-plane and the vector $\overrightarrow{i}$ (positive x axis); and
- ϕ be the angle between the vector $\overrightarrow{OP}$ and the vector $\overrightarrow{k}$ (positive z-axis).



The conversion equations are $\underline{x = r\cos\theta\sin\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\phi.}$

The Jacobian determinant can be computed to be $\boxed{J = r^2\sin\phi.}$ Thus,

$$dx\ dy\ dz = r^2\ \sin\phi\ dr\ d\phi\ d\theta.$$

Note that the angle θ is the same in cylindrical and spherical coordinates.

It is important to remember that the distance r is *different in cylindrical and in spherical coordinates.*

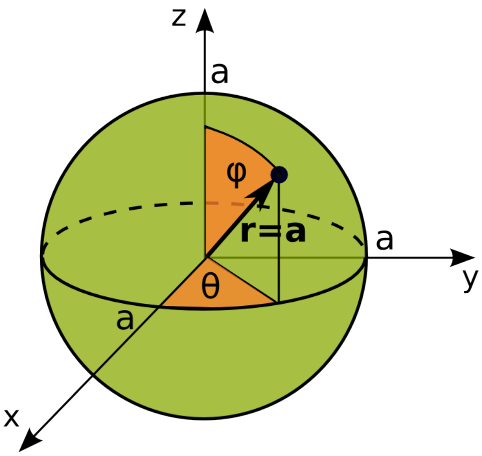
	Meaning of r	Relation to x, y, z
Cylindrical	distance from (x,y,z) to z-axis	$x^2 + y^2 = r^2$
Spherical	distance from (x,y,z) to the origin	$x^2 + y^2 + z^2 = r^2$

Spherical coordinates parametrization of a sphere. If a is a positive constant and a point (x, y, z) is on the sphere centered at the origin of radius a, then the coordinates satisfy the equation

$$x^2 + y^2 + z^2 = a^2.$$

So, the distance from the origin r is exactly a for every such point. In other words, r is *constant and equal to* a. Thus, the equation of the sphere in spherical coordinates become simple and short

$$r = a.$$



The equations $x = a\cos\theta\sin\phi,\ y = a\sin\theta\sin\phi,\ z = a\cos\phi$

parametrize the sphere. When these equations are substituted in the expression $x^2 + y^2 + z^2$, it simplifies to a^2 (you should convince yourself of this fact).

Practice problems.

1. Evaluate the triple integral

   a) $\int\int\int_E \sqrt{x^2 + y^2}\ dx\ dy\ dz$ where E is the region that lies inside the cylinder $x^2 + y^2 = 16$ and between the planes $z = -5$ and $z = 4$.

   b) $\int\int\int_E 2\ dx\ dy\ dz$ where E is the solid that lies between the cylinders $x^2+y^2 = 1$ $x^2+y^2 = 4$ and between the xy-plane and the plane $z = x + 2$.

   c) $\int\int\int_E (x^2 + y^2 + z^2)\ dx\ dy\ dz$ where E is the unit ball $x^2 + y^2 + z^2 \leq 1$.

   d) $\int\int\int_E z\ \ dx\ \ dy\ \ dz$ where E is the region between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ in the first octant.

2. Find the volume of the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 36 - 3x^2 - 3y^2$.

3. Find the volume of the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 18 - x^2 - y^2$.

4. Find the volume of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1$ by using the transformation $x = 2u,\ y = 3v$ $z = 5w$.

Solutions.

1. a) Use cylindrical coordinates. The interior of the circle $x^2+y^2 = 16$ can be described by $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 4$. The bounds for z are given by $z = -5$ and $z = 4$. $\int\int\int_E \sqrt{x^2 + y^2}\ dx\ dy\ dz = \int_0^{2\pi}\int_0^4\int_{-5}^4 \sqrt{r^2} r\ dr\ d\theta\ dz = \int_0^{2\pi} d\theta\ \int_0^4 r^2\ dr \int_{-5}^4 dz = 2\pi\frac{64}{3}(4+5) = 384\pi$.

   b) The region between the circles $x^2 + y^2 = 1$ $x^2 + y^2 = 4$ has $0 \leq \theta \leq 2\pi$ and $1 \leq r \leq 2$. The bounds for z are xy-plane $z = 0$ and the plane $z = x + 2$ which in polar coordinates has the equation $z = r\cos\theta + 2$. Thus, using the cylindrical coordinates, $\int\int\int_E 2\ dx\ dy\ dz = \int_0^{2\pi}\int_1^2\int_0^{r\cos\theta+2} 2r\ \ dr\ \ d\theta\ \ dz = \int_0^{2\pi} d\theta\ \ \int_1^2 2r\ \ dr\ \ (r\cos\theta + 2) = \int_0^{2\pi} d\theta\ \ (2\frac{r^3}{3}\cos\theta + 2r^2)|_1^2 = \int_0^{2\pi} d\theta\ (\frac{14}{3}\cos\theta + 6) = 12\pi$.

c) Using spherical coordinates, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$, and $0 \leq r \leq 1$. $\int\int\int_E x^2 + y^2 + z^2\, dx\; dy\; dz = \int_0^{2\pi} \int_0^{\pi} \int_0^1 r^2\, r^2 \sin\phi dr\; d\theta\; d\phi = \int_0^{2\pi} d\theta\; \int_0^{\pi} \sin\phi d\phi\; \int_0^1 r^4\, dr = 2\pi(-\cos\phi)|_0^{\pi} \frac{r^5}{5}|_0^1 = 2\pi(2)\frac{1}{5} = \frac{4\pi}{5}$.

d) Use spherical coordinates. Since the region is in the first octant, $0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \phi \leq \frac{\pi}{2}$. The bounds for r are determined by the radii of the spheres, so $1 \leq r \leq 2$. $\int\int\int_E z\; dx\; dy\; dz = \int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 r\cos\phi\; r^2 \sin\phi dr\; d\theta\; d\phi = \int_0^{\pi/2} d\theta\; \int_0^{\pi/2} \cos\phi \sin\phi d\phi\; \int_1^2 r^3 dr = \frac{\pi}{2}\; \frac{1}{2}\; \frac{r^4}{4}|_1^2 = \frac{15\pi}{16}$

2. Use cylindrical coordinates. The paraboloids have the equations $z = x^2 + y^2 = r^2$ and $z = 36 - 3x^2 - 3y^2 = 36 - 3r^2$. The first is the lower z-bound and the second is the upper. The bounds for θ are $0 \leq \theta \leq 2\pi$. The paraboloids intersect in a circle. The projection of the circle in xy-plane determines the r-bounds. The intersection is when $36 - 3r^2 = r^2 \Rightarrow 36 = 4r^2 \Rightarrow 9 = r^2 \Rightarrow r = 3$ (negative solution is not relevant). Thus, the r-bounds are $0 \leq r \leq 3$. The volume is $V = \int\int\int\; dxdydz = \int_0^{2\pi} \int_0^3 \int_{r^2}^{36-3r^2}\; r\, dr\, d\theta\, dz = \int_0^{2\pi} d\theta\; \int_0^3 r\, dr(36 - 3r^2 - r^2) = 2\pi(18r^2 - r^4)|_0^3 = 2\pi(162 - 81) = 162\pi$.

3. Very similar to the previous problem. The z-bounds are $x^2+y^2 = r^2 \leq z \leq 18-x^2-y^2 = 18-r^2$. The bounds for θ are $0 \leq \theta \leq 2\pi$. The intersection of paraboloids is when $18 - r^2 = r^2 \Rightarrow 18 = 2r^2 \Rightarrow 9 = r^2 \Rightarrow r = 3$ (negative solution is not relevant). Thus, the r-bounds are $0 \leq r \leq 3$. The volume is $V = \int\int\int\; dxdydz = \int_0^{2\pi} \int_0^3 \int_{r^2}^{18-r^2}\; r\, dr\, d\theta\, dz = \int_0^{2\pi} d\theta\; \int_0^3 r\, dr(18 - r^2 - r^2) = 2\pi(9r^2 - \frac{r^4}{2})|_0^3 = 2\pi(81 - \frac{81}{2}) = 81\pi$.

4. The substitution $x = 2u$, $y = 3v$ and $z = 5w$ converts the ellipsoid into a sphere of radius 1. The Jacobian of the substitution is $J = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{vmatrix} = 30$. Thus, the volume is equal to $V = \int\int\int\; dx\, dy\, dz = \int\int\int 30\; du\, dv\, dw$. Use the spherical coordinates now. The bounds are $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$, and $0 \leq r \leq 1$ and the Jacobian is $r^2 \sin\phi$. Thus, the volume is $V = \int\int\int 30\; r^2 \sin\phi\; dr\, d\phi\, d\theta = 30\; \int_0^{2\pi} d\theta\; \int_0^{\pi} \sin\phi\, d\phi\; \int_0^1 r^2 dr = 30\; 2\pi\,(-\cos\phi)|_0^{\pi}\; \frac{r^3}{3}|_0^1 = 120\pi\frac{1}{3} = 40\pi$.