

Review for Exam 2

1. Let $A = \{1\}$ and $B = \{2, 3\}$. Determine the following sets.

$$\mathcal{P}(A), \mathcal{P}(B), \mathcal{P}(\mathcal{P}(A)), A \times B, \mathcal{P}(A \times B), \mathcal{P}(A) \times B, A \times \mathcal{P}(B), \mathcal{P}(A) \times \mathcal{P}(B).$$

2. Show the following identities or statements in which A, B, C , and D stand for arbitrary sets. In part (g), I is an arbitrary set and A_i are sets for $i \in I$.

(a) $A \cap B \subseteq A$

(b) $A \subseteq A \cup B$

(c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(d) $(A - B) - C = A - (B \cup C)$

(e) $A - (B \cup C) = (A - B) \cap (A - C)$

(f) $A \subseteq B \Leftrightarrow \overline{B} \subseteq \overline{A}$

(g) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

(h) $\overline{\bigcap_{i \in I} A_i} = \bigcup_{i \in I} \overline{A_i}$

(i) $A \subseteq B \wedge C \subseteq D \Rightarrow A \times C \subseteq B \times D$

3. Determine $\bigcap_{n=1}^{\infty} A_n$ and $\bigcup_{n=1}^{\infty} A_n$ for given sets A_n where $n = 1, 2, \dots$

(a) $A_n = \{1, 2, \dots, n\}$,

(b) $A_n = \{n, n + 1, \dots\}$

(c) $A_n = [0, n]$

4. For a given set A and a relation \sim on it, check whether the given equation \sim is an equivalence relation. If it is, determine the quotient set.

(a) $A = \{1, 2, 3\}$ and \sim consists of the ordered pairs $(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)$.

(b) $A = \{1, 2, 3\}$ and \sim consists of the ordered pairs $(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (1, 3), (3, 1)$.

(c) A is the set of real numbers and \sim is given by $a \sim b \Leftrightarrow a^2 = b^2$.

(d) A is the set of integers and \equiv is given by $m \equiv n \Leftrightarrow m - n$ is divisible by 5. Recall that this condition is equivalent with $(\exists k \in \mathbb{Z}) m - n = 5k$.

5. For a given set A and a relation \preceq on it, determine whether \preceq is a partial order. If it is, represent it by a Hasse diagram and determine whether it is a total order. Then, determine the greatest, the smallest elements, minimal and maximal elements, if any of those exist.

(a) $A = \{1, 2, 3\}$ and \preceq consists of the pairs $(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (3, 2), (1, 3)$.

(b) $A = \mathcal{P}(\{1, 2, 3\})$ and \preceq consists of the pairs $(\{1\}, \{1\}), (\{2\}, \{2\}), (\{3\}, \{3\})$.

(c) $A = \{\{1\}, \{2\}, \{3\}\}$ and \preceq consists of the pairs $(\{1\}, \{1\}), (\{2\}, \{2\}), (\{3\}, \{3\})$.

(d) $A = \{\{1\}, \{2\}, \{3\}\}$ and \preceq consists of the pairs $(\{1\}, \{1\}), (\{2\}, \{2\}), (\{1\}, \{2\}), (\{3\}, \{3\})$.

6. If R is a relation on a set A which is reflexive and transitive, show that the relation \sim given by

$$a \sim b \Leftrightarrow aRb \text{ and } bRa$$

is an equivalence relation.

7. Consider the set of integers \mathbb{Z} with the relation \equiv given by $m \equiv n \Leftrightarrow 2$ divides $m - n$ and the set of planes P with the relation \parallel given by $\alpha \parallel \beta \Leftrightarrow \alpha$ and β are parallel. Define a relation \sim on $\mathbb{Z} \times P$ by

$$(m, \alpha) \sim (n, \beta) \Leftrightarrow m \equiv n \wedge \alpha \parallel \beta.$$

Using that \equiv and \parallel are equivalence relations, show that \sim is an equivalence relation.

8. Let A and B be any sets and let \preceq_A be a partial order on A and \preceq_B is a partial order on B . Let us define \preceq on $A \times B$ by

$$(a, b) \preceq (c, d) \Leftrightarrow a \preceq_A c \wedge b \preceq_B d.$$

Show that \preceq is a partial order on $A \times B$.

9. For the given poset A of the set of real numbers \mathbb{R} , consider both A and \mathbb{R} to be partially ordered by the relation \leq . Determine the greatest, the smallest elements, minimal and maximal elements, and suprema and infima of A , if any of those exist.

(a) $A = [0, 1)$

(b) $A = (0, 1) \cup (1, 2)$

(c) $A = \bigcup_{n=1}^{\infty} [0, n)$

10. If $f : A \rightarrow B$, $g : B \rightarrow C$, and $h : C \rightarrow D$ are functions, show the following properties.

(a) Associativity holds for the composite: $(h \circ g) \circ f = h \circ (g \circ f)$.

(b) The identity function is a neutral element for the composite: $f \circ \text{id}_A = f$ and $\text{id}_B \circ f = f$.

(c) If f and g are injections, then $g \circ f$ is an injection.

(d) If f and g are surjections, then $g \circ f$ is a surjection.

(e) If $g \circ f$ is an injection, then f is an injection.

(f) If $g \circ f$ is a surjection, then g is a surjection.

(g) If f is onto, show that $g_1 \circ f = g_2 \circ f$ implies that $g_1 = g_2$ for every $C \neq \emptyset$ and every functions $g_1, g_2 : B \rightarrow C$.

11. Show the following properties of a function $f : A \rightarrow B$, $C, C_1, C_2 \subseteq A$, and $D, D_1, D_2 \subseteq B$.

(a) $C \subseteq f^{-1}(f(C))$

(b) $f(f^{-1}(D)) \subseteq D$

(c) $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$.

(d) $f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$.

(e) $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$

(f) $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$. Show that the converse holds if f is injective.

Solutions

1. If $A = \{1\}$, and $B = \{2, 3\}$, then

$$\mathcal{P}(A) = \{\emptyset, \{1\}\}, \quad \mathcal{P}(B) = \{\emptyset, \{2\}, \{3\}, \{2, 3\}\}, \quad \mathcal{P}(\mathcal{P}(A)) = \{\emptyset, \{\emptyset\}, \{\{1\}\}, \{\emptyset, \{1\}\}\}.$$

$$A \times B = \{(1, 2), (1, 3)\}, \quad \mathcal{P}(A \times B) = \{\emptyset, \{(1, 2)\}, \{(1, 3)\}, \{(1, 2), (1, 3)\}\},$$

$$\mathcal{P}(A) \times B = \{(\emptyset, 2), (\emptyset, 3), (\{1\}, 2), (\{1\}, 3)\}, \quad A \times \mathcal{P}(B) = \{(1, \emptyset), (1, \{2\}), (1, \{3\}), (1, \{2, 3\})\}.$$

$$\mathcal{P}(A) \times \mathcal{P}(B) = \{(\emptyset, \emptyset), (\emptyset, \{2\}), (\emptyset, \{3\}), (\emptyset, \{2, 3\}), (\{1\}, \emptyset), (\{1\}, \{2\}), (\{1\}, \{3\}), (\{1\}, \{2, 3\})\}.$$

2. (a) Show that $x \in A \cap B$ implies $x \in A$.

$$\begin{aligned} x \in A \cap B &\Leftrightarrow x \in A \wedge x \in B && \text{(by the definition of } \cap) \\ &\Rightarrow x \in A && \text{(since } p \wedge q \Rightarrow p \text{ is a tautology)} \end{aligned}$$

(b) Show that $x \in A$ implies $x \in A \cup B$.

$$\begin{aligned} x \in A &\Rightarrow x \in A \vee x \in B && \text{(since } p \Rightarrow p \vee q \text{ is a tautology)} \\ &\Leftrightarrow x \in A \cup B && \text{(by the definition of } \cup) \end{aligned}$$

(c) Show the equivalence $x \in A \cap (B \cup C) \Leftrightarrow x \in (A \cap B) \cup (A \cap C)$.

$$\begin{aligned} x \in A \cap (B \cup C) &\Leftrightarrow x \in A \wedge x \in B \cup C && \text{(by the definition of } \cap) \\ &\Leftrightarrow x \in A \wedge (x \in B \vee x \in C) && \text{(by the definition of } \cup) \\ &\Leftrightarrow (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C) && \text{(by distributivity)} \\ &\Leftrightarrow x \in A \cap B \vee x \in A \cap C && \text{(by the definition of } \cap) \\ &\Leftrightarrow x \in (A \cap B) \cup (A \cap C) && \text{(by the definition of } \cup) \end{aligned}$$

(d) Show the equivalence $x \in (A - B) - C \Leftrightarrow x \in A - (B \cup C)$.

$$\begin{aligned} x \in (A - B) - C &\Leftrightarrow x \in A - B \wedge \neg x \in C && \text{(by the definition of } -) \\ &\Leftrightarrow x \in A \wedge \neg x \in B \wedge \neg x \in C && \text{(by the definition of } \cap) \\ &\Leftrightarrow x \in A \wedge \neg(x \in B \vee x \in C) && \text{(by De Morgan's law)} \\ &\Leftrightarrow x \in A \wedge \neg(x \in B \cup C) && \text{(by the definition of } \cup) \\ &\Leftrightarrow x \in A - (B \cup C) && \text{(by the definition of } -) \end{aligned}$$

(e) Show the equivalence $x \in A - (B \cup C) \Leftrightarrow x \in (A - B) \cap (A - C)$.

$$\begin{aligned} x \in A - (B \cup C) &\Leftrightarrow x \in A \wedge \neg x \in B \cup C && \text{(by the definition of } -) \\ &\Leftrightarrow x \in A \wedge \neg(x \in B \vee x \in C) && \text{(by the definition of } \cup) \\ &\Leftrightarrow x \in A \wedge \neg x \in B \wedge \neg x \in C && \text{(by De Morgan's law)} \\ &\Leftrightarrow x \in A \wedge x \in A \wedge \neg x \in B \wedge \neg x \in C && \text{(by idempotence } p \wedge p \Leftrightarrow p) \\ &\Leftrightarrow x \in A \wedge \neg x \in B \wedge x \in A \wedge \neg x \in C && \text{(by commutativity of } \wedge) \\ &\Leftrightarrow x \in A - B \wedge x \in A - C && \text{(by the definition of } -) \\ &\Leftrightarrow x \in (A - B) \cap (A - C) && \text{(by the definition of } \cap) \end{aligned}$$

(f)

$$\begin{aligned} A \subseteq B &\Leftrightarrow (\forall x)(x \in A \Rightarrow x \in B) && \text{(by the definition of } \subseteq) \\ &\Leftrightarrow (\forall x)(\neg x \in B \Rightarrow \neg x \in A) && \text{(by Contrapositive law)} \\ &\Leftrightarrow (\forall x)x \in \overline{B} \Rightarrow x \in \overline{A} && \text{(by the definition of the complement)} \\ &\Leftrightarrow \overline{B} \subseteq \overline{A} && \text{(by the definition of } \subseteq) \end{aligned}$$

(g) Show the equivalence $x \in \overline{A \cap B} \Leftrightarrow x \in \overline{A} \cup \overline{B}$.

$$\begin{aligned}
 x \in \overline{A \cap B} &\Leftrightarrow \neg x \in A \cap B && \text{(by the definition of the complement)} \\
 &\Leftrightarrow \neg (x \in A \wedge x \in B) && \text{(by the definition of } \cap \text{)} \\
 &\Leftrightarrow \neg x \in A \vee \neg x \in B && \text{(by De Morgan's law)} \\
 &\Leftrightarrow x \in \overline{A} \vee x \in \overline{B} && \text{(by the definition of the complement)} \\
 &\Leftrightarrow x \in \overline{A} \cup \overline{B} && \text{(by the definition of } \cup \text{)}
 \end{aligned}$$

(h) Show the equivalence $x \in \overline{\bigcap_{i \in I} A_i} \Leftrightarrow x \in \bigcup_{i \in I} \overline{A_i}$.

$$\begin{aligned}
 x \in \overline{\bigcap_{i \in I} A_i} &\Leftrightarrow \neg x \in \bigcap_{i \in I} A_i && \text{(by the definition of the complement)} \\
 &\Leftrightarrow \neg (\forall i \in I) x \in A_i && \text{(by the definition of } \bigcap \text{)} \\
 &\Leftrightarrow (\exists i \in I) \neg x \in A_i && \text{(Distributing } \neg \text{ through } \forall \text{)} \\
 &\Leftrightarrow (\exists i \in I) x \in \overline{A_i} && \text{(by the definition of the complement)} \\
 &\Leftrightarrow x \in \bigcup_{i \in I} \overline{A_i} && \text{(by the definition of } \bigcup \text{)}
 \end{aligned}$$

(i) Assume that $A \subseteq B$ and $C \subseteq D$ and show that $A \times C \subseteq B \times D$. So, we need to show the implication $(x, y) \in A \times C \Rightarrow (x, y) \in B \times D$.

$$\begin{aligned}
 (x, y) \in A \times C &\Leftrightarrow x \in A \wedge y \in C && \text{(by the definition of } \times \text{)} \\
 &\Rightarrow x \in B \wedge y \in D && \text{(since } A \subseteq B \text{ and } C \subseteq D \text{)} \\
 &\Leftrightarrow (x, y) \in B \times D && \text{(by the definition of } \times \text{)}
 \end{aligned}$$

3. (a) $\bigcap_{n=1}^{\infty} A_n = \{1\} \cap \{1, 2\} \cap \{1, 2, 3\} \cap \dots = \{1\}$ and $\bigcup_{n=1}^{\infty} A_n = \{1\} \cup \{1, 2\} \cup \dots = \{1, 2, 3, \dots\}$.
 (b) $\bigcap_{n=1}^{\infty} A_n = \{1, 2, 3, \dots\} \cap \{2, 3, 4, \dots\} \cap \dots = \emptyset$ and $\bigcup_{n=1}^{\infty} A_n = \{1, 2, 3, \dots\} \cup \{2, 3, 4, \dots\} \cup \dots = \{1, 2, 3, \dots\}$.
 (c) $\bigcap_{n=1}^{\infty} A_n = [0, 1) \cap [0, 2) \cap [0, 3) \cap \dots = [0, 1)$ and $\bigcup_{n=1}^{\infty} A_n = [0, 1) \cup [0, 2) \cup [0, 3) \cup \dots [0, n) \cup \dots = [0, \infty)$.

4. (a) The relation is reflexive ($1 \sim 1, 2 \sim 2$, and $3 \sim 3$ all hold) and symmetric ($1 \sim 2$ and $2 \sim 1$ both hold, $2 \sim 3$ and $3 \sim 2$ both hold) but not transitive: $1 \sim 2$ and $2 \sim 3$ hold, but not $1 \sim 3$.

(b) The relation is reflexive ($1 \sim 1, 2 \sim 2$, and $3 \sim 3$ all hold), but neither symmetric nor transitive. It is not symmetric since $2 \sim 3$ holds but not $3 \sim 2$. It is not transitive since $3 \sim 1$ and $1 \sim 2$ hold, but not $3 \sim 2$.

(c) The relation is reflexive since $a^2 = a^2$ holds. It is symmetric since $a^2 = b^2$ implies that $b^2 = a^2$ and transitive since $a^2 = b^2$ and $b^2 = c^2$ imply that $a^2 = c^2$.

Note that $a^2 = b^2$ if and only if $b = \pm a$. So, the equivalence class $[a]$ of any real number a consists of two elements a and $-a$ for $a \neq 0$ and $[0] = \{0\}$. Thus, the quotient set is the set of the sets $\{a, -a\}$ where $a \in \mathbb{R}$. As each negative number $-a$ is "identified" to its opposite a , the quotient set can be represented as the set of nonnegative real numbers.

(d) We are given that \equiv is defined as follows.

$$m \equiv n \Leftrightarrow (\exists k \in \mathbb{Z}) m - n = 5k$$

Check that this relation is an equivalence.

Reflexivity. We need to show that $m \equiv m$ for any $m \in \mathbb{Z}$. This is so since

$$\begin{aligned} m \equiv m &\Leftrightarrow (\exists k \in \mathbb{Z})m - m = 5k && \text{(by the definition of } \equiv) \\ &\Leftrightarrow \top && \text{(since one can take } k = 0) \end{aligned}$$

Symmetry. Assume that $m \equiv n$ for some $m, n \in \mathbb{Z}$ and show that $n \equiv m$.

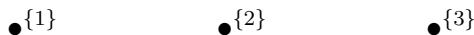
$$\begin{aligned} m \equiv n &\Leftrightarrow (\exists k \in \mathbb{Z})m - n = 5k && \text{(by the definition of } \equiv) \\ &\Leftrightarrow (\exists k \in \mathbb{Z})n - m = -5k && \text{(by multiplying both sides by } -1) \\ &\Leftrightarrow (\exists l \in \mathbb{Z})n - m = 5l && \text{(by taking } l = -k) \\ &\Leftrightarrow n \equiv m && \text{(by the definition of } \equiv) \end{aligned}$$

Transitivity. Assume that $m \equiv n$ and $n \equiv o$ for some $m, n, o \in \mathbb{Z}$ and show that $m \equiv o$.

$$\begin{aligned} m \equiv n \wedge n \equiv o &\Leftrightarrow (\exists k \in \mathbb{Z})m - n = 5k \wedge (\exists l \in \mathbb{Z})n - o = 5l && \text{(by the definition of } \equiv) \\ &\Rightarrow (\exists k \in \mathbb{Z})(\exists l \in \mathbb{Z})m - n + n - o = 5k + 5l && \text{(by adding the two equations)} \\ &\Rightarrow (\exists p \in \mathbb{Z})m - o = 5p && \text{(cancel } n \text{ and let } p = k + l) \\ &\Leftrightarrow m \equiv o && \text{(by the definition of } \equiv) \end{aligned}$$

The quotient set: two integers are in relation, if they have the same remainder when dividing by 5. As the possible remainders are 0, 1, 2, 3, and 4, there are five different equivalence classes $[0], [1], [2], [3],$ and $[4]$ (the class $[2]$, for example, consists of all integers of the form $5k + 2$ for $k \in \mathbb{Z}$). The quotient set consists of five elements $A/\equiv = \{[0], [1], [2], [3], [4]\}$.

5. (a) The relation \preceq is reflexive and transitive but not antisymmetric as we have that $2 \preceq 3$ and $3 \preceq 2$ but $2 \neq 3$.
- (b) The relation is not reflexive: $\{1, 2\}$ is an element of A but $(\{1, 2\}, \{1, 2\})$ is not an element of \preceq .
- (c) The relation is reflexive since every element of A is in the relation with itself. The relation is antisymmetric: the premise of the implication $(a \preceq b \text{ and } b \preceq a \Rightarrow a = b)$ is never true if $a \neq b$. The implication is also transitive since the premise of the implication $(a \preceq b \text{ and } b \preceq c \Rightarrow a \preceq c)$ is never true if $a \neq b$ and $b \neq c$ and it trivially holds when $a = b$ or $b = c$. The Hasse diagram of \preceq is below. The partial order is not total since there are incomparable elements (actually any two different elements are incomparable with each other). There are no greatest or smallest elements and every element of A is both maximal and minimal element.



- (d) The relation is reflexive since every element of A is in the relation with itself. The relation is antisymmetric: the premise of the implication $(a \preceq b \text{ and } b \preceq a \Rightarrow a = b)$ is never true if $a \neq b$. The implication is also transitive since the premise of the implication $(a \preceq b \text{ and } b \preceq c \Rightarrow a \preceq c)$ is never true if $a \neq b$ or $b \neq c$ and it trivially holds when $a = b$ or $b = c$. The Hasse diagram of \preceq is below. The partial order is not total since $\{1\}$ and $\{3\}$ are incomparable (as are $\{2\}$ and $\{3\}$). There are no greatest or smallest elements, $\{1\}$ and $\{3\}$ are minimal and $\{2\}$ and $\{3\}$ are maximal elements.



6. *Reflexivity.* We need to show that $a \sim a$ holds for any $a \in A$.

$$\begin{aligned} a \sim a &\Leftrightarrow aRa \wedge aRa && \text{(by the definition of } \sim) \\ &\Leftrightarrow aRa && \text{(by idempotence of } \wedge) \\ &\Leftrightarrow \top && \text{(by reflexivity of } R) \end{aligned}$$

Symmetry. Assume that $a \sim b$ holds and show that $b \sim a$ holds.

$$\begin{aligned} a \sim b &\Leftrightarrow aRb \wedge bRa && \text{(by the definition of } \sim) \\ &\Leftrightarrow bRa \wedge aRb && \text{(by commutativity of } \wedge) \\ &\Leftrightarrow b \sim a && \text{(by the definition of } \sim) \end{aligned}$$

Transitivity. Assume that $a \sim b$ and $b \sim c$ hold and show that $a \sim c$ holds,

$$\begin{aligned} a \sim b \wedge b \sim c &\Leftrightarrow (aRb \wedge bRa) \wedge (bRc \wedge cRb) && \text{(by the definition of } \sim) \\ &\Leftrightarrow (aRb \wedge bRc) \wedge (cRb \wedge bRa) && \text{(by commutativity of } \wedge) \\ &\Leftrightarrow aRc \wedge cRa && \text{(by transitivity of } R) \\ &\Leftrightarrow a \sim c && \text{(by the definition of } \sim) \end{aligned}$$

7. *Reflexivity.* We need to show that $(m, \alpha) \sim (m, \alpha)$ for any $m \in \mathbb{Z}$ and any $\alpha \in P$.

$$\begin{aligned} (m, \alpha) \sim (m, \alpha) &\Leftrightarrow m \equiv m \wedge \alpha \parallel \alpha && \text{(by the definition of } \sim) \\ &\Leftrightarrow \top \wedge \top && \text{(since } \equiv \text{ and } \parallel \text{ are reflexive)} \\ &\Leftrightarrow \top && \text{(by the definition of } \wedge) \end{aligned}$$

Symmetry. Assume that $(m, \alpha) \sim (n, \beta)$ for some $m, n \in \mathbb{Z}$ and $\alpha, \beta \in P$ and show that $(n, \beta) \sim (m, \alpha)$.

$$\begin{aligned} (m, \alpha) \sim (n, \beta) &\Leftrightarrow m \equiv n \wedge \alpha \parallel \beta && \text{(by the definition of } \sim) \\ &\Rightarrow n \equiv m \wedge \beta \parallel \alpha && \text{(since } \equiv \text{ and } \parallel \text{ are symmetric)} \\ &\Leftrightarrow (n, \beta) \sim (m, \alpha) && \text{(by the definition of } \sim) \end{aligned}$$

Transitivity. Assume that $(m, \alpha) \sim (n, \beta)$ and $(n, \beta) \sim (o, \gamma)$ for some $m, n, o \in \mathbb{Z}$ and $\alpha, \beta, \gamma \in P$ and show that $(m, \alpha) \sim (o, \gamma)$.

$$\begin{aligned} (m, \alpha) \sim (n, \beta) \wedge (n, \beta) \sim (o, \gamma) &\Leftrightarrow (m \equiv n \wedge \alpha \parallel \beta) \wedge (n \equiv o \wedge \beta \parallel \gamma) && \text{(by the definition of } \sim) \\ &\Leftrightarrow (m \equiv n \wedge n \equiv o) \wedge (\alpha \parallel \beta \wedge \beta \parallel \gamma) && \text{(by commutativity of } \wedge) \\ &\Rightarrow m \equiv o \wedge \alpha \parallel \gamma && \text{(since } \equiv \text{ and } \parallel \text{ are transitive)} \\ &\Leftrightarrow (m, \alpha) \sim (o, \gamma) && \text{(by the definition of } \sim) \end{aligned}$$

8. *Reflexivity.* We need to show that $(a, b) \preceq (a, b)$ holds for any $a \in A$ and any $b \in B$.

$$\begin{aligned} (a, b) \preceq (a, b) &\Leftrightarrow a \preceq_A a \wedge b \preceq_B b && \text{(by the definition of } \preceq) \\ &\Leftrightarrow \top \wedge \top && \text{(since } \preceq_A \text{ and } \preceq_B \text{ are reflexive)} \\ &\Leftrightarrow \top && \text{(by the definition of } \wedge) \end{aligned}$$

Antisymmetry. Assume that $(a, b) \preceq (c, d)$ and that $(c, d) \preceq (a, b)$ for some $a, c \in A$ and $b, d \in B$ and show that $(a, b) = (c, d)$.

$$\begin{aligned}
 (a, b) \preceq (c, d) \wedge (c, d) \preceq (a, b) &\Leftrightarrow (a \preceq_A c \wedge b \preceq_B d) \wedge (c \preceq_A a \wedge d \preceq_B b) && \text{(by the definition of } \preceq) \\
 &\Leftrightarrow (a \preceq_A c \wedge c \preceq_A a) \wedge (b \preceq_B d \wedge d \preceq_B b) && \text{(by commutativity of } \wedge) \\
 &\Rightarrow a = c \wedge b = d && \text{(since } \preceq_A \text{ and } \preceq_B \text{ are antisymmetric)} \\
 &\Leftrightarrow (a, b) = (c, d) && \text{(by the definition of an ordered pair)}
 \end{aligned}$$

Transitivity. Assume that $(a, b) \preceq (c, d)$ and $(c, d) \preceq (e, f)$ for some $a, c, e \in A$ and $b, d, f \in B$ and show that $(a, b) \preceq (e, f)$.

$$\begin{aligned}
 (a, b) \preceq (c, d) \wedge (c, d) \preceq (e, f) &\Leftrightarrow (a \preceq_A c \wedge b \preceq_B d) \wedge (c \preceq_A e \wedge d \preceq_B f) && \text{(by the definition of } \preceq) \\
 &\Leftrightarrow (a \preceq_A c \wedge c \preceq_A e) \wedge (b \preceq_B d \wedge d \preceq_B f) && \text{(by commutativity of } \wedge) \\
 &\Rightarrow a \preceq_A e \wedge b \preceq_B f && \text{(since } \preceq_A \text{ and } \preceq_B \text{ are transitive)} \\
 &\Leftrightarrow (a, b) \preceq (e, f) && \text{(by the definition of } \preceq)
 \end{aligned}$$

9. (a) If $A = [0, 1)$, neither the greatest element nor a maximal element exist. The supremum exist and it is 1. 0 is the smallest, a minimal element and the infimum of A .
- (b) If $A = (0, 1) \cup (1, 2)$, there are no smallest nor greatest elements, no minimal and maximal elements, 0 is the infimum, and 2 is the supremum of A .
- (c) Note that A is the interval $[0, \infty)$ (see problem 2(c)). Thus, 0 is the smallest element, a (unique) minimal element and the infimum. There is no greatest element, no maximum, and no supremum.
10. (a) Let $a \in A$ be arbitrary. We have that $((h \circ g) \circ f)(a) = (h \circ g)(f(a)) = h(g(f(a)))$ and that $(h \circ (g \circ f))(a) = h((g \circ f)(a)) = h(g(f(a)))$. This shows that $((h \circ g) \circ f)(a) = (h \circ (g \circ f))(a)$ for any $a \in A$ and so $(h \circ g) \circ f = h \circ (g \circ f)$.
- (b) Let $a \in A$ be arbitrary. We have that $(f \circ \text{id}_A)(a) = f(\text{id}_A(a)) = f(a)$. Thus $f \circ \text{id}_A = f$. To show the second identity, note that $\text{id}_B(f(a)) = f(a)$ by the definition of id_B . Thus, for any $a \in A$, $(\text{id}_B \circ f)(a) = \text{id}_B(f(a)) = f(a)$, which shows that $\text{id}_B \circ f = f$.
- (c) Assume that f and g are injections. To show that $g \circ f$ is injective, assume that $g \circ f(a_1) = g \circ f(a_2)$ for $a_1, a_2 \in A$, and show that $a_1 = a_2$.

$$\begin{aligned}
 g \circ f(a_1) = g \circ f(a_2) &\Leftrightarrow g(f(a_1)) = g(f(a_2)) && \text{(by the definition of } \circ) \\
 &\Rightarrow f(a_1) = f(a_2) && \text{(since } g \text{ is injective)} \\
 &\Rightarrow a_1 = a_2 && \text{(since } f \text{ is injective)}
 \end{aligned}$$

- (d) Assume that f and g are surjections. We need to show that $g \circ f$ is a surjection, i.e. that for every $c \in C$, there is $a \in A$ such that $(g \circ f)(a) = c$.

Let $c \in C$ be arbitrary. As g is a surjection, there is $b \in B$ such that $g(b) = c$. Since f is also surjective, for b there is $a \in A$ such that $f(a) = b$. Hence, $(g \circ f)(a) = g(f(a)) = g(b) = c$.

- (e) Assume that $g \circ f$ is an injection. To show that f is an injection, we need to show the implication $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$ for arbitrary $a_1, a_2 \in A$.

$$\begin{aligned}
 f(a_1) = f(a_2) &\Rightarrow g(f(a_1)) = g(f(a_2)) && \text{(since } g \text{ is a function)} \\
 &\Leftrightarrow g \circ f(a_1) = g \circ f(a_2) && \text{(by the definition of } \circ) \\
 &\Rightarrow a_1 = a_2 && \text{(since } g \circ f \text{ is injective)}
 \end{aligned}$$

(f) Assume that $g \circ f$ is surjective. To show that g is surjective, we need to show that $(\forall c \in C)(\exists b \in B)g(b) = c$. So, let $c \in C$. As $g \circ f$ is surjective, there is $a \in A$ such that $(g \circ f)(a) = c$. Thus, $g(f(a)) = c$. By taking b to be $f(a)$, we have that $g(b) = c$.

(g) Assume that f is onto and that $g_1 \circ f = g_2 \circ f$ for some $C \neq \emptyset$ and $g_1, g_2 : B \rightarrow C$. We need to show that $g_1 = g_2$ which means that we have to show that $g_1(b) = g_2(b)$ for every $b \in B$. Let $b \in B$. Since f is onto, there is $a \in A$ such that $b = f(a)$.

Since $g_1 \circ f = g_2 \circ f$, we have that $g_1 \circ f(a) = g_2 \circ f(a)$ and so $g_1(b) = g_1(f(a)) = g_1 \circ f(a) = g_2 \circ f(a) = g_2(f(a)) = g_2(b)$.

11. (a) Assume that $c \in C$. Then $f(c) \in f(C)$ by the definition of $f(C)$ so $c \in f^{-1}(f(C))$ by the definition of the inverse image of $f(C)$.

(b) Assume that $d \in f(f^{-1}(D))$ and show that $d \in D$. As $d \in f(f^{-1}(D))$, there is $a \in f^{-1}(D)$ such that $d = f(a)$. Since $a \in f^{-1}(D)$, we have that $f(a)$ is in D . Hence $d = f(a) \in D$.

(c) Let $a \in A$.

$$\begin{aligned} a \in f^{-1}(D_1 \cap D_2) &\Leftrightarrow f(a) \in D_1 \cap D_2 && \text{(by the definition of the inverse image)} \\ &\Leftrightarrow f(a) \in D_1 \wedge f(a) \in D_2 && \text{(by the definition of the intersection)} \\ &\Leftrightarrow a \in f^{-1}(D_1) \wedge a \in f^{-1}(D_2) && \text{(by the definition of the inverse image)} \\ &\Leftrightarrow a \in f^{-1}(D_1) \cap f^{-1}(D_2) && \text{(by the definition of the intersection)} \end{aligned}$$

(d) Let $a \in A$.

$$\begin{aligned} a \in f^{-1}(D_1 \cup D_2) &\Leftrightarrow f(a) \in D_1 \cup D_2 && \text{(by the definition of the inverse image)} \\ &\Leftrightarrow f(a) \in D_1 \vee f(a) \in D_2 && \text{(by the definition of the union)} \\ &\Leftrightarrow a \in f^{-1}(D_1) \vee a \in f^{-1}(D_2) && \text{(by the definition of the inverse image)} \\ &\Leftrightarrow a \in f^{-1}(D_1) \cup f^{-1}(D_2) && \text{(by the definition of the union).} \end{aligned}$$

(e) Let $b \in B$.

$$\begin{aligned} b \in f(C_1 \cup C_2) &\Leftrightarrow (\exists a \in A)(b = f(a) \wedge a \in C_1 \cup C_2) && \text{(by the definition of the image)} \\ &\Leftrightarrow (\exists a \in A)(b = f(a) \wedge (a \in C_1 \vee a \in C_2)) && \text{(by the definition of } \cup) \\ &\Leftrightarrow (\exists a \in A)((b = f(a) \wedge a \in C_1) \vee (b = f(a) \wedge a \in C_2)) && \text{(by the distributive law)} \\ &\Leftrightarrow (\exists a \in A)(b = f(a) \wedge a \in C_1) \vee (\exists a \in A)(b = f(a) \wedge a \in C_2) \\ &\quad \text{(by passing } \exists \text{ through } \vee) \\ &\Leftrightarrow b \in f(C_1) \vee b \in f(C_2) && \text{(by the definition of the inverse image)} \\ &\Leftrightarrow b \in f(C_1) \cup f(C_2) && \text{(by the definition of } \cup). \end{aligned}$$

(f) Let $b \in B$.

$$\begin{aligned} b \in f(C_1 \cap C_2) &\Leftrightarrow (\exists a \in A)(b = f(a) \wedge a \in C_1 \cap C_2) && \text{(by the definition of the image)} \\ &\Leftrightarrow (\exists a \in A)(b = f(a) \wedge a \in C_1 \wedge a \in C_2) && \text{(by the definition of } \cap) \\ &\Leftrightarrow (\exists a \in A)(b = f(a) \wedge a \in C_1 \wedge b = f(a) \wedge a \in C_2) && \text{(by idempotence of } \wedge) \\ &\Rightarrow (\exists a \in A)(b = f(a) \wedge a \in C_1) \wedge (\exists a \in A)(b = f(a) \wedge a \in C_2) \\ &\quad \text{(by passing } \exists \text{ through } \wedge) \\ &\Leftrightarrow b \in f(C_1) \wedge b \in f(C_2) && \text{(by the definition of the inverse image)} \\ &\Leftrightarrow b \in f(C_1) \cap f(C_2) && \text{(by the definition of } \cap). \end{aligned}$$

Let us assume now that f is injective and let us show the converse. So, let us assume that $b \in f(C_1) \cap f(C_2)$ so that $b = f(a_1)$ for some $a_1 \in C_1$ and $b = f(a_2)$ for some $a_2 \in C_2$. Thus, we have that $f(a_1) = b = f(a_2)$ and from these relations we can deduce that $a_1 = a_2$ because f is injective. So, as $a_1 \in C_1$, $a_2 \in C_2$, and $a_1 = a_2$, we have that $a_1 \in C_1 \cap C_2$. Since $b = f(a_1)$, we obtain that $b \in f(C_1 \cap C_2)$.