

## Formulas for Exam 4

1. **Integer numbers.** Equivalence classes for  $\sim$  on  $\mathbb{N} \times \mathbb{N}$  given by

$$(k, l) \sim (m, n) \Leftrightarrow k + n = m + l$$

Recall that

$$[(k, l)] = [(m, n)] \Leftrightarrow (k, l) \sim (m, n)$$

Operations  $+$  and  $\cdot$  and the order  $\leq$ .

$$[(k, l)] + [(m, n)] = [(k + m, l + n)]$$

$$[(k, l)] \cdot [(m, n)] = [(km + ln, lm + kn)]$$

$$[(k, l)] \leq [(m, n)] \Leftrightarrow k + n \leq l + m.$$

With these operations and the order, the notation  $[(m, n)]$  can be shortened to  $m - n$ .

Some properties: both  $+$  and  $\cdot$  are associative, commutative, the distributivity holds,  $\leq$  is compatible with both  $+$  and  $\cdot$ , there are no zero divisors for  $\cdot$ ,  $[(0, 0)]$  is the identity element for  $+$  and  $[(1, 0)]$  for  $\cdot$ ,  $[(n, m)]$  is the inverse of  $[(m, n)]$  for  $+$ . Moreover,

$$[(m, n)] = [(m + k, n + k)]$$

for every  $k$ .

2. **Rational numbers.** Equivalence classes for  $\sim$  on  $\mathbb{Z} \times \mathbb{Z} - \{0\}$  given by

$$(k, l) \sim (m, n) \Leftrightarrow kn = ml$$

Recall that

$$[(k, l)] = [(m, n)] \Leftrightarrow (k, l) \sim (m, n)$$

Operations  $+$  and  $\cdot$  and the order  $\leq$ .

$$[(k, l)] + [(m, n)] = [(kn + ml, ln)]$$

$$[(k, l)] \cdot [(m, n)] = [(km, ln)]$$

$$\text{if } ln > 0, \text{ then } [(k, l)] \leq [(m, n)] \Leftrightarrow kn \leq lm.$$

With these operations and the order, the notation  $[(m, n)]$  can be shortened to  $\frac{m}{n}$ .

Some properties: both  $+$  and  $\cdot$  are associative, commutative, the distributivity holds,  $\leq$  is compatible with both  $+$  and  $\cdot$ , there are no zero divisors for  $\cdot$ ,  $[(0, 1)]$  is the identity element for  $+$  and  $[(1, 1)]$  for  $\cdot$ ,  $[(-m, n)]$  is the inverse of  $[(m, n)]$  for  $+$  and  $[(n, m)]$  is the inverse for  $\cdot$  when  $m \neq 0$ . Moreover,

$$[(m, n)] = [(mk, nk)]$$

for every nonzero  $k$ .

3. **Real numbers.** A sequence  $a_n$  is Cauchy if

$$(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n, m > n_0) |a_n - a_m| < \varepsilon$$

$R$  = set of all Cauchy sequences  $(a_n)$  of rational numbers.  $\mathbb{R}$  is the quotient set  $R/\sim$  for

$$(a_n) \sim (b_n) \Leftrightarrow (a_n - b_n) \text{ has limit zero.}$$

Operations  $+$  and  $\cdot$  and the order  $\leq$ .

$$[(a_n)] + [(b_n)] = [(a_n + b_n)]$$

$$[(a_n)] \cdot [(b_n)] = [(a_n \cdot b_n)].$$

$$[(a_n)] \leq [(b_n)] \Leftrightarrow \text{the limit of } b_n - a_n \text{ is not negative.}$$

With these operations and the order, the notation  $[(a_n)]$  can be shortened to  $\lim_{n \rightarrow \infty} a_n$ .

Some properties: both  $+$  and  $\cdot$  are associative, commutative, the distributivity holds,  $\leq$  is compatible with both  $+$  and  $\cdot$ , there are no zero divisors for  $\cdot$ ,  $[(0)]$  is the identity element for  $+$  and  $[(1)]$  for  $\cdot$ ,  $[(-a_n)]$  is the inverse of  $[(a_n)]$  for  $+$  and  $[(\frac{1}{a_n})]$  is the inverse for  $\cdot$  when  $\lim_{n \rightarrow \infty} a_n \neq 0$ .

Geometric series and their sums. For  $-1 < r < 1$ ,

$$\sum_{n=k}^{\infty} ar^n = \frac{ar^k}{1-r}$$

4. **Cardinality.**  $2^{\aleph_0} > \aleph_0$  and

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| = \aleph_0$$

$$|\mathbb{C}| = |\mathbb{R}| = \mathfrak{c} = 2^{\aleph_0}$$

The following intervals where  $a, b \in \mathbb{R}$  and  $a < b$  also have cardinality  $|\mathbb{R}|$

$$(a, b), [a, b), (a, b], [a, b], (a, \infty), [a, \infty), (-\infty, a), (-\infty, a]$$

5. **Complex numbers.**  $i$  is a solution of  $z^2 = -1$  and  $\mathbb{C}$  is the set of elements  $x + iy$  where  $x, y \in \mathbb{R}$ .  $x - iy$  is the complex conjugate of the number  $x + iy$ .

Polar coordinate representation

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r = |x + iy| = \sqrt{x^2 + y^2}, \quad \tan \theta = \frac{y}{x}$$

Polar representation of complex numbers and Euler's formula.

$$z = x + iy = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta) = r e^{i\theta}.$$

Finding  $n$ -solutions of the equation  $z^n = a$ . If  $a = r(\cos(\theta) + i \sin(\theta)) = r e^{i\theta}$ , then the solutions are

$$\sqrt[n]{r} e^{\frac{(\theta+2k\pi)i}{n}} \quad \text{for } k = 0, 1, \dots, n-1.$$

Complex functions.

$$e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y) \quad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) \quad \cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$