

## Review of vectors. The dot and cross products

**Review of vectors in two and three dimensions.** A **two-dimensional vector** is an ordered pair  $\vec{a} = \langle a_1, a_2 \rangle$  of real numbers. The coordinate representation of the vector  $\vec{a}$  corresponds to the arrow from the origin  $(0, 0)$  to the point  $(a_1, a_2)$ . Thus, the **length** of  $\vec{a}$  is  $|\vec{a}| = \sqrt{a_1^2 + a_2^2}$ . Analogously, we have the following.

A **three-dimensional vector** is an ordered triple

$$\vec{a} = \langle a_1, a_2, a_3 \rangle$$

of real numbers. The coordinate representation of the vector  $\vec{a}$  corresponds to the arrow from the origin  $(0, 0, 0)$  to the point  $(a_1, a_2, a_3)$ .

The **length** of  $\vec{a}$  is

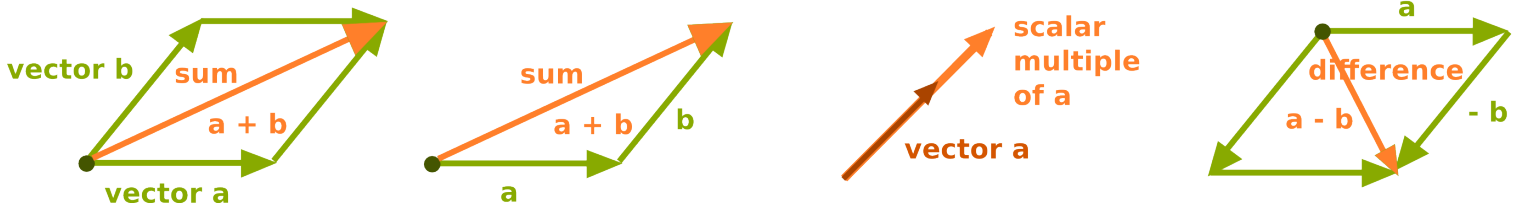
$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Using the coordinate representation the vector addition and scalar multiplication can be realized as follows.

**Vector Addition** - by coordinates  $\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

**Scalar multiplication** - by coordinates  $k\langle a_1, a_2, a_3 \rangle = \langle ka_1, ka_2, ka_3 \rangle$

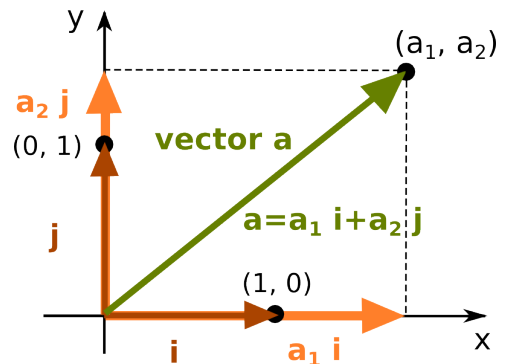
This corresponds to the geometrical representation illustrated in the figure below.



Using its coordinates, a vector  $\vec{a} = \langle a_1, a_2 \rangle$  in  $xy$ -plane can be represented as a linear combination of vectors  $\vec{i} = \langle 1, 0 \rangle$  and  $\vec{j} = \langle 0, 1 \rangle$  as follows.

$$\vec{a} = a_1\vec{i} + a_2\vec{j}$$

The coordinates of a vector and geometrical representation have analogous relation in three dimensional space.



If  $\vec{i} = \langle 1, 0, 0 \rangle$ ,  $\vec{j} = \langle 0, 1, 0 \rangle$ , and  $\vec{k} = \langle 0, 0, 1 \rangle$  and a vector  $\vec{a}$  can be represented as  $\vec{a} = \langle a_1, a_2, a_3 \rangle$ , then

$$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}.$$

In the next section, it will be relevant to determine the coordinates of the vector from one point to the other. Let  $P = (a_1, a_2, a_3)$  and  $Q = (b_1, b_2, b_3)$ , be two points in space. If  $O$  denotes the origin  $(0, 0, 0)$ , then the vector  $\vec{OP}$  can be represented as  $\langle a_1, a_2, a_3 \rangle$ , and the vector  $\vec{OQ}$  as  $\langle b_1, b_2, b_3 \rangle$ .

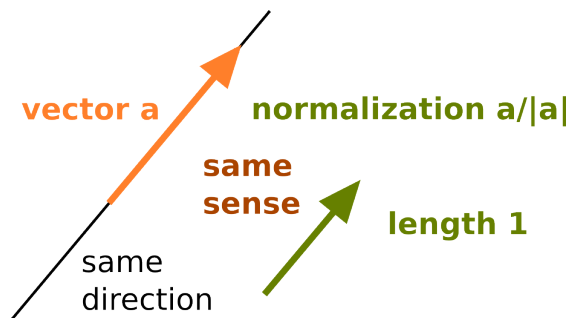
Since  $\vec{OP} + \vec{PQ} = \vec{OQ}$  we have that

$$\vec{PQ} = \vec{OQ} - \vec{OP} = \langle b_1, b_2, b_3 \rangle - \langle a_1, a_2, a_3 \rangle = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle.$$

In some cases, we may need to find the vector with same direction and sense as a nonzero vector  $\vec{a}$  but of length 1. Such vector is called the **normalization** of  $\vec{a}$ .

The **normalization** of  $\vec{a}$  is the vector of length 1 in the direction of  $\vec{a}$ ,  

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}.$$



### Practice problems.

- Let  $P$  be the point  $(2, -1)$  and  $Q$  be the point  $(1, 3)$ . Determine and sketch the vector  $\vec{PQ}$ .
- Let  $\vec{a} = \langle 2, -1 \rangle$  and  $\vec{b} = \langle 1, 3 \rangle$ . Sketch  $\vec{a} + \vec{b}$ ,  $\vec{a} - \vec{b}$ ,  $2\vec{a}$ ,  $2\vec{a} - 3\vec{b}$ .
- Let  $\vec{a} = \langle 3, 4, 0 \rangle$  and  $\vec{b} = \langle -1, 4, 2 \rangle$ . Determine  $|\vec{a}|$ ,  $2\vec{a} + 3\vec{b}$ ,  $3\vec{a} - 2\vec{b}$ .
- Let  $\vec{a} = \vec{i} + 4\vec{j} - 8\vec{k}$  and  $\vec{b} = -2\vec{i} + \vec{j} + 2\vec{k}$ . Determine  $|\vec{a}|$ ,  $\vec{a} + \vec{b}$ ,  $2\vec{a} - 3\vec{b}$ .
- Find the normalization of the vector  $\vec{a} = \vec{i} + 4\vec{j} + 8\vec{k}$ .
- Find the normalization of the vector  $\vec{a} = \langle 3, 4, 0 \rangle$ .

**Solutions.** 1.  $\vec{PQ} = \langle -1, 4 \rangle$       3.  $|\vec{a}| = 5$ ,  $2\vec{a} + 3\vec{b} = \langle 3, 20, 6 \rangle$ ,  $3\vec{a} - 2\vec{b} = \langle 11, 4, -4 \rangle$   
 4.  $|\vec{a}| = 9$ ,  $\vec{a} + \vec{b} = \langle -1, 5, -6 \rangle$ ,  $2\vec{a} - 3\vec{b} = \langle 8, 5, -22 \rangle$       5. The length of  $\vec{a}$  is  $|\vec{a}| = \sqrt{1 + 4^2 + 8^2} = \sqrt{81} = 9$  so  $\hat{a} = \frac{1}{9}\vec{i} + \frac{4}{9}\vec{j} + \frac{8}{9}\vec{k}$       6. The length of  $\vec{a}$  is  $|\vec{a}| = \sqrt{3^2 + 4^2 + 0^2} = \sqrt{25} = 5$  so  $\hat{a} = \langle \frac{3}{5}, \frac{4}{5}, 0 \rangle$ .

## The Dot Product

If one is to define a meaningful product of two vectors,  $\vec{a} = \langle a_1, a_2, a_3 \rangle$  and  $\vec{b} = \langle b_1, b_2, b_3 \rangle$ , the first idea that comes to mind would probably be to consider coordinate-wise multiplication  $\langle a_1b_1, a_2b_2, a_3b_3 \rangle$ . However, since this type of product is geometrically not very meaningful nor applicable, one consider two other types of multiplication, the dot and the cross product.

The **dot product** of vectors  $\vec{a} = \langle a_1, a_2, a_3 \rangle$  and  $\vec{b} = \langle b_1, b_2, b_3 \rangle$  is defined to be the scalar obtained by adding the coordinates of our first attempt to define the product,  $\langle a_1b_1, a_2b_2, a_3b_3 \rangle$ . The notation used for such product is  $\vec{a} \cdot \vec{b}$ . Thus

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

This product can be used to determine the angle between the vectors and, in particular, to test whether two vectors are perpendicular to each other. If  $\theta$  is the angle between two nonzero vectors  $\vec{a}$  and  $\vec{b}$ , then

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$$

As a consequence,  $\vec{a}$  and  $\vec{b}$  are perpendicular (or orthogonal) exactly when  $\cos \theta = 0$  which, in turn, happens exactly when  $\vec{a} \cdot \vec{b} = 0$ . Thus,

$\vec{a}$  and  $\vec{b}$  are **perpendicular** if and only if  $\vec{a} \cdot \vec{b} = 0$ .

In case when  $\vec{a} = \vec{b}$ ,  $\theta = 0$  and  $\cos \theta = 1$ , and the formula  $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$  becomes

$$\vec{a} \cdot \vec{a} = |\vec{a}|^2$$

which relates the dot product and the length of a vector  $\vec{a}$ .

**Projection of one vector to another.** In many physics applications, it is relevant to determine the coordinates of a projection of one vector onto the other.

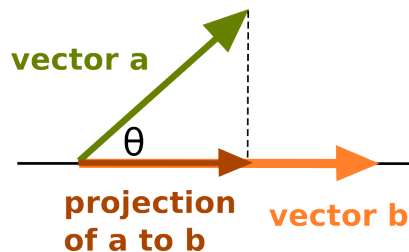
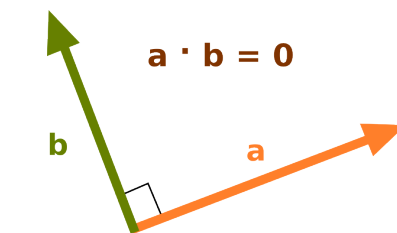
Let  $\text{proj}_{\vec{b}}\vec{a}$  denote the **projection of  $\vec{a}$  onto  $\vec{b}$**  for given nonzero vectors  $\vec{a}$  and  $\vec{b}$ . The projection of  $\vec{a}$  onto  $\vec{b}$  has the same direction and sense as vector  $\vec{b}$ . The length of  $\text{proj}_{\vec{b}}\vec{a}$  satisfies

$$|\text{proj}_{\vec{b}}\vec{a}| = |a| \cos \theta.$$

Thus,  $\text{proj}_{\vec{b}}\vec{a}$  can be obtained by multiplying its length with the normalization of  $\vec{b}$ .

$$\text{proj}_{\vec{b}}\vec{a} = |\text{proj}_{\vec{b}}\vec{a}| \hat{b} = |a| \cos \theta \hat{b} = |a| \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} \frac{\vec{b}}{|\vec{b}|} \Rightarrow$$

$$\text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b}.$$



**length =  $|a| \cos \theta$**       **direction & sense = those of b**

**Practice problems.**

1. Find the dot product of the vectors  $\vec{a} = \langle 1, 3, -4 \rangle$  and  $\vec{b} = \langle -2, 3, 1 \rangle$ .
2. Find the angle between the vectors  $\vec{a} = \langle 3, 4 \rangle$  and  $\vec{b} = \langle 5, 12 \rangle$ .

- Find the angle between the vectors  $\vec{a} = \langle 3, -1, 2 \rangle$  and  $\vec{b} = \langle 2, 4, -1 \rangle$ .
- Find the projection of  $\vec{a}$  onto  $\vec{b}$  if  $\vec{a} = \langle 1, -1, 0 \rangle$  and  $\vec{b} = \langle 1, 0, 1 \rangle$ .

**Solutions.** 1.  $\vec{a} \cdot \vec{b} = -2 + 9 - 4 = 3$ . 2. If  $\theta$  denotes the angle between the vectors, then  $\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{15+48}{(5)(13)} = \frac{63}{65} \approx .969$ .  $\theta = \cos^{-1}(.969) = .249$  radians or 14.25 degrees.

- $\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{0}{|\vec{a}||\vec{b}|} = 0$ . So,  $\theta = 90$  degrees and the vectors are perpendicular.
- $\text{proj}_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b} = \frac{1}{(\sqrt{2})^2} \langle 1, 0, 1 \rangle = \frac{1}{2} \langle 1, 0, 1 \rangle = \langle \frac{1}{2}, 0, \frac{1}{2} \rangle$ .

## The Cross Product

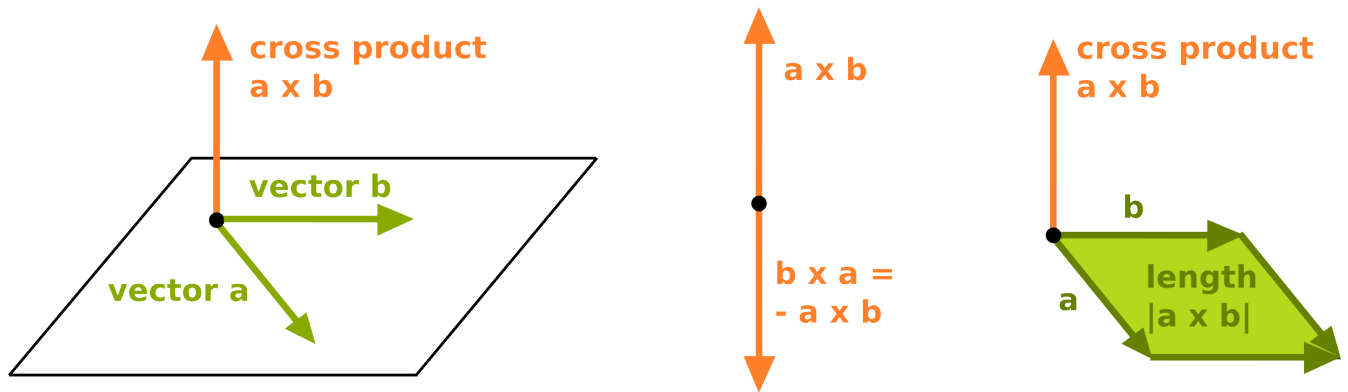
As opposed to the dot product which results in a scalar, the cross product of two vectors is again a *vector*. If  $\vec{a}$  and  $\vec{b}$  are two vectors, their **cross product** is denoted by  $\vec{a} \times \vec{b}$ .

The vector  $\vec{a} \times \vec{b}$  is perpendicular to the plane determined by  $\vec{a}$  and  $\vec{b}$ . This determines the **direction** of  $\vec{a} \times \vec{b}$ . The **sense** of  $\vec{a} \times \vec{b}$  is determined by the right hand rule: if  $\vec{a}$  and is the thumb and  $\vec{b}$  the middle finger, the index finger has the same sense as  $\vec{a} \times \vec{b}$ . Using the right hand rule, you can see that the cross product is not **not** commutative,  $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$  in general, and that

$$\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}.$$

The **length** of  $\vec{a} \times \vec{b}$  is the same as the **area of the parallelogram determined by  $\vec{a}$  and  $\vec{b}$** . If  $\theta$  is the angle between  $\vec{a}$  and  $\vec{b}$ , then

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \theta.$$



The cross product of  $\vec{a} = \langle a_1, a_2, a_3 \rangle$  and  $\vec{b} = \langle b_1, b_2, b_3 \rangle$  can be computed using the coordinates as follows.

$$\vec{a} \times \vec{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Since  $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \theta$ ,  $\vec{a}$  and  $\vec{b}$  are parallel exactly when  $\sin \theta = 0$  which happens exactly when  $\vec{a} \times \vec{b} = \vec{0}$ . Thus,

$$\vec{a} \text{ and } \vec{b} \text{ are } \mathbf{parallel} \text{ if and only if } \vec{a} \times \vec{b} = \vec{0}.$$

Another way to check if the two vectors are parallel is to check if one is a scalar multiple of the other (i.e. if  $\vec{a} = k\vec{b}$  for some  $k$ ). In this case, for  $\vec{b} \neq \vec{0}$ , the coordinates are such that  $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$ .

### Practice Problems.

1. Let  $\vec{a} = \langle 1, 2, 0 \rangle$  and  $\vec{b} = \langle 0, 3, 1 \rangle$ . Find  $\vec{a} \times \vec{b}$ .
2. Do the same for  $\vec{a} = 2\vec{i} + \vec{j} - \vec{k}$  and  $\vec{b} = \vec{j} + 2\vec{k}$ .
3. Let  $\vec{a} = \langle -5, 3, 7 \rangle$  and  $\vec{b} = \langle 6, -8, 2 \rangle$ . Determine if the vectors are parallel, perpendicular or neither.
4. Do the same for  $\vec{a} = -\vec{i} + 2\vec{j} + 5\vec{k}$  and  $\vec{b} = 3\vec{i} + 4\vec{j} - \vec{k}$ .
5. Find a vector perpendicular to the plane through the points  $P(1, 0, 0)$ ,  $Q(0, 2, 0)$  and  $R(0, 0, 3)$  and find the area of the triangle  $PQR$ .
6. Do the same for  $P(0, 0, 0)$ ,  $Q(1, -1, 1)$  and  $R(4, 3, 7)$ .

**Solutions.** 1.  $\langle 2, -1, 3 \rangle$       2.  $3\vec{i} - 4\vec{j} + 2\vec{k}$

3.  $\vec{a} \cdot \vec{b} = -40 \neq 0$  so the vectors are not perpendicular. Also, the coordinates are not proportional ( $-\frac{5}{6} \neq \frac{3}{-8} \neq \frac{7}{2}$ ) so the vectors are not parallel either. Alternatively, find that the cross product is  $\vec{a} \times \vec{b} = \langle 62, 52, 22 \rangle \neq \langle 0, 0, 0 \rangle$  so the vectors are not parallel.

4.  $\vec{a} \cdot \vec{b} = 0$ , thus the vectors are perpendicular.

5. Since vectors  $\vec{PQ}$  and  $\vec{PR}$  are in the plane, their cross product  $\vec{PQ} \times \vec{PR}$  is perpendicular to the plane. Calculate  $\vec{PQ} = \langle -1, 2, 0 \rangle$  and  $\vec{PR} = \langle -1, 0, 3 \rangle$ ,  $\vec{PQ} \times \vec{PR} = \langle 6, 3, 2 \rangle$ . The area of the triangle determined by  $P$ ,  $Q$ , and  $R$  is half of the area of the parallelogram determined by the vectors  $\vec{PQ}$  and  $\vec{PR}$  which is the magnitude of  $\vec{PQ} \times \vec{PR}$ . Thus the triangle area is  $\frac{1}{2}\sqrt{36 + 9 + 4} = \frac{7}{2}$ .

6. Similarly to previous problem, find a vector perpendicular to the plane to be  $\langle -10, -3, 7 \rangle$  and the area of the triangle to be  $\sqrt{158}/2 = 6.28$ .