

Formulas for Exam 2

1. Sets.

$$A = B \Leftrightarrow (\forall x) (x \in A \Leftrightarrow x \in B)$$

$$A \subseteq B \Leftrightarrow (\forall x) (x \in A \Rightarrow x \in B)$$

Operations on sets.

$$A \cap B = \{x : x \in A \wedge x \in B\}$$

$$x \in A \cap B \Leftrightarrow x \in A \wedge x \in B$$

$$A \cup B = \{x : x \in A \vee x \in B\}$$

$$x \in A \cup B \Leftrightarrow x \in A \vee x \in B$$

$$A - B = \{x : x \in A \wedge \neg x \in B\}$$

$$x \in A - B \Leftrightarrow x \in A \wedge \neg x \in B$$

$$\bar{A} = \{x \in U : \neg x \in A\}$$

$$x \in \bar{A} \Leftrightarrow \neg x \in A$$

$$A \times B = \{(a, b) : a \in A \wedge b \in B\}$$

$$(a, b) \in A \times B \Leftrightarrow a \in A \wedge b \in B$$

$$\mathcal{P}(A) = \{B : B \subseteq A\}$$

$$B \in \mathcal{P}(A) \Leftrightarrow B \subseteq A$$

$$A = \emptyset \Leftrightarrow \neg(\exists x) x \in A$$

Generalized union and intersection

$$\bigcap_{i \in I} A_i = \{x : (\forall i \in I) x \in A_i\}$$

$$x \in \bigcap_{i \in I} A_i \Leftrightarrow (\forall i \in I) x \in A_i$$

$$\bigcup_{i \in I} A_i = \{x : (\exists i \in I) x \in A_i\}$$

$$x \in \bigcup_{i \in I} A_i \Leftrightarrow (\exists i \in I) x \in A_i$$

2. **Relations.** A relation on A is any subset of $A \times A$. A relation \sim on A is an **equivalence** on A if \sim is

reflexive: $(\forall a \in A) a \sim a$

symmetric: $(\forall a, b \in A) (a \sim b \Rightarrow b \sim a)$

transitive: $(\forall a, b, c \in A) (a \sim b \wedge b \sim c \Rightarrow a \sim c)$

The **equivalence class** $[a]$ for $a \in A$ is $[a] = \{b \in A : a \sim b\}$ and

$$[a] = [b] \Leftrightarrow a \sim b$$

The **quotient set** A/\sim is the set of equivalence classes $A/\sim = \{[a] : a \in A\}$.

A relation \preceq on A is a **partial order** on A if \preceq is reflexive $((\forall a \in A) a \preceq a)$, transitive $((\forall a, b, c \in A) (a \preceq b \wedge b \preceq c \Rightarrow a \preceq c))$ and

antisymmetric: $(\forall a, b, c \in A) (a \preceq b \wedge b \preceq a \Rightarrow a = b)$

A partial order \preceq on A is a **total order** if $(\forall a, b \in A) (a \preceq b \vee b \preceq a)$.

Let \preceq be a partial order on A .

- $a \in A$ is the **greatest element** if $(\forall b \in A) b \preceq a$.
 $a \in A$ is the **least element** if $(\forall b \in A) a \preceq b$.
- $a \in A$ is a **maximal element** of A if $\neg(\exists b \in A) (a \preceq b \wedge a \neq b)$ (equivalently, $(\forall b \in A) (a \preceq b \Rightarrow a = b)$).
 $a \in A$ is a **minimal element** of A if $\neg(\exists b \in A) (b \preceq a \wedge a \neq b)$ (equivalently, $(\forall b \in A) (b \preceq a \Rightarrow a = b)$).
- Let $B \subseteq A$. $a \in A$ is an **upper bound** of B if $(\forall b \in B) b \preceq a$, and $a \in A$ is a **supremum** of B if a is the least element of the set of the upper bounds of B .
 $a \in A$ is a **lower bound** of B if $(\forall b \in B) a \preceq b$, and $a \in A$ is an **infimum** of B if a is the greatest element of the set of the lower bounds of B .

3. **Functions.** A function $f : A \rightarrow B$ is a subset of $A \times B$ for which $(a, b) \in f$ is written by $f(a) = b$ and such that

- (a) $(\forall a \in A) (\exists b \in B) f(a) = b$
- (b) $(\forall a_1, a_2 \in A) (a_1 = a_2 \Rightarrow f(a_1) = f(a_2))$

A function $f : A \rightarrow B$ is

- (a) is **onto** or **surjective** if

$$(\forall b \in B) (\exists a \in A) f(a) = b$$

- (b) A function $f : A \rightarrow B$ is **one-to-one** or **injective** if

$$(\forall a_1, a_2 \in A) (f(a_1) = f(a_2) \Rightarrow a_1 = a_2)$$

(contrapositive: $(\forall a_1, a_2 \in A) (a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2))$)

- (c) A function $f : A \rightarrow B$ is **bijective** if it is one-to-one and onto.

If $f : A \rightarrow B$ and $g : B \rightarrow C$ are two functions, a **composition** $g \circ f : A \rightarrow C$ is the function given by

$$(g \circ f)(a) = g(f(a))$$

for $a \in A$.

The **identity function** on A is $\text{id}_A : A \rightarrow A$ defined by $\text{id}_A(a) = a$ for every $a \in A$. Useful identities for $f : A \rightarrow B$: $f \circ \text{id}_A = f$ and $\text{id}_B \circ f = f$.

A function $f : A \rightarrow B$ has the **inverse** f^{-1} if $f \circ f^{-1} = \text{id}_B$ and $f^{-1} \circ f = \text{id}_A$.

If $f : A \rightarrow B$ is a function, $C \subseteq A$ and $D \subseteq B$, the **image of C** is

$$f(C) = \{b \in B : (\exists c \in C) b = f(c)\}. \quad \text{So, } b \in f(C) \Leftrightarrow (\exists c \in C) b = f(c).$$

The **inverse image of D** is

$$f^{-1}(D) = \{a \in A : f(a) \in D\}. \quad \text{So, } a \in f^{-1}(D) \Leftrightarrow f(a) \in D.$$