Fundamentals of Mathematics
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Review for Exam 1

1. Let p stand for "the number larger than 2 is prime" and q stand for "the number larger than 2 is odd". Recall that every prime number larger than 2 is odd. The converse is false (for example, 9 is an odd number which is not prime).

   Represent the next statements as sentences of propositional logic and determine their validity.

   (a) A number larger than 2 is prime if it is odd.
   (b) A number larger than 2 is prime only if it is odd.
   (c) For a number larger than 2 to be prime, it is necessary that it is odd.
   (d) For a number larger than 2 to be prime, it is sufficient that it is odd.

2. Show that the sentences below are tautologies without using the truth tables.

   (a) $\neg(p \vee q) \Rightarrow \neg p \wedge \neg q$
   (b) $\neg p \Rightarrow (\neg q \Rightarrow \neg(\neg p \Rightarrow q))$
   (c) $((p \Rightarrow q) \Rightarrow q) \Leftrightarrow p \vee q$

3. Show that the sentences below are not tautologies without using the truth tables.

   (a) $(p \Rightarrow q) \Rightarrow p \wedge \neg q$ (b) $\neg p \Rightarrow (\neg q \Rightarrow p)$ (c) $p \vee q \Leftrightarrow (p \Rightarrow r)$

4. Find a sentence which is logically equivalent to the given one and which contains only the specified connectives.

   (a) $p \Rightarrow (q \Rightarrow r)$ using $\neg$ and $\vee$. (b) $\neg p \Leftrightarrow q \vee r$ using $\neg$ and $\vee$.

5. Check whether the following sets of sentences are consistent or inconsistent.

   (a) The set containing p and $p \wedge \neg q$.
   (b) The set containing p, q and $p \wedge \neg q$.
   (c) The set containing $\neg(p \Rightarrow q)$ and $q \Rightarrow p$.

6. Consider the set of positive integers to be the domain of interpretation in which

   $A(x, y)$ stands for "y is divisible by x", $B(x)$ stands for "x is even", and
   $C(x)$ stands for "x is prime". (Constants $1, 2, 3, \ldots$ have the obvious interpretations.)

   Represent the following statements as formulas of predicate logic and determine whether the sentences are true in this interpretation.

   (a) Every positive integer divisible by 6 is divisible by 3.

(b) There is a positive integer divisible by 3 and not divisible by 6.

(c) Every positive integer divisible by 2 is even.

(d) Every positive integer which is prime is odd.

(e) Every prime positive integer divisible by 3 is such that 3 is divisible by it.

7. Assess the validity of the given formulas in the given interpretations.

   (a) $(\exists x)(\forall y)A(x, y)$, the domain is the set of positive integers and $A(x, y)$ is interpreted as x divides y.

   (b) $(\exists x)(\forall y)\neg A(x, y)$, the domain is the set of positive integers and $A(x, y)$ is interpreted as x divides y.

   (c) $(\exists x)A(x) \Rightarrow (\forall x)A(x)$, the domain is the set of three blue and two red balls and $A(x)$ is interpreted as "x is red".

   (d) $(\exists x)(\exists y)A(x, y) \Rightarrow (\exists x)A(x, x)$ where the domain is the set of positive integers and $A(x, y)$ is interpreted as $x < y$.

   (e) $(\exists x)(\forall y)A(x, y)$, the domain is the interval $[0, 1]$ and $A(x, y)$ is interpreted as $x \leq y$.

   (f) $(\exists x)(\forall y)A(x, y)$, the domain is the interval $(0, 1)$ and $A(x, y)$ is interpreted as $x \leq y$.

8. Show that the sentences below are tautologies. For practice, try to do (a) to (c) "directly" and do (d) and (e) using existing tautologies.

   (a) $(\forall x)(\forall y)P(x, y) \Rightarrow (\forall x)P(x, x)$

   (b) $(\forall x)(P(x) \Rightarrow Q(x)) \Rightarrow ((\forall x)P(x) \Rightarrow (\forall x)Q(x))$

   (c) $((\exists x)P(x) \Rightarrow (\forall x)Q(x)) \Rightarrow (\forall x)(P(x) \Rightarrow Q(x))$

   (d) $(\exists x)(\forall y)\neg P(x, y) \Leftrightarrow \neg(\forall y)(\exists x)P(y, x)$

   (e) $((\exists x)(P(x) \Rightarrow Q(x)) \Leftrightarrow ((\forall x)P(x) \Rightarrow (\exists x)Q(x))$

9. Show that the sentences below are not tautologies.

   (a) $(\forall x)P(x, x) \Rightarrow (\forall x)(\forall y)P(x, y)$

   (b) $(\exists x)P(x) \Rightarrow P(x)$

   (c) $((\forall x)P(x) \Rightarrow (\forall x)Q(x)) \Rightarrow (\forall x)(P(x) \Rightarrow Q(x))$

10. Determine whether the sets consisting of each of the following groups of sentences are satisfiable.

   (a) $(\exists x)P(x), (\exists x)Q(x), (\forall x)(P(x) \Rightarrow Q(x))$

   (b) $(\exists x)P(x, x)$, $(\forall x)(\forall y)(\forall z)(P(x, y) \wedge P(y, z) \Rightarrow P(x, z))$

Solutions

1. The statement $p \Rightarrow q$ is true and the statement $q \Rightarrow p$ may not be true. Part (a) states that $q \Rightarrow p$ so it is not true. Part (b) states that $p \Rightarrow q$ so it is true. Part (c) states that $p \Rightarrow q$ so it is true. Part (d) states that $q \Rightarrow p$, so it is false.

2. (a) Assume that $\neg(p \vee q)$ is true. Thus, $p \vee q$ is false, so both p and q are false. This makes both $\neg p$ and $\neg q$ true, so the conclusion $\neg p \wedge \neg q$ is true.

   (b) There are several ways to show that $\neg p \Rightarrow (\neg q \Rightarrow \neg(\neg p \Rightarrow q))$ is a tautology.

   One way to do this. Assume that $\neg p$ is true and show that the conclusion of the statement is true. Since the conclusion is also an implication, assume also that the assumption $\neg q$ is true. So, $\neg p$ and $\neg q$ are true, so p and q are false. This makes the conclusion $\neg(\neg p \Rightarrow q)$ be $\neg(\neg\bot \Rightarrow \bot) = \neg(\top \Rightarrow \bot) = \neg\bot = \top$. Hence, the conclusion is true.

   Another argument. Using Exportation, the given sentence is equivalent to $\neg p \wedge \neg q \Rightarrow \neg(\neg p \Rightarrow q))$. Assuming that $\neg p \wedge \neg q$ is true, we have that both p and q are false. This makes the conclusion $\neg(\neg p \Rightarrow q)$ be $\neg(\top \Rightarrow \bot) = \neg\bot = \top$.

   The third option. Using Material Implication for all three implications, the given formula is equivalent to $p \vee (q \vee \neg(p \vee q)$. Using associativity, this is equivalent to $(p \vee q) \vee \neg(p \vee q)$. Letting $P = p \vee q$, this formula has the format $P \vee \neg P$ and it is true by the Law of Excluded Middle.

   (c) There are several ways to show that $((p \Rightarrow q) \Rightarrow q) \Leftrightarrow p \vee q$ is a tautology. Maybe the shortest way is to use Material Implication to express the implications in $(p \Rightarrow q) \Rightarrow q$ in terms of $\neq$ and $\vee$.

$$
\begin{array}{rcll}
(p \Rightarrow q) \Rightarrow q & \Leftrightarrow & \neg(\neg p \vee q) \vee q & \text{(by Material Implications for both } \Rightarrow\text{)} \\
& \Leftrightarrow & (\neg\neg p \wedge \neg q) \vee q & \text{(by De Morgan's law)} \\
& \Leftrightarrow & (p \wedge \neg q) \vee q & \text{(by Double Negation)} \\
& \Leftrightarrow & (p \vee q) \wedge (\neg q \vee q) & \text{(by the Distributivity Law)} \\
& \Leftrightarrow & (p \vee q) \wedge \top & \text{(by the Law of Excluded Middle)} \\
& \Leftrightarrow & p \vee q & \text{(by the truth table for } \wedge\text{)}
\end{array}
$$

3. (a) Look for the truth values which make $p \Rightarrow q$ true and $p \wedge \neg q$ false. As $p \wedge \neg q$ is false, p is false and q is true. As this indeed makes $p \Rightarrow q$ true, the sentence is false for these truth value assignments.

   (b) Look for the truth values which make $\neg p$ true and $\neg q \Rightarrow p$ false. Thus, $\neg q$ should be true and p false (so $\neg p$ is indeed true). Thus, if both p and q are false, the sentence is false.

   (c) Look for the truth values which make one side of the equivalence true and the other false. For example, if both p and q are false, $p \vee q$ is false and $p \Rightarrow r$ is true (regardless of the value of r). **Another option** would be to consider p and q to be true and r to be false. In this case, $p \vee q$ is true and $p \Rightarrow r$ is false.

4. (a)

$$
\begin{array}{rcll}
p \Rightarrow (q \Rightarrow r) & \Leftrightarrow & \neg p \vee (q \Rightarrow r) & \text{(by Material Implication)} \\
& \Leftrightarrow & \neg p \vee (\neg q \vee r) & \text{(by Material Implication)}
\end{array}
$$

(b)

$$
\begin{aligned}
\neg p \Leftrightarrow q \vee r \quad &\Leftrightarrow \quad (\neg p \Rightarrow q \vee r) \wedge (q \vee r \Rightarrow \neg p) && \text{(by Biconditional Law)} \\
&\Leftrightarrow \quad (\neg\neg p \vee (q \vee r)) \wedge (\neg(q \vee r) \vee \neg p) && \text{(by Material Implication for both } \Rightarrow\text{)} \\
&\Leftrightarrow \quad (p \vee (q \vee r)) \wedge (\neg(q \vee r) \vee \neg p) && \text{(by Double Negation)} \\
&\Leftrightarrow \quad \neg(\neg(p \vee (q \vee r)) \vee \neg(\neg(q \vee r) \vee \neg p)) && \text{(by De Morgan's law)}
\end{aligned}
$$

5. (a) The set is consistent since the values $\top$ for p and $\bot$ for q make both sentences true.
   (b) The set is inconsistent since if p and q are true, then $p \wedge \neg q$ is false so no truth value of p and q can make both sentences *simultaneously* true.
   (c) The set is consistent since if p is true and q is false, then both sentences $\neg(p \Rightarrow q)$ and $q \Rightarrow p$ are true.

6. (a) $(\forall x)(A(6, x) \Rightarrow A(3, x)$. This sentence is true in the given interpretation.
   (b) $(\exists x)(A(3, x) \wedge \neg A(6, x))$. This sentence is also true in the given interpretation because taking 3 for x, for example, we have that $A(3, 3)$ is true and $A(6, 3)$ is false.
   (c) $(\forall x)(A(2, x) \Rightarrow B(x))$. This sentence is true in the given interpretation.
   (d) $(\forall x)(C(x) \Rightarrow \neg B(x))$. This sentence is not true in the given interpretation because 2 is a positive integer which is prime but it is not odd.
   (e) $(\forall x)(C(x) \Rightarrow (A(3, x) \Rightarrow A(x, 3)))$ or $(\forall x : C(x))(A(3, x) \Rightarrow A(x, 3))$. This sentence is true in the given interpretation because if integer is prime and divisible by 3, then it is equal to 3 (so $x = 3$). As $A(3, 3)$ holds, the conclusion of the implication holds.

7. (a) $(\exists x)(\forall y)A(x, y)$ is true because $x = 1$ divides any positive integer y.
   (b) $(\exists x)(\forall y)\neg A(x, y)$ is false because it is not the case that there is a positive integer which does not divide any positive integer: any positive integer divides itself.
   (c) $(\exists x)A(x) \Rightarrow (\forall x)A(x)$, is false because the premise is true (there is a red ball) but the conclusion is not (it is not the case that all balls are red).
   (d) $(\exists x)(\exists y)A(x, y) \Rightarrow (\exists x)A(x, x)$ is false because the premise is true (there are positive integers, 2 and 3 for example, such that $2 < 3$) and the conclusion is false: $x < x$ holds for no positive integer x.
   (e) $(\exists x)(\forall y)A(x, y)$ is true because $x = 0$ is less than or equal to any other number in $[0, 1]$.
   (f) $(\exists x)(\forall y)A(x, y)$ is false because the the interval $(0, 1)$ does not contain any number x with the property that x is smaller than or equal to any y in $(0, 1)$ (zero is not in $(0, 1)$ so $(0, 1)$ does not have the minimal element.

8. (a) Assume that the premise $(\forall x)(\forall y)P(x, y)$ holds. So, $P(a, b)$ holds for every element a and every element b of the domain. In particular, by taking $b = a$, we have that $P(a, a)$ holds for every a in the domain. Thus the conclusion $(\forall x)P(x, x)$ holds.
   (b) Assume that the premise $(\forall x)(P(x) \Rightarrow Q(x))$ holds and that the premise $(\forall x)P(x)$ of the conclusion we need to show also holds. Thus, both $P(a)$ and $P(a) \Rightarrow Q(a)$ hold for every element a of the domain, and, since the assumption $P(a)$ of $P(a) \Rightarrow Q(a)$ is true for every a, the conclusion $Q(a)$ holds for every a in the domain. Thus, $(\forall x)Q(x)$ holds.

(c) Assume that the assumption $((\exists x)P(x) \Rightarrow (\forall x)Q(x))$ is true. To show $(\forall x)(P(x) \Rightarrow Q(x))$, let a be an arbitrary element of the domain and let the premise $P(a)$ be true. This shows that the premise $(\exists x)P(x)$ of the implication $((\exists x)P(x) \Rightarrow (\forall x)Q(x))$ is true, so the conclusion $(\forall x)Q(x))$ is also true. Thus, $Q(a)$ holds. This shows that $(\forall x)(P(x) \Rightarrow Q(x))$ holds.

(d)

$$\begin{aligned}(\exists x)(\forall y)\neg P(x,y) &\Leftrightarrow \neg(\forall x)\neg(\exists y)P(x,y) && \text{(by moving } \neg \text{ through } \forall)\\ &\Leftrightarrow \neg(\forall x)(\exists y)P(x,y) && \text{(by moving } \neg \text{ through } \exists)\\ &\Leftrightarrow \neg(\forall y)(\exists x)P(y,x) && \text{(by renaming, } x \text{ to } y\text{, and } y \text{ to } x)\end{aligned}$$

(e) This tautology is on moving $\exists$ through $\Rightarrow$. Since we have analogous tautologies available for $\exists$ and $\vee$ and $\Rightarrow$ can be expressed in terms of $\vee$ by using Material Implication, start by using this rule.

$$\begin{aligned}(\exists x)(P(x) \Rightarrow Q(x)) &\Leftrightarrow (\exists x)(\neg P(x) \vee Q(x)) && \text{(by Material Implication)}\\ &\Leftrightarrow (\exists x)\neg P(x) \vee (\exists x)Q(x) && \text{(by moving } \exists \text{ through } \vee)\\ &\Leftrightarrow \neg(\forall x)P(x) \vee (\exists x)Q(x) && \text{(by moving } \neg \text{ through } \exists)\\ &\Leftrightarrow (\forall x)P(x) \Rightarrow (\exists x)Q(x) && \text{(by Material Implication)}\end{aligned}$$

9. (a) You can use a number set with a relation like $\leq$ or divisibility which fails to hold for every two elements. For example, with $\leq$ on the set of integers, $x \leq x$ holds for every integer x but $20 \leq 3$ fails, so the conclusion $(\forall x)(\forall y)P(x,y)$ fails.

You can also use some non-mathematical interpretation. For example, consider the set of red and blue balls and let $P(x,y)$ stand for "x has the same color as y". The premise $(\forall x)P(x,x)$ holds because every ball has the same color as itself. The conclusion $(\forall x)(\forall y)P(x,y)$ does not hold because not all the balls are of the same color.

(b) Any domain and its property which holds for some but not all of its elements can be used here. For example, the set of red and blue balls and $P(x)$ being "x is red". The premise $(\exists x)P(x)$ holds because there are some red balls. The conclusion $P(x)$ fails when x is taken to be one of the blue balls.

(c) We need an interpretation in which $(\forall x)P(x) \Rightarrow (\forall x)Q(x)$ is true and $(\forall x)(P(x) \Rightarrow Q(x))$ is not. So, let us choose a domain and two properties of its elements such that if $P(a)$ holds but not $Q(a)$ for some a. For example, let D be the set of blue and red balls, $P(x)$ be "x is red" and $Q(x)$ be "x is blue". As $(\forall x)P(x)$ fails, the implication $(\forall x)P(x) \Rightarrow (\forall x)Q(x)$ is true. The conclusion $(\forall x)(P(x) \Rightarrow Q(x))$ is false, because if a ball is red, then it is not the case that it is blue.

10. (a) Any set some of which elements have properties P and Q such that P implies Q can make this set satisfiable. For example, set of positive integers with $P(x)$ being "x is divisible by 4" and $Q(x)$ being "x is divisible by 2" makes the given sentences satisfiable. Or, for example, a set of red balls and $P(x)$ being "x is red and x is a ball" and $Q(x)$ being "x is red".

(b) A number set with a relation like $=$, $\leq$ or $\geq$ makes the given sentences satisfiable. A set of positive integers with relation $|$ (divisibility) also makes the given sentences satisfiable.