Fundamentals of Mathematics
Lia Vas

Formulas for Exam 4

1. **Integer numbers.** Equivalence classes for $\sim$ on $\mathbb{N} \times \mathbb{N}$ given by

$$(k,l) \sim (m,n) \quad \Leftrightarrow \quad k+n = m+l$$

Recall that

$$[(k,l)] = [(m,n)] \quad \Leftrightarrow \quad (k,l) \sim (m,n)$$

Operations $+$ and $\cdot$ and the order $\leq$.

$$[(k,l)] + [(m,n)] = [(k+m, l+n)]$$

$$[(k,l)] \cdot [(m,n)] = [(km+ln, lm+kn)]$$

$$[(k,l)] \leq [(m,n)] \quad \Leftrightarrow \quad k+n \leq l+m.$$

With these operations and the order, the notation $[(m,n)]$ can be shortened to $m-n$.

Some properties: both $+$ and $\cdot$ are associative, commutative, the distributivity holds, $\leq$ is compatible with both $+$ and $\cdot$, there are no zero divisors for $\cdot$, $[(0,0)]$ is the identity element for $+$ and $[(1,0)]$ for $\cdot$, $[(n,m)]$ is the inverse of $[(m,n)]$ for $+$. Moreover,

$$[(m,n)] = [(m+k, n+k)]$$

for every k.

2. **Rational numbers.** Equivalence classes for $\sim$ on $\mathbb{Z} \times \mathbb{Z} - \{0\}$ given by

$$(k,l) \sim (m,n) \quad \Leftrightarrow \quad kn = ml$$

Recall that

$$[(k,l)] = [(m,n)] \quad \Leftrightarrow \quad (k,l) \sim (m,n)$$

Operations $+$ and $\cdot$ and the order $\leq$.

$$[(k,l)] + [(m,n)] = [(kn+ml,\ ln)]$$

$$[(k,l)] \cdot [(m,n)] = [(km, ln)]$$

$$\text{if } \ ln > 0, \text{ then } \quad [(k,l)] \leq [(m,n)] \ \Leftrightarrow \ kn \leq lm.$$

With these operations and the order, the notation $[(m,n)]$ can be shortened to $\frac{m}{n}$.

Some properties: both $+$ and $\cdot$ are associative, commutative, the distributivity holds, $\leq$ is compatible with both $+$ and $\cdot$, there are no zero divisors for $\cdot$, $[(0,1)]$ is the identity element for $+$ and $[(1,1)]$ for $\cdot$, $[(-m,n)]$ is the inverse of $[(m,n)]$ for $+$ and $[(n,m)]$ is the inverse for $\cdot$ when $m \neq 0$. Moreover,

$$[(m,n)] = [(mk, nk)]$$

for every nonzero k.

3. **Real numbers.** A sequence a_n is Cauchy if

$$(\forall\varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n, m > n_0)\ |a_n - a_m| < \varepsilon$$

$R =$ set of all Cauchy sequences (a_n) of rational numbers. $\mathbb{R}$ is the quotient set $R/\sim$ for

$$(a_n) \sim (b_n) \quad \Leftrightarrow \quad (a_n - b_n) \text{ has limit zero.}$$

Operations $+$ and $\cdot$ and the order $\leq$.

$$[(a_n)] + [(b_n)] = [(a_n + b_n)]$$

$$[(a_n)] \cdot [(b_n)] = [(a_n \cdot b_n)].$$

$$[(a_n)] \leq [(b_n)] \ \Leftrightarrow \ \text{the limit of } b_n - a_n \text{ is not negative.}$$

With these operations and the order, the notation $[(a_n)]$ can be shortened to $\lim_{n\to\infty} a_n$.

Some properties: both $+$ and $\cdot$ are associative, commutative, the distributivity holds, $\leq$ is compatible with both $+$ and $\cdot$, there are no zero divisors for $\cdot$, $[(0)]$ is the identity element for $+$ and $[(1)]$ for $\cdot$, $[(-a_n)]$ is the inverse of $[(a_n)]$ for $+$ and $[(\frac{1}{a_n})]$ is the inverse for $\cdot$ when $\lim_{n\to\infty} a_n \neq 0$.

Geometric series and their sums. For $-1 < r < 1$,

$$\sum_{n=k}^{\infty} ar^n = \frac{ar^k}{1-r}$$

4. **Cardinality.** $2^{\aleph_0} > \aleph_0$ and

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| = \aleph_0$$

$$|\mathbb{C}| = |\mathbb{R}| = \mathfrak{c} = 2^{\aleph_0}$$

The following intervals where $a, b \in \mathbb{R}$ and $a < b$ also have cardinality $|\mathbb{R}|$

$$(a,b), \quad [a,b), \quad (a,b], \quad [a,b], \quad (a,\infty), \quad [a,\infty), \quad (-\infty,a), \quad (-\infty,a]$$

5. **Complex numbers.** i is a solution of $z^2 = -1$ and $\mathbb{C}$ is the set of elements $x + iy$ where $x, y \in \mathbb{R}$. $x - iy$ is the complex conjugate of the number $x + iy$.

Polar coordinate representation

$$x = r\cos\theta, \ \ y = r\sin\theta, \quad r = |x + iy| = \sqrt{x^2 + y^2}, \ \ \tan\theta = \frac{y}{x}$$

Polar representation of complex numbers and Euler's formula.

$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = r\ e^{\theta i}.$$

Finding n-solutions of the equation $z^n = a$. If $a = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}$, then the solutions are

$$\sqrt[n]{r}\ \ e^{\frac{(\theta+2k\pi)i}{n}} \qquad \text{for } k = 0, 1, \ldots n-1.$$