Review of vectors. The dot and cross products

Review of vectors in two and three dimensions. A two-dimensional vector is an ordered pair \( \vec{a} = \langle a_1, a_2 \rangle \) of real numbers. The coordinate representation of the vector \( \vec{a} \) corresponds to the arrow from the origin \((0, 0)\) to the point \((a_1, a_2)\). Thus, the length of \( \vec{a} \) is \(|\vec{a}| = \sqrt{a_1^2 + a_2^2}\). Analogously, we have the following.

A three-dimensional vector is an ordered triple
\[
\vec{a} = \langle a_1, a_2, a_3 \rangle
\]
of real numbers. The coordinate representation of the vector \( \vec{a} \) corresponds to the arrow from the origin \((0, 0, 0)\) to the point \((a_1, a_2, a_3)\).

The length of \( \vec{a} \) is
\[
|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.
\]

Using the coordinate representation the vector addition and scalar multiplication can be realized as follows.

**Vector Addition** - by coordinates
\[
\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle
\]

**Scalar multiplication** - by coordinates
\[
k\langle a_1, a_2, a_3 \rangle = \langle ka_1, ka_2, ka_3 \rangle
\]

This corresponds to the geometrical representation illustrated in the figure below.

Using its coordinates, a vector \( \vec{a} = \langle a_1, a_2 \rangle \) in \( xy \)-plane can be represented as a linear combination of vectors \( \vec{i} = \langle 1, 0 \rangle \) and \( \vec{j} = \langle 0, 1 \rangle \) as follows.
\[
\vec{a} = a_1 \vec{i} + a_2 \vec{j}
\]

The coordinates of a vector and geometrical representation have analogous relation in three dimensional space.
If \( \mathbf{i} = \langle 1, 0, 0 \rangle \), \( \mathbf{j} = \langle 0, 1, 0 \rangle \), and \( \mathbf{k} = \langle 0, 0, 1 \rangle \) and a vector \( \mathbf{a} \) can be represented as \( \mathbf{a} = \langle a_1, a_2, a_3 \rangle \), then

\[
\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}.
\]

In the next section, it will be relevant to determine the coordinates of the vector from one point to the other. Let \( P = (a_1, a_2, a_3) \) and \( Q = (b_1, b_2, b_3) \), be two points in space. If \( O \) denotes the origin \((0, 0, 0)\), then the vector \( \overrightarrow{OP} \) can be represented as \( \langle a_1, a_2, a_3 \rangle \), and the vector \( \overrightarrow{OQ} \) as \( \langle b_1, b_2, b_3 \rangle \).

Since \( \overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ} \) we have that

\[
\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \langle b_1, b_2, b_3 \rangle - \langle a_1, a_2, a_3 \rangle = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle.
\]

In some cases, we may need to find the vector with same direction and sense as a nonzero vector \( \mathbf{a} \) but of length 1. Such vector is called the normalization of \( \mathbf{a} \).

**Practice problems.**

1. Let \( P \) be the point \((2, -1)\) and \( Q \) be the point \((1, 3)\). Determine and sketch the vector \( \overrightarrow{PQ} \).

2. Let \( \mathbf{a} = \langle 2, -1 \rangle \) and \( \mathbf{b} = \langle 1, 3 \rangle \). Sketch \( \mathbf{a} + \mathbf{b}, \mathbf{a} - \mathbf{b}, 2\mathbf{a}, 2\mathbf{a} - 3\mathbf{b} \).

3. Let \( \mathbf{a} = \langle 3, 4, 0 \rangle \) and \( \mathbf{b} = \langle -1, 4, 2 \rangle \). Determine \( |\mathbf{a}| \), \( 2\mathbf{a} + 3\mathbf{b}, 3\mathbf{a} - 2\mathbf{b} \).

4. Let \( \mathbf{a} = \mathbf{i} + 4\mathbf{j} - 8\mathbf{k} \) and \( \mathbf{b} = -2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \). Determine \( |\mathbf{a}|, \mathbf{a} + \mathbf{b}, 2\mathbf{a} - 3\mathbf{b} \).

5. Find the normalization of the vector \( \mathbf{a} = \mathbf{i} + 4\mathbf{j} + 8\mathbf{k} \).

6. Find the normalization of the vector \( \mathbf{a} = \langle 3, 4, 0 \rangle \).

**Solutions.**

1. \( \overrightarrow{PQ} = \langle -1, 4 \rangle \)

2. \( |\mathbf{a}| = 5, 2\mathbf{a} + 3\mathbf{b} = \langle 3, 20, 6 \rangle, 3\mathbf{a} - 2\mathbf{b} = \langle 11, 4, -4 \rangle \)

3. \( |\mathbf{a}| = 9, \mathbf{a} + \mathbf{b} = \langle -1, 5, -6 \rangle, 2\mathbf{a} - 3\mathbf{b} = \langle 8, 5, -22 \rangle \)

4. \( |\mathbf{a}| = \sqrt{1 + 4^2 + 8^2} = \sqrt{81} = 9 \) so \( \mathbf{a} = \frac{\mathbf{i}}{9} + \frac{4\mathbf{j}}{9} + \frac{8\mathbf{k}}{9} \)

5. The length of \( \mathbf{a} \) is \( |\mathbf{a}| = \sqrt{3^2 + 4^2 + 0^2} = \sqrt{25} = 5 \) so \( \mathbf{a} = \langle \frac{3}{5}, \frac{4}{5}, 0 \rangle \).

6. The length of \( \mathbf{a} \) is \( |\mathbf{a}| = \sqrt{3^2 + 4^2 + 0^2} = \sqrt{25} = 5 \) so \( \mathbf{a} = \langle \frac{3}{5}, \frac{4}{5}, 0 \rangle \).

**The Dot Product**

If one is to define a meaningful product of two vectors, \( \mathbf{a} = \langle a_1, a_2, a_3 \rangle \) and \( \mathbf{b} = \langle b_1, b_2, b_3 \rangle \), the first idea that comes to mind would probably be to consider coordinate-wise multiplication \( \langle a_1b_1, a_2b_2, a_3b_3 \rangle \). However, since this type of product is geometrically not very meaningful nor applicable, one consider two other types of multiplication, the dot and the cross product.
The **dot product** of vectors \( \vec{a} = \langle a_1, a_2, a_3 \rangle \) and \( \vec{b} = \langle b_1, b_2, b_3 \rangle \) is defined to be the scalar obtained by adding the coordinates of our first attempt to define the product, \( \langle a_1 b_1, a_2 b_2, a_3 b_3 \rangle \). The notation used for such product is \( \vec{a} \cdot \vec{b} \). Thus

\[
\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.
\]

This product can be used to determine the angle between the vectors and, in particular, to test whether two vectors are perpendicular to each other. If \( \theta \) is the angle between two nonzero vectors \( \vec{a} \) and \( \vec{b} \), then

\[
\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta
\]

As a consequence, \( \vec{a} \) and \( \vec{b} \) are perpendicular (or orthogonal) exactly when \( \cos \theta = 0 \) which, in turn, happens exactly when \( \vec{a} \cdot \vec{b} = 0 \). Thus,

\( \vec{a} \) and \( \vec{b} \) are **perpendicular** if and only if \( \vec{a} \cdot \vec{b} = 0 \).

In case when \( \vec{a} = \vec{b}, \theta = 0 \) and \( \cos \theta = 1 \), and the formula \( \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta \) becomes

\[
\vec{a} \cdot \vec{a} = |\vec{a}|^2
\]

which relates the dot product and the length of a vector \( \vec{a} \).

**Projection of one vector to another.** In many physics applications, it is relevant to determine the coordinates of a projection of one vector onto the other.

Let \( \text{proj}_b \vec{a} \) denote the **projection of \( \vec{a} \) onto \( \vec{b} \)** for given nonzero vectors \( \vec{a} \) and \( \vec{b} \). The projection of \( \vec{a} \) onto \( \vec{b} \) has the same direction and sense as vector \( \vec{b} \). The length of \( \text{proj}_b \vec{a} \) satisfies

\[
|\text{proj}_b \vec{a}| = |\vec{a}| \cos \theta.
\]

Thus, \( \text{proj}_b \vec{a} \) can be obtained by multiplying its length with the normalization of \( \vec{b} \).

\[
\text{proj}_b \vec{a} = |\text{proj}_b \vec{a}| \hat{b} = |\vec{a}| \cos \theta \hat{b} = |\vec{a}| \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} \hat{b} \Rightarrow \\
\text{proj}_b \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \hat{b}.
\]

**Practice problems.**

1. Find the dot product of the vectors \( \vec{a} = \langle 1, 3, -4 \rangle \) and \( \vec{b} = \langle -2, 3, 1 \rangle \).
2. Find the angle between the vectors \( \vec{a} = \langle 3, 4 \rangle \) and \( \vec{b} = \langle 5, 12 \rangle \).
3. Find the angle between the vectors \( \vec{a} = \langle 3, -1, 2 \rangle \) and \( \vec{b} = \langle 2, 4, -1 \rangle \).

4. Find the projection of \( \vec{a} \) onto \( \vec{b} \) if \( \vec{a} = \langle 1, -1, 0 \rangle \) and \( \vec{b} = \langle 1, 0, 1 \rangle \).

**Solutions.**

1. \( \vec{a} \cdot \vec{b} = -2 + 9 - 4 = 3 \).

2. If \( \theta \) denotes the angle between the vectors, then

\[
\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{63}{65} = 0.969.
\]

So, \( \theta = \cos^{-1}(0.969) \approx 14.25 \text{ degrees} \) or \( 0.249 \text{ radians} \).

3. \( \cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = 0 \). So, \( \theta = 90 \text{ degrees} \) and the vectors are perpendicular.

4. \( \text{proj}_b \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b} = \frac{1}{(\sqrt{2})^2} \langle 1, 0, 1 \rangle = \frac{1}{2} \langle 1, 0, 1 \rangle = \frac{1}{2} \vec{b} \).

**The Cross Product**

As opposed to the dot product which results in a scalar, the cross product of two vectors is again a vector. If \( \vec{a} \) and \( \vec{b} \) are two vectors, their cross product is denoted by \( \vec{a} \times \vec{b} \).

The vector \( \vec{a} \times \vec{b} \) is perpendicular to the plane determined by \( \vec{a} \) and \( \vec{b} \). This determines the direction of \( \vec{a} \times \vec{b} \). The sense of \( \vec{a} \times \vec{b} \) is determined by the right hand rule: if \( \vec{a} \) and is the thumb and \( \vec{b} \) the middle finger, the index finger has the same sense as \( \vec{a} \times \vec{b} \). Using the right hand rule, you can see that the cross product is not \textbf{not} commutative, \( \vec{a} \times \vec{b} \neq \vec{b} \times \vec{a} \) in general, and that

\[
\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}.
\]

The length of \( \vec{a} \times \vec{b} \) is the same as the area of the parallelogram determined by \( \vec{a} \) and \( \vec{b} \). If \( \theta \) is the angle between \( \vec{a} \) and \( \vec{b} \), then

\[
|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \theta.
\]

The cross product of \( \vec{a} = \langle a_1, a_2, a_3 \rangle \) and \( \vec{b} = \langle b_1, b_2, b_3 \rangle \) can be computed using the coordinates as follows.

\[
\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 
\end{vmatrix}
\]

Since \( |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \theta \), \( \vec{a} \) and \( \vec{b} \) are parallel exactly when \( \sin \theta = 0 \) which happens exactly when \( \vec{a} \times \vec{b} = \vec{0} \). Thus,

\( \vec{a} \) and \( \vec{b} \) are \textbf{parallel} if and only if \( \vec{a} \times \vec{b} = \vec{0} \).
Another way to check if the two vectors are parallel is to check if one is a scalar multiple of the other (i.e. if \( \vec{a} = k\vec{b} \) for some \( k \)). In this case, for \( \vec{b} \neq \vec{0} \), the coordinates are such that \( \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} \).

**Practice Problems.**

1. Let \( \vec{a} = \langle 1, 2, 0 \rangle \) and \( \vec{b} = \langle 0, 3, 1 \rangle \). Find \( \vec{a} \times \vec{b} \).

2. Do the same for \( \vec{a} = 2\vec{i} + \vec{j} - \vec{k} \) and \( \vec{b} = \vec{j} + 2\vec{k} \).

3. Let \( \vec{a} = \langle -5, 3, 7 \rangle \) and \( \vec{b} = \langle 6, -8, 2 \rangle \). Determine if the vectors are parallel, perpendicular or neither.

4. Do the same for \( \vec{a} = -\vec{i} + 2\vec{j} + 5\vec{k} \) and \( \vec{b} = 3\vec{i} + 4\vec{j} - \vec{k} \).

5. Find a vector perpendicular to the plane through the points \( P(1, 0, 0) \), \( Q(0, 2, 0) \) and \( R(0, 0, 3) \) and find the area of the triangle \( PQR \).

6. Do the same for \( P(0, 0, 0) \), \( Q(1, -1, 1) \) and \( R(4, 3, 7) \).

**Solutions.**

1. \( \langle 2, -1, 3 \rangle \)

2. \( 3\vec{i} - 4\vec{j} + 2\vec{k} \)

3. \( \vec{a} \cdot \vec{b} = -40 \neq 0 \) so the vectors are not perpendicular. Also, the coordinates are not proportional \( (\frac{-5}{6} \neq \frac{3}{-8} \neq \frac{7}{2}) \) so the vectors are not parallel either. Alternatively, find that the cross product is \( \vec{a} \times \vec{b} = \langle 62, 52, 22 \rangle \neq \langle 0, 0, 0 \rangle \) so the vectors are not parallel.

4. \( \vec{a} \cdot \vec{b} = 0 \), thus the vectors are perpendicular.

5. Since vectors \( \vec{PQ} \) and \( \vec{PR} \) are in the plane, their cross product \( \vec{PQ} \times \vec{PR} \) is perpendicular to the plane. Calculate \( \vec{PQ} = \langle -1, 2, 0 \rangle \) and \( \vec{PR} = \langle -1, 0, 3 \rangle \), \( \vec{PQ} \times \vec{PR} = \langle 6, 3, 2 \rangle \). The area of the triangle determined by \( P, Q, \) and \( R \) is half of the area of the parallelogram determined by the vectors \( \vec{PQ} \) and \( \vec{PR} \) which is the magnitude of \( \vec{PQ} \times \vec{PR} \). Thus the triangle area is \( \frac{1}{2} \sqrt{36 + 9 + 4} = \frac{7}{2} \).

6. Similarly to previous problem, find a vector perpendicular to the plane to be \( \langle -10, -3, 7 \rangle \) and the area of the triangle to be \( \sqrt{158}/2 = 6.28 \).