Fundamentals of Mathematics
Lia Vas

Review for Exam 2

1. Let $A = \{1\}$ and $B = \{2, 3\}$. Determine the following sets.

$$\mathcal{P}(A),\ \ \mathcal{P}(B),\ \ \mathcal{P}(\mathcal{P}(A)),\ \ A \times B,\ \ \mathcal{P}(A \times B),\ \ \mathcal{P}(A) \times B,\ \ A \times \mathcal{P}(B),\ \ \mathcal{P}(A) \times \mathcal{P}(B).$$

2. Show the following identities or statements in which A, B, C, and D stand for arbitrary sets. In part (g), I is an arbitrary set and A_i are sets for $i \in I$.

   (a) $\emptyset \subseteq A$

   (b) $A \cap B \subseteq A$ and $A \subseteq A \cup B$

   (c) $A \subseteq B \ \Leftrightarrow\ A \cap B = A$

   (d) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

   (e) $A \subseteq B \ \Leftrightarrow\ \overline{B} \subseteq \overline{A}$

   (f) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

   (g) $\overline{\bigcap_{i \in I} A_i} = \bigcup_{i \in I} \overline{A_i}$

   (h) $A \subseteq B \ \wedge\ C \subseteq D \ \ \Rightarrow\ \ A \times C \subseteq B \times D$

   (i) $A \cup B = \emptyset \ \ \Leftrightarrow\ A = \emptyset \wedge B = \emptyset$

3. Determine $\bigcap_{n=1}^{\infty} A_n$ and $\bigcup_{n=1}^{\infty} A_n$ for given sets A_n where $n = 1, 2, \ldots$.

   (a) $A_n = \{1, 2, \ldots, n\}$, (b) $A_n = \{n, n+1, \ldots\}$ (c) $A_n = [0, n)$

4. For a given set A and a relation $\sim$ on it, check whether the given equation $\sim$ is an equivalence relation. If it is, determine the quotient set.

   (a) $A = \{1, 2, 3\}$ and $\sim$ consists of the ordered pairs $(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)$, $(2, 3)$, $(3, 2)$.

   (b) $A = \{1, 2, 3\}$ and $\sim$ consists of the ordered pairs $(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)$, $(2, 3)$, $(1, 3)$, $(3, 1)$.

   (c) A is the set of real numbers and $\sim$ is given by $a \sim b$ if $a^2 = b^2$.

   (d) A is the set of integers and $\equiv$ is given by $m \equiv n$ if $m - n$ is divisible by 5.

5. For a given set A and a relation $\preceq$ on it, determine whether $\preceq$ is a partial order. If it is, represent it by a Hasse diagram and determine whether it is a total order. Then, determine the greatest, the smallest elements, minimal and maximal elements, if any of those exist.

   (a) $A = \{1, 2, 3\}$ and $\preceq$ consists of the pairs $(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)$, $(3, 2)$, $(1, 3)$.

   (b) $A = \mathcal{P}(\{1, 2, 3\})$ and $\preceq$ consists of the pairs $(\{1\}, \{1\})$, $(\{2\}, \{2\})$, $(\{3\}, \{3\})$.

   (c) $A = \{\{1\}, \{2\}, \{3\}\}$ and $\preceq$ consists of the pairs $(\{1\}, \{1\})$, $(\{2\}, \{2\})$, $(\{3\}, \{3\})$.

(d) $A = \{\{1\}, \{2\}, \{3\}\}$ and $\preceq$ consists of the pairs $(\{1\}, \{1\})$, $(\{2\}, \{2\})$, $(\{1\}, \{2\})$, $(\{3\}, \{3\})$.

6. If R is a relation on a set A which is reflexive and transitive, show that the relation $\sim$ given by

$$a \sim b \quad \text{if} \quad aRb \text{ and } bRa$$

is an equivalence relation.

7. Let A and B be any sets and let $\preceq$ be a partial order on A and $\precsim$ is a partial order on B. Let us define $\precapprox$ on $A \times B$ by

$$(a, b) \precapprox (c, d) \quad \text{if and only if} \quad a \preceq c \text{ and } b \precsim d.$$

Show that $\precapprox$ is a partial order on $A \times B$.

8. For the given poset A of the set of real numbers $\mathbb{R}$, consider both A and $\mathbb{R}$ to be partially ordered by the relation $\leq$. Determine the greatest, the smallest elements, minimal and maximal elements, and suprema and infima of A, if any of those exist.

(a) $A = [0, 1)$ (b) $A = (0, 1) \cup (1, 2)$ (c) $A = \bigcup_{n=1}^{\infty} [0, n)$

9. Show that if a binary relation R defined on a nonempty set A is both symmetric and antisymmetric, then it is the equality relation, that is

$$aRb \quad \Rightarrow \quad a = b$$

for every $a, b \in A$. If R is also reflexive, then the converse $a = b \Rightarrow aRb$ also holds.

10. If $f : A \to B$, $g : B \to C$, and $h : C \to D$ are functions, show the following properties.

(a) Associativity holds for the composite: $(h \circ g) \circ f = h \circ (g \circ f)$.

(b) The identity function is a neutral element for the composite: $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_B \circ f = f$.

(c) If f and g are injections, then $g \circ f$ is an injection.

(d) If f and g are surjections, then $g \circ f$ is a surjection.

(e) If $g \circ f$ is an injection, then f is an injection.

(f) If $g \circ f$ is a surjection, then g is a surjection.

(g) If f is onto, show that $g_1 \circ f = g_2 \circ f$ implies that $g_1 = g_2$ for every $C \neq \emptyset$ and every functions $g_1, g_2 : B \to C$.

11. Show the following properties of a function $f : A \to B$, $C, C_1, C_2 \subseteq A$, and $D, D_1, D_2 \subseteq B$.

(a) $C \subseteq f^{-1}(f(C))$

(b) $f(f^{-1}(D)) \subseteq D$

(c) $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$.

(d) $f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$.

(e) $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$

(f) $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$. Show that the converse holds if f is injective.

Solutions

1. If $A = \{1\}$, and $B = \{2,3\}$, then

$$\mathcal{P}(A) = \{\emptyset, \{1\}\}, \ \ P(B) = \{\emptyset, \{2\}, \{3\}, \{2,3\}\}, \ \ \mathcal{P}(\mathcal{P}(A)) = \{\emptyset, \{\emptyset\}, \{\{1\}\}, \{\emptyset, \{1\}\}\}.$$

$$A \times B = \{(1,2), (1,3)\}, \quad \mathcal{P}(A \times B) = \{\emptyset, \{(1,2)\}, \{(1,3)\}, \{(1,2), (1,3)\}\},$$

$$\mathcal{P}(A) \times B = \{(\emptyset, 2), (\emptyset, 3)(\{1\}, 2), (\{1\}, 3)\}, \quad A \times \mathcal{P}(B) = \{(1, \emptyset), (1, \{2\}), (1, \{3\}), (1, \{2,3\})\}.$$

$$\mathcal{P}(A) \times \mathcal{P}(B) = \{(\emptyset, \emptyset), (\emptyset, \{2\}), (\emptyset, \{3\}), (\emptyset, \{2,3\}), (\{1\}, \emptyset), (\{1\}, \{2\}), (\{1\}, \{3\}), (\{1\}, \{2,3\})\}.$$

2. (a) One needs to show that the implication $x \in \emptyset \Rightarrow x \in A$ holds for every x. Since the premise $x \in \emptyset$ is always false, the implication holds (recall that $\perp \Rightarrow p$ is true for any value of p).

   (b) To show the first relation, assume that $x \in A \cap B$. Then $x \in A$ and $x \in B$, so $x \in A$ holds. This shows that the premise $x \in A \cap B$ implies $x \in A$ so $A \cap B \subseteq A$.

   To show the second relation, assume that $x \in A$. Then the disjunction $x \in A$ or $x \in B$ is true, so $x \in A \cup B$. This shows that the premise $x \in A$ implies $x \in A \cup B$ so $A \subseteq A \cup B$.

   (c) We can show the direction ($\Rightarrow$) first. Assume that $A \subseteq B$. Since $A \cap B \subseteq A$ (see the previous problem), to show (2) it is sufficient to show that $A \subseteq A \cap B$. Assume that $x \in A$. Then x is also in B as $A \subseteq B$. So, the conjunction $x \in A$ and $x \in B$ is true and so $x \in A \cap B$.

   Let us show the direction ($\Leftarrow$) next. Assume that $A = A \cap B$ holds and let us show $A \subseteq B$. So, assuming $x \in A$, we need to show $x \in B$. If $x \in A$ then $x \in A \cap B$ since $A = A \cap B$. So, $x \in A$ and $x \in B$ both hold. In particular, $x \in B$ holds.

   (d)

$$\begin{array}{rcll} x \in A \cap (B \cup C) & \Leftrightarrow & x \in A \wedge x \in B \cup C & \text{(by the definition of } \cap) \\ & \Leftrightarrow & x \in A \wedge (x \in B \vee x \in C) & \text{(by the definition of } \cup) \\ & \Leftrightarrow & (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C) & \text{(Distributivity of } \wedge \text{ and } \vee) \\ & \Leftrightarrow & x \in A \cap B \vee x \in A \cap C & \text{(by the definition of } \cap) \\ & \Leftrightarrow & x \in (A \cap B) \cup (A \cap C) & \text{(by the definition of } \cup) \end{array}$$

   (e)

$$\begin{array}{rcll} \text{(c) } A \subseteq B & \Leftrightarrow & (\forall x)(x \in A \Rightarrow x \in B) & \text{(by the definition of } \subseteq) \\ & \Leftrightarrow & (\forall x)(\neg x \in B \Rightarrow \neg x \in A) & \text{(by Contrapositive law)} \\ & \Leftrightarrow & (\forall x) x \in \overline{B} \Rightarrow x \in \overline{A} & \text{(by the definition of the complement)} \\ & \Leftrightarrow & \overline{B} \subseteq \overline{A} & \text{(by the definition of } \subseteq) \end{array}$$

   (f)

$$\begin{array}{rcll} x \in \overline{A \cap B} & \Leftrightarrow & \neg\, x \in A \cap B & \text{(by the definition of the complement)} \\ & \Leftrightarrow & \neg\, (x \in A \wedge x \in B) & \text{(by the definition of } \cap) \\ & \Leftrightarrow & \neg\, x \in A \vee \neg\, x \in B & \text{(by De Morgan's law)} \\ & \Leftrightarrow & x \in \overline{A} \vee x \in \overline{B} & \text{(by the definition of the complement)} \\ & \Leftrightarrow & x \in \overline{A} \cup \overline{B} & \text{(by the definition of } \cup) \end{array}$$

(g)

$$\begin{array}{rcll} x \in \overline{\bigcap_{i\in I} A_i} & \Leftrightarrow & \neg\, x \in \bigcap_{i\in I} A_i & \text{(by the definition of the complement)} \\ & \Leftrightarrow & \neg\, (\forall i \in I)\; x \in A_i & \text{(by the definition of } \bigcap) \\ & \Leftrightarrow & (\exists i \in I)\; \neg\, x \in A_i & \text{(Distributing } \neg \text{ through } \forall) \\ & \Leftrightarrow & (\exists i \in I)\; x \in \overline{A_i} & \text{(by the definition of the complement)} \\ & \Leftrightarrow & x \in \bigcup_{i\in I} \overline{A_i} & \text{(by the definition of } \bigcup) \end{array}$$

(h) Assume that $A \subseteq B$ and $C \subseteq D$ and show that $A \times C \subseteq B \times D$. So, we need to show that if $(a, c) \in A \times C$, then $(a, c) \in B \times D$. Assume that $(a, c) \in A \times C$, then $a \in A$ and $c \in C$. If $a \in A$ then $a \in B$ since $A \subseteq B$. If $c \in C$, then $c \in D$ since $C \subseteq D$. So, we have that $a \in B$ and $c \in D$ which implies that $(a, c) \in B \times D$.

(i)

$$\begin{array}{rcll} A \cup B = \emptyset & \Leftrightarrow & \neg x \in A \cup B & \text{(by the definition of } \emptyset) \\ & \Leftrightarrow & \neg(x \in A \vee x \in B) & \text{(by the definition of } \cup) \\ & \Leftrightarrow & \neg x \in A \wedge \neg x \in B & \text{(by De Morgan's law)} \\ & \Leftrightarrow & A = \emptyset \wedge B = \emptyset & \text{(by the definition of } \emptyset) \end{array}$$

3. (a) $\bigcap_{n=1}^{\infty} A_n = \{1\} \cap \{1, 2\} \cap \{1, 2, 3\} \cap \ldots = \{1\}$ and $\bigcup_{n=1}^{\infty} A_n = \{1\} \cup \{1, 2\} \cup \ldots = \{1, 2, 3, \ldots\}$.

   (b) $\bigcap_{n=1}^{\infty} A_n = \{1, 2, 3, \ldots\} \cap \{2, 3, 4, \ldots\} \cap \ldots = \emptyset$ and $\bigcup_{n=1}^{\infty} A_n = \{1, 2, 3, \ldots\} \cup \{2, 3, 4, \ldots\} \cup \ldots = \{1, 2, 3, \ldots\}$.

   (c) $\bigcap_{n=1}^{\infty} A_n = [0, 1) \cap [0, 2) \cap [0, 3) \cap \ldots = [0, 1)$ and $\bigcup_{n=1}^{\infty} A_n = [0, 1) \cup [0, 2) \cup [0, 3) \cup \ldots [0, n) \cup \ldots = [0, \infty)$.

4. (a) The relation is reflexive ($1 \sim 1, 2 \sim 2$, and $3 \sim 3$ all hold) and symmetric ($1 \sim 2$ and $2 \sim 1$ both holds an $2 \sim 3$ and $3 \sim 2$ both hold) but not transitive: $1 \sim 2$ and $2 \sim 3$ hold, but not $1 \sim 3$.

   (b) The relation is reflexive ($1 \sim 1, 2 \sim 2$, and $3 \sim 3$ all hold), but neither symmetric nor transitive. It is not symmetric since $2 \sim 3$ holds but not $3 \sim 2$. It is not transitive since $3 \sim 1$ and $1 \sim 2$ hold, but not $3 \sim 2$.

   (c) The relation is reflexive since $a^2 = a^2$ holds. It is symmetric since $a^2 = b^2$ implies that $b^2 = a^2$ and transitive since $a^2 = b^2$ and $b^2 = c^2$ imply that $a^2 = c^2$.

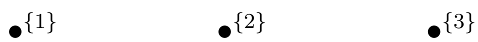
   Note that $a^2 = b^2$ if and only if $b = \pm a$. So, the equivalence class $[a]$ of any real number a consists of two elements a and $-a$ for $a \neq 0$ and $[0] = \{0\}$. Thus, the quotient set is the set of the sets $\{a, -a\}$ where $a \in \mathbb{R}$. As each negative number $-a$ is "identified" to its opposite a, the quotient set can be represented as the set of nonnegative real numbers.

   (d) *Reflexivity.* Since $m - m = 0$ and 0 is divisible by 5, $m \equiv m$ holds.
   Symmetry. If $n - m$ is divisible by 5, then $m - n = -(n - m)$ is also divisible by 5, so $m \equiv n$ implies that $n \equiv m$.
   Transitivity. If $m \equiv n$ and $n \equiv k$, then both $m - n$ and $n - k$ are divisible by 5. Then, their sum $(m - n) + (n - k) = m - k$ is also divisible by 5. This shows that $m \equiv k$.
   Quotient set: two integers are in relation, if they have the same remainder when dividing by 5. As the possible remainders are $0, 1, 2, 3$, and 4, there are five different equivalence classes $[0], [1], [2], [3]$, and $[4]$ (the class $[2]$, for example, consists of all integers of the form $5k + 2$ for $k \in \mathbb{Z}$). The quotient set consists of five elements $A/{\equiv} = \{[0], [1], [2], [3], [4]\}$.

5. (a) The relation $\preceq$ is reflexive and transitive but not antisymmetric as we have that $2 \preceq 3$ and $3 \preceq 2$ but $2 \neq 3$.

   (b) The relation is not reflexive: $\{1,2\}$ is an element of A but $(\{1,2\},\{1,2\})$ is not an element of $\preceq$.

   (c) The relation is reflexive since every element of A is in the relation with itself. The relation is antisymmetric: the premise of the implication ($a \preceq b$ and $b \preceq a \Rightarrow a = b$) is never true if $a \neq b$. The implication is also transitive since the premise of the implication ($a \preceq b$ and $b \preceq c \Rightarrow a \preceq c$) is never true if $a \neq b$ and $b \neq c$ and it trivially holds when $a = b$ or $b = c$. The Hasse diagram of $\preceq$ is below. The partial order is not total since there are incomparable elements (actually any two different elements are incomparable with each other). There are no greatest or smallest elements and every element of A is both maximal and minimal element.



   (d) The relation is reflexive since every element of A is in the relation with itself. The relation is antisymmetric: the premise of the implication ($a \preceq b$ and $b \preceq a \Rightarrow a = b$) is never true if $a \neq b$. The implication is also transitive since the premise of the implication ($a \preceq b$ and $b \preceq c \Rightarrow a \preceq c$) is never true if $a \neq b$ or $b \neq c$ and it trivially holds when $a = b$ or $b = c$. The Hasse diagram of $\preceq$ is below. The partial order is not total since $\{1\}$ and $\{3\}$ are incomparable (as are $\{2\}$ and $\{3\}$). There are no greatest or smallest elements, $\{1\}$ and $\{3\}$ are minimal and $\{2\}$ and $\{3\}$ are maximal elements.

6. *Reflexivity.* We need to show that $a \sim a$ holds for any $a \in A$.

$$\begin{array}{rcll} a \sim a & \Leftrightarrow & aRa \ \wedge \ aRa & \text{(by the definition of } \sim) \\ & \Leftrightarrow & aRa & \text{(by idempotence of } \wedge) \\ & \Leftrightarrow & \top & \text{(by reflexivity of } R) \end{array}$$

Symmetry. Assume that $a \sim b$ holds and show that $b \sim a$ holds.

$$\begin{array}{rcll} a \sim b & \Leftrightarrow & aRb \ \wedge \ bRa & \text{(by the definition of } \sim) \\ & \Leftrightarrow & bRa \ \wedge \ aRb & \text{(by commutativity of } \wedge) \\ & \Leftrightarrow & b \sim a & \text{(by the definition of } \sim) \end{array}$$

Transitivity. Assume that $a \sim b$ and $b \sim c$ hold and show that $a \sim c$ holds,

$$\begin{array}{rcll} a \sim b \wedge b \sim c & \Leftrightarrow & (aRb \ \wedge \ bRa) \wedge (bRc \ \wedge \ cRb) & \text{(by the definition of } \sim) \\ & \Leftrightarrow & (aRb \ \wedge \ bRc) \wedge (cRb \ \wedge \ bRa) & \text{(by commutativity of } \wedge) \\ & \Leftrightarrow & aRc \ \wedge \ cRa & \text{(by transitivity of } R) \\ & \Leftrightarrow & a \sim c & \text{(by the definition of } \sim) \end{array}$$

7. *Reflexivity.* We need to show that $(a,b) \precapprox (a,b)$ holds for any $a \in A$ and any $b \in B$.

$$\begin{array}{rcll} (a,b) \precapprox (a,b) & \Leftrightarrow & a \preceq a \;\wedge\; b \precsim b & \text{(by the definition of } \precapprox\text{)} \\ & \Leftrightarrow & \top \wedge \top & \text{(since } \preceq \text{ and } \precsim \text{ are reflexive)} \\ & \Leftrightarrow & \top & \text{(by the definition of } \wedge\text{)} \end{array}$$

Antisymmetry. Assume that $(a,b) \precapprox (c,d)$ and that $(c,d) \precapprox (a,b)$ for some $a, c \in A$ and $b, d \in B$ and show that $(a,b) = (c,d)$.

$$\begin{array}{rcl} (a,b) \precapprox (c,d) \wedge (c,d) \precapprox (a,b) & \Leftrightarrow & (a \preceq c \;\wedge\; b \precsim d) \wedge (c \preceq a \;\wedge\; d \precsim b) \quad \text{(by the definition of } \precapprox\text{)} \\ & \Leftrightarrow & (a \preceq c \;\wedge\; c \preceq a) \wedge (b \precsim d \;\wedge\; d \precsim b) \quad \text{(by commutativity of } \wedge\text{)} \\ & \Rightarrow & a = c \;\;\wedge\;\; b = d \quad \text{(since } \preceq \text{ and } \precsim \text{ are antisymmetric)} \\ & \Leftrightarrow & (a,b) = (c,d) \quad \text{(by the definition of an ordered pair)} \end{array}$$

Transitivity. Assume that $(a,b) \precapprox (c,d)$ and $(c,d) \precapprox (e,f)$ for some $a, c, e \in A$ and $b, d, f \in B$ and show that $(a,b) \precapprox (e,f)$.

$$\begin{array}{rcl} (a,b) \precapprox (c,d) \wedge (c,d) \precapprox (e,f) & \Leftrightarrow & (a \preceq c \;\wedge\; b \precsim d) \wedge (c \preceq e \;\wedge\; d \precsim f) \quad \text{(by the definition of } \precapprox\text{)} \\ & \Leftrightarrow & (a \preceq c \;\wedge\; c \preceq e) \wedge (b \precsim d \;\wedge\; d \precsim f) \quad \text{(by commutativity of } \wedge\text{)} \\ & \Rightarrow & a \preceq e \;\;\wedge\;\; b \precsim f \quad \text{(since } \preceq \text{ and } \precsim \text{ are transitive)} \\ & \Leftrightarrow & (a,b) \precapprox (e,f) \quad \text{(by the definition of } \precapprox\text{)} \end{array}$$

8. (a) If $A = [0,1)$, neither the greatest element nor a maximal element exist. The supremum exist and it is 1. 0 is the smallest, a minimal element and the infimum of A.

   (b) If $A = (0,1) \cup (1,2)$, there are no smallest nor greatest elements, no minimal and maximal elements, 0 is the infimum, and 2 is the supremum of A.

   (c) Note that A is the interval $[0, \infty)$ (see problem 2(c)). Thus, 0 is the smallest element, a (unique) minimal element and the infimum. There is no greatest element, no maximum, and no supremum.

9. The problem is asking us to show the implication $aRb \Rightarrow a = b$ for any $a, b \in A$. So, assume that a and b are elements of A such that aRb holds. As R is symmetric, we have that bRa holds. Thus, the premise of the implication $aRb \wedge bRa \Rightarrow a = b$ is true and the implication itself is true because R is antisymmetric. Hence, the conclusion $a = b$ is also true.

   If R is reflexive, then aRa holds for any $a \in A$. So, if $a = b$ holds, then aRa is aRb and it holds. This shows the converse implication $a = b \Rightarrow aRb$.

10. (a) Let $a \in A$ be arbitrary. We have that $((h \circ g) \circ f)(a) = (h \circ g)(f(a)) = h(g(f(a)))$ and that $(h \circ (g \circ f))(a) = h((g \circ f)(a)) = h(g(f(a)))$. This shows that $((h \circ g) \circ f)(a) = (h \circ (g \circ f))(a)$ for any $a \in A$ and so $(h \circ g) \circ f = h \circ (g \circ f)$.

    (b) Let $a \in A$ be arbitrary. We have that $(f \circ \mathrm{id}_A)(a) = f(\mathrm{id}_A(a)) = f(a)$. Thus $f \circ \mathrm{id}_A = f$. To show the second identity, note that $\mathrm{id}_B(f(a)) = f(a)$ by the definition of id_B. Thus, for any $a \in A$, $(\mathrm{id}_B \circ f)(a) = \mathrm{id}_B(f(a)) = f(a)$, which shows that $\mathrm{id}_B \circ f = f$.

(c) Assume that f and g are injections. To show that $g \circ f$ is injective, assume that $g \circ f(a_1) = g \circ f(a_2)$ for $a_1, a_2 \in A$, and show that $a_1 = a_2$.

$$\begin{array}{rcll} g \circ f(a_1) = g \circ f(a_2) & \Leftrightarrow & g(f(a_1)) = g(f(a_2)) & \text{(by the definition of } \circ) \\ & \Rightarrow & f(a_1) = f(a_2) & \text{(since } g \text{ is injective)} \\ & \Rightarrow & a_1 = a_2 & \text{(since } f \text{ is injective)} \end{array}$$

(d) Assume that f and g are surjections. We need to show that $g \circ f$ is a surjection, i.e. that for every $c \in C$, there is $a \in A$ such that $(g \circ f)(a) = c$.

Let $c \in C$ be arbitrary. As g is a surjection, there is $b \in B$ such that $g(b) = c$. Since f is also surjective, for b there is $a \in A$ such that $f(a) = b$. Hence, $(g \circ f)(a) = g(f(a)) = g(b) = c$.

(e) Assume that $g \circ f$ is an injection. To show that f is an injection, we need to show the implication $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$ for arbitrary $a_1, a_2 \in A$.

$$\begin{array}{rcll} f(a_1) = f(a_2) & \Rightarrow & g(f(a_1)) = g(f(a_2)) & \text{(since } g \text{ is a function)} \\ & \Leftrightarrow & g \circ f(a_1) = g \circ f(a_2) & \text{(by the definition of } \circ) \\ & \Rightarrow & a_1 = a_2 & \text{(since } g \circ f \text{ is injective)} \end{array}$$

(f) Assume that $g \circ f$ is surjective. To show that g is surjective, we need to show that $(\forall c \in C)(\exists b \in B) g(b) = c$. So, let $c \in C$. As $g \circ f$ is surjective, there is $a \in A$ such that $(g \circ f)(a) = c$. Thus, $g(f(a)) = c$. By taking b to be $f(a)$, we have that $g(b) = c$.

(g) Assume that f is onto and that $g_1 \circ f = g_2 \circ f$ for some $C \neq \emptyset$ and $g_1, g_2 : B \to C$. We need to show that $g_1 = g_2$ which means that we have to show that $g_1(b) = g_2(b)$ for every $b \in B$. Let $b \in B$. Since f is onto, there is $a \in A$ such that $b = f(a)$.

Since $g_1 \circ f = g_2 \circ f$, we have that $g_1 \circ f(a) = g_2 \circ f(a)$ and so $g_1(b) = g_1(f(a)) = g_1 \circ f(a) = g_2 \circ f(a) = g_2(f(a)) = g_2(b)$.

11. (a) Assume that $c \in C$. Then $f(c) \in f(C)$ by the definition of $f(C)$ so $c \in f^{-1}(f(C))$ by the definition of the inverse image of $f(C)$.

(b) Assume that $d \in f(f^{-1}(D))$ and show that $d \in D$. As $d \in f(f^{-1}(D))$, there is $a \in f^{-1}(D)$ such that $d = f(a)$. Since $a \in f^{-1}(D)$, we have that $f(a)$ is in D. Hence $d = f(a) \in D$.

(c) Let $a \in A$.

$$\begin{array}{rcll} a \in f^{-1}(D_1 \cap D_2) & \Leftrightarrow & f(a) \in D_1 \cap D_2 & \text{(by the definition of the inverse image)} \\ & \Leftrightarrow & f(a) \in D_1 \ \wedge \ f(a) \in D_2 & \text{(by the definition of the intersection)} \\ & \Leftrightarrow & a \in f^{-1}(D_1) \wedge a \in f^{-1}(D_2) & \text{(by the definition of the inverse image)} \\ & \Leftrightarrow & a \in f^{-1}(D_1) \cap f^{-1}(D_2) & \text{(by the definition of the intersection)} \end{array}$$

(d) Let $a \in A$.

$$\begin{array}{rcll} a \in f^{-1}(D_1 \cup D_2) & \Leftrightarrow & f(a) \in D_1 \cup D_2 & \text{(by the definition of the inverse image)} \\ & \Leftrightarrow & f(a) \in D_1 \vee f(a) \in D_2 & \text{(by the definition of the union)} \\ & \Leftrightarrow & a \in f^{-1}(D_1) \vee a \in f^{-1}(D_2) & \text{(by the definition of the inverse image)} \\ & \Leftrightarrow & a \in f^{-1}(D_1) \cup f^{-1}(D_2) & \text{(by the definition of the union).} \end{array}$$

(e) Let $b \in B$.

$$\begin{aligned}
b \in f(C_1 \cup C_2) &\Leftrightarrow (\exists a \in A)(b = f(a) \wedge a \in C_1 \cup C_2) \text{ (by the definition of the image)} \\
&\Leftrightarrow (\exists a \in A)(b = f(a) \wedge (a \in C_1 \vee a \in C_2)) \text{ (by the definition of } \cup) \\
&\Leftrightarrow (\exists a \in A)((b = f(a) \wedge a \in C_1) \vee (b = f(a) \wedge a \in C_2)) \text{ (by the distributive law)} \\
&\Leftrightarrow (\exists a \in A)(b = f(a) \wedge a \in C_1) \vee (\exists a \in A)(b = f(a) \wedge a \in C_2) \\
& \text{ (by passing } \exists \text{ through } \vee) \\
&\Leftrightarrow b \in f(C_1) \vee b \in f(C_2) \text{ (by the definition of the inverse image)} \\
&\Leftrightarrow b \in f(C_1) \cup f(C_2) \text{ (by the definition of } \cup).
\end{aligned}$$

(f) Let $b \in B$.

$$\begin{aligned}
b \in f(C_1 \cap C_2) &\Leftrightarrow (\exists a \in A)(b = f(a) \wedge a \in C_1 \cap C_2) \text{ (by the definition of the image)} \\
&\Leftrightarrow (\exists a \in A)(b = f(a) \wedge a \in C_1 \wedge a \in C_2) \text{ (by the definition of } \cap) \\
&\Leftrightarrow (\exists a \in A)(b = f(a) \wedge a \in C_1 \wedge b = f(a) \wedge a \in C_2) \text{ (by idempotence of } \wedge) \\
&\Rightarrow (\exists a \in A)(b = f(a) \wedge a \in C_1) \wedge (\exists a \in A)(b = f(a) \wedge a \in C_2) \\
& \text{ (by passing } \exists \text{ through } \wedge) \\
&\Leftrightarrow b \in f(C_1) \wedge b \in f(C_2) \text{ (by the definition of the inverse image)} \\
&\Leftrightarrow b \in f(C_1) \cap f(C_2) \text{ (by the definition of } \cap).
\end{aligned}$$

Let us assume now that f is injective and let us show the converse. So, let us assume that $b \in f(C_1) \cap f(C_2)$ so that $b = f(a_1)$ for some $a_1 \in C_1$ and $b = f(a_2)$ for some $a_2 \in C_2$. Thus, we have that $f(a_1) = b = f(a_2)$ and from these relations we can deduce that $a_1 = a_2$ because f is injective. So, as $a_1 \in C_1$, $a_2 \in C_2$, and $a_1 = a_2$, we have that $a_1 \in C_1 \cap C_2$. Since $b = f(a_1)$, we obtain that $b \in f(C_1 \cap C_2)$.