Fundamentals of Mathematics
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Formulas for Exam 2

1. **Sets.**

$$A = B \iff (\forall x)(x \in A \Leftrightarrow x \in B)$$
$$A \subseteq B \iff (\forall x)(x \in A \Rightarrow x \in B)$$

Operations on sets.

$$A \cap B = \{x : x \in A \wedge x \in B\} \qquad x \in A \cap B \iff x \in A \wedge x \in B$$
$$A \cup B = \{x : x \in A \vee x \in B\} \qquad x \in A \cup B \iff x \in A \vee x \in B$$
$$A - B = \{x : x \in A \wedge \neg x \in B\} \qquad x \in A - B \iff x \in A \wedge \neg x \in B$$
$$\overline{A} = \{x \in U : \neg x \in A\} \qquad x \in \overline{A} \iff x \in U \wedge \neg x \in A$$
$$A \times B = \{(a, b) : a \in A \wedge b \in B\} \qquad (a, b) \in A \times B \iff a \in A \wedge b \in B$$
$$\mathcal{P}(A) = \{B : B \subseteq A\} \qquad B \in \mathcal{P}(A) \iff B \subseteq A$$
$$A = \emptyset \iff \neg x \in A$$

Generalized union and intersection

$$\bigcap_{i \in I} A_i = \{x : (\forall i \in I)\, x \in A_i\} \qquad x \in \bigcap_{i \in I} A_i \iff (\forall i \in I)\, x \in A_i$$
$$\bigcup_{i \in I} A_i = \{x : (\exists i \in I)\, x \in A_i\} \qquad x \in \bigcap_{i \in I} A_i \iff (\forall i \in I)\, x \in A_i$$

2. **Relations.** A relation on A is any subset of $A \times A$. A relation $\sim$ on A is an **equivalence** on A if $\sim$ is

reflexive: $(\forall a \in A)\ a \sim a$

symmetric: $(\forall a, b \in A)\ (a \sim b \Rightarrow b \sim a)$

transitive: $(\forall a, b, c \in A)\ (a \sim b \wedge b \sim c \Rightarrow a \sim c)$

The **equivalence class** $[a]$ for $a \in A$ is $[a] = \{b \in A : a \sim b\}$.
The **quotient set** $A/\!\sim$ is the set of equivalence classes $A/\!\sim\, = \{[a] : a \in A\}$.

A relation $\preceq$ on A is a **partial order** on A if $\preceq$ is reflexive ($(\forall a \in A)\ a \preceq a$), transitive ($(\forall a, b, c \in A)\ (a \preceq b \wedge b \preceq c \Rightarrow a \preceq c)$) and

antisymmetric: $(\forall a, b, c \in A)\ (a \preceq b \wedge b \preceq a \Rightarrow a = b)$

A partial order $\preceq$ on A is a **total order** if $(\forall a, b \in A)\, (a \preceq b \vee b \preceq a)$.

Let $\preceq$ be a partial order on A.

- $a \in A$ is **the greatest element** if $(\forall b \in A)\, b \preceq a$.
  $a \in A$ is **the least element** if $(\forall b \in A)\, a \preceq b$.
- $a \in A$ is **a maximal element** of A if $\neg(\exists b \in A)\,(a \preceq b \wedge a \neq b)$ (equivalently, $(\forall b \in A)(a \preceq b \Rightarrow a = b)$).
  $a \in A$ is **a minimal element** of A if $\neg(\exists b \in A)\,(b \preceq a \wedge a \neq b)$ (equivalently, $(\forall b \in A)(b \preceq a \Rightarrow a = b)$).
- Let $B \subseteq A$. $a \in A$ is an **upper bound** of B if $(\forall b \in B)\, b \preceq a$, and $a \in A$ is a **supremum** of B if a is the least element of the set of the upper bounds of B.

  $a \in A$ is a **lower bound** of B if $(\forall b \in B)\, a \preceq b$, and $a \in A$ is an **infimum** of B if a is the greatest element of the set of the lower bounds of B.

3. **Functions.** A function $f : A \to B$ is a subset of $A \times B$ for which $(a, b) \in f$ is written by $f(a) = b$ and such that

   (a) $(\forall a \in A)(\exists b \in B)\ f(a) = b$

   (b) $(\forall a_1, a_2 \in A)\ (a_1 = a_2 \Rightarrow f(a_1) = f(a_2))$

   A function $f : A \to B$ is

   (a) is **onto** or **surjective** if
   $$(\forall b \in B)(\exists a \in A)\ f(a) = b$$

   (b) A function $f : A \to B$ is **one-to-one** or **injective** if
   $$(\forall a_1, a_2 \in A)\ (f(a_1) = f(a_2) \Rightarrow a_1 = a_2)$$
   (contrapositive: $(\forall a_1, a_2 \in A)\ (a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2))$

   (c) A function $f : A \to B$ is **bijective** if it is one-to-one and onto.

   If $f : A \to B$ and $g : B \to C$ are two functions, a **composition** $g \circ f : A \to C$ is the function given by
   $$(g \circ f)(a) = g(f(a))$$
   for $a \in A$.

   The **identity function** on A is $\mathrm{id}_A : A \to A$ defined by $\mathrm{id}_A(a) = a$ for every $a \in A$. Useful identities for $f : A \to B$: $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_B \circ f = f$.

   A function $f : A \to B$ has the **inverse** f^{-1} if $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$.

   If $f : A \to B$ is a function, $C \subseteq A$ and $D \subseteq B$, the **image of** C is

   $f(C) = \{b \in B : (\exists c \in C)\ b = f(c)\}$. So, $b \in f(C) \Leftrightarrow (\exists c \in C)\ b = f(c)$.

   The **inverse image of** D is

   $f^{-1}(D) = \{a \in A : f(a) \in D\}$. So, $a \in f^{-1}(D) \Leftrightarrow f(a) \in D$.