Fundamentals of Mathematics
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Review for Exam 3

1. Show that the following pairs of sets have the same cardinality by explicitly producing a bijection between them.

   (a) The set of all positive integers and the set of even positive integers.

   (b) The interval $(5, 9)$ and the interval $(1, 7)$.

   (c) The interval $[5, 9]$ and the interval $[1, 7]$.

2. Consider the following sets.

$$\mathcal{P}(A),\ \ \mathcal{P}(B),\ \ A \times B,\ \ \mathcal{P}(A \times B),\ \ \mathcal{P}(A) \times B,\ \ A \times \mathcal{P}(B),\ \ \text{and}\ \ \mathcal{P}(A) \times \mathcal{P}(B)$$

   Determine the cardinality of the above sets given the cardinalities of A and B. Express your answers in terms of the given cardinalities of A and B.

   (a) $|A| = 3$ and $|B| = 2$.

   (b) $|A| = \aleph_0$ and $|B| = 2$.

3. Let $A_n = \omega - \{0, 1, 2, \ldots, n\}$ for $n \in \omega$. Determine the cardinality of the following sets.

$$A_n, \qquad A_n - A_{n+1}, \qquad \omega - A_n, \qquad \bigcap_{n\in\omega} A_n \qquad , \bigcup_{n\in\omega} A_n$$

4. Show that the relation $\approx$ (given by $A \approx B \Leftrightarrow (\exists f : A \to B) f$ is a bijection) is reflexive, symmetric and transitive.

5. If $A, B, C,$ and D are sets such that $|A| = |C|$ and $|B| = |D|$, show that $|A| \cdot |B| = |C| \cdot |D|$.

6. Show the following properties of the cardinal addition and multiplication.

   (a) $|A| + 0 = 0 + |A| = |A|$

   (b) $|A| + |B| = |B| + |A|$

   (c) $|A| \cdot 1 = 1 \cdot |A| = |A|$

   (d) $|A| \cdot |B| = |B| \cdot |A|$

7. Use induction to show that the following formulas hold for every natural number.

   (a)
$$0 + 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

   (b)
$$1 + 3 + 5 + \ldots + (2n + 1) = (n + 1)^2$$

(c) For every real number $x \neq 1$,

$$1 + x + x^2 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

8. Show the following statements on divisibility using induction.

   (a) $n^3 + 2n$ is divisible by 3 for any natural number n.

   (b) $6^n - 1$ is divisible by 5 for any natural number $n > 0$.

9. Use induction to show that the inequality

$$n! > 2^n$$

   holds for all $n \geq 4$.

10. If m and n are natural numbers, let $P(m, n)$ be the statement below.

    If $m \leq n$, then there is k such that $m + k = n$.

    Show that $P(m, n)$ is true for any m and n.

11. Show that the given formulas of the form $a_n = f(n)$ are closed forms of the given recursive sequences.

    (a) Recursive definition: $a_{n+1} = a_n + 5, a_1 = 3$. Closed form: $a_n = 5n - 2$.

    (b) Recursive definition: $a_{n+1} = 2a_n - a_{n-1}, a_0 = 2, a_1 = 5$. Closed form: $a_n = 3n + 2$.

    (c) Recursive definition: $a_{n+1} = 4a_n - 4a_{n-1}, a_0 = 0, a_1 = 2$. Closed form: $a_n = n2^n$.

Solutions

1. (a) Let $A = \{1, 2, 3, \ldots\}$ and $B = \{2, 4, 6, \ldots\}$ be the two sets as required. Then $f : A \to B$ mapping n onto $2n$ is a function. Its inverse $g : B \to A$ can be defined by mapping an even positive integer of the form $2n$ onto n. Then for any n, we have that $g(f(n)) = g(2n) = n$ so $g \circ f$ is the identity on A and $f(g(2n)) = f(n) = 2n$, so $f \circ g$ is the identity on B. Thus, f is invertible and, hence, a bijection.

   (b) Any linear function mapping the endpoints of the interval onto the endpoints of the interval can be used. For example, we can take the linear function with the slope $\frac{7-1}{9-5} = \frac{6}{4} = \frac{3}{2}$ such that $y = 1$ when $x = 5$. Thus, $y - 1 = \frac{3}{2}(x - 5) \Rightarrow y = \frac{3}{2}x - \frac{13}{2}$. Thus, let $f : (5, 9) \to (7, 1)$ be given by $f(x) = \frac{3}{2}x - \frac{13}{2}$. The formula for the inverse can be obtained by solving

$y = \frac{3}{2}x - \frac{13}{2}$ for $x : y + \frac{13}{2} = \frac{3}{2}x \Rightarrow x = \frac{2}{3}y + \frac{13}{3}$, so let $g : (7,1) \to (5,9)$ be given by $g(x) = \frac{2}{3}x + \frac{13}{3}$. Both compositions $g \circ f$ and $f \circ g$ are identity maps:

$$g(f(x)) = g\left(\frac{3}{2}x - \frac{13}{2}\right) = \frac{2}{3}\left(\frac{3}{2}x - \frac{13}{2}\right) + \frac{13}{3} = x - \frac{13}{3} + \frac{13}{3} = x \text{ and}$$

$$f(g(x)) = f\left(\frac{2}{3}x + \frac{13}{3}\right) = \frac{3}{2}\left(\frac{2}{3}x + \frac{13}{3}\right) - \frac{13}{2} = x + \frac{13}{2} - \frac{13}{2} = x$$

(c) Since f and g from the previous solution map the endpoints of the intervals onto the endpoints of the intervals, the same functions can be used.

2. (a) If $|A| = 3$ and $|B| = 2$, then $|\mathcal{P}(A)| = 2^3 = 8$, $|\mathcal{P}(B)| = 2^2 = 4$, $|A \times B| = 3 \cdot 2 = 6$, $|\mathcal{P}(A \times B)| = 2^6 = 64$, $|\mathcal{P}(A) \times B| = 8 \cdot 2 = 16$, $|A \times \mathcal{P}(B)| = 3 \cdot 4 = 12$, and $|\mathcal{P}(A) \times \mathcal{P}(B)| = 8 \cdot 4 = 32$.

   (b) If $|A| = \aleph_0$ and $|B| = 2$, then $|\mathcal{P}(A)| = 2^{\aleph_0}$, $|\mathcal{P}(B)| = 2^2 = 4$, $|A \times B| = \aleph_0 \cdot 2 = \aleph_0$, $|\mathcal{P}(A \times B)| = 2^{\aleph_0}$, $|\mathcal{P}(A) \times B| = 2^{\aleph_0} \cdot 2 = 2^{\aleph_0}$, $|A \times \mathcal{P}(B)| = \aleph_0 \cdot 4 = \aleph_0$, and $|\mathcal{P}(A) \times \mathcal{P}(B)| = 2^{\aleph_0} \cdot 4 = 2^{\aleph_0}$.

3. If $A_n = \omega - \{0, 1, 2, \ldots, n\} = \{n+1, n+2, \ldots\}$, then $|A_n| = \aleph_0$. $A_n - A_{n+1} = \{n+1, n+2, \ldots\} - \{n+2, n+3, \ldots\}) = \{n+1\}$ so $|A_n - A_{n+1}| = 1$. $\omega - A_n = \omega - \{n+1, n+2, \ldots\} = \{0, 1, \ldots, n\}$, so $|\omega - A_n| = n + 1$.

   Note that $A_0 = \{1, 2, \ldots\}$, $A_1 = \{2, 3, \ldots\}$, $A_2 = \{3, 4, \ldots\} \ldots$, so $\bigcap_{n\in\omega} A_n = \emptyset$ and $|\bigcap_{n\in\omega} A_n| = 0$. We also have that $\bigcup_{n\in\omega} A_n = \{1, 2, 3, \ldots\}$ so $|\bigcup_{n\in\omega} A_n| = |\omega| = \aleph_0$.

4. Since id_A is a bijection $A \to A$, we have that $A \approx A$, so $\approx$ is reflexive.

   If $A \approx B$, then there is a bijection $f : A \to B$. As f is a bijection, there is the inverse $f^{-1} : B \to A$ which is also a bijection. This shows that $B \approx A$ and so $\approx$ is symmetric.

   If $A \approx B$ and $B \approx C$, then there are bijections $f : A \to B$ and $g : B \to C$. By a problem from the previous review sheet, $g \circ f : A \to C$ is a bijection, so $A \approx C$. Thus, $\approx$ is transitive.

5. As $|A| = |C|$ and $|B| = |D|$, there are bijections $f : A \to C$ and $g : B \to D$.

   Since $|A| \cdot |B|$ is defined as $|A \times B|$ and $|C| \cdot |D|$ is defined as $|C \times D|$, we need to show that $|A \times B| = |C \times D|$. This means that we need to define a function $F : A \times B \to C \times D$ which will turn out to be a bijection. Let us define such a function $F : A \times B \to C \times D$ by $(a, b) \mapsto (f(a), g(b))$. If f^{-1} and g^{-1} are the inverses of f and g respectively, then let us also define $G : C \times D \to A \times B$ by $(c, d) \mapsto (f^{-1}(c), g^{-1}(d))$. Check that both $G \circ F$ and $F \circ G$ are the identities.

$$(G \circ F)(a, b) = G(F(a, b)) = G(f(a), g(b)) = (f^{-1}(f(a)), g^{-1}(g(b))) = (a, b) \text{ and}$$

$$(F \circ G)(c, d) = F(G(c, d)) = F(f^{-1}(c), g^{-1}(d)) = (f(f^{-1}(c)), g(g^{-1}(d))) = (c, d).$$

   Thus, F and G are bijections, so $|A \times B| = |C \times D|$ holds.

6. (a) Note that $|A| + 0$ is the cardinality of the set $(A \times \{\Box\}) \cup (\emptyset \times \{\triangle\})$. Since $\emptyset \times \{\triangle\} = \emptyset$, the above union is $A \times \{\Box\}$. This set has the same cardinality as A since the function

$f : A \to A \times \{\square\}$ given by $a \mapsto (a, \square)$ is one-to-one ($(a_1, \square) = (a_2, \square)$ implies $a_1 = a_2$) and onto (a is the original of $(a, \square)$).

One can show $0 + |A| = |A|$ similarly or, after having part (b), this relation follows from (b) and $|A| + 0 = |A.|$

(b) The function $f : (A \times \{\square\}) \cup (B \times \{\triangle\}) \to (B \times \{\square\}) \cup (A \times \{\triangle\})$ given by $(a, \square) \mapsto (a, \triangle)$ and $(b, \triangle) \mapsto (b, \square)$ is inverse to itself (check that $f(f(a, \square) = (a, \square)$ and $f(f(b, \triangle)) = (b, \triangle))$, so this shows that it is a bijection

(c) Let us use $\{0\}$ to represent 1. Checking that the function $f : A \to A \times \{0\}$ given by $a \mapsto (a, 0)$ is a bijection since $(a_1, 0) = (a_2, 0)$ implies $a_1 = a_2$ and a is the original of $(a, 0)$. This shows that $|A| \cdot 1 = |A|$. The relation $1 \cdot |A| = |A|$ can be shown analogously. Alternatively, it follows from part (b) and the relation $|A| \cdot 1 = |A|$.

(d) The function $f : A \times B \to B \times A$ given by $(a, b) \mapsto (b, a)$ is inverse to itself since

$$f(f(a, b)) = f(b, a) = (a, b).$$

So, it is invertible and, hence, a bijection. This shows that $|A \times B| = |B \times A|$ and so $|A| \cdot |B| = |B| \cdot |A|$.

7. (a) The formula holds for $n = 0$ since the left side consists of a single term 0 and the right side is $\frac{0(0+1)}{2} = 0$. Assume the formula holds for n and let us show it for $n + 1$. By induction hypothesis, the first equality below holds.

$$0 + 1 + 2 + \ldots + n + (n + 1) = \frac{n(n + 1)}{2} + (n + 1) = \frac{n(n + 1)}{2} + \frac{2(n + 1)}{2} =$$

$$\frac{n(n + 1) + 2(n + 1)}{2} = \frac{(n + 1)(n + 2)}{2}.$$

(b) The formula holds for $n = 0$ since the left side consists of a single term $2(0) + 1 = 1$ and the right side is $(0 + 1)^2 = 1$. Assume the formula holds for n and let us show it for $n + 1$. By induction hypothesis, the first equality below holds.

$$1 + 3 + 5 + \ldots + (2n + 1) + 2(n + 1) + 1 = (n + 1)^2 + 2(n + 1) + 1 =$$

$$n^2 + 2n + 1 + 2n + 2 + 1 = n^2 + 4n + 4 = (n + 2)(n + 2) = (n + 2)^2.$$

(c) Let x be a real number $x \neq 1$. The formula holds for $n = 0$ since the left side consists of a single term 1 and the right side is $\frac{1-x^{0+1}}{1-x} = \frac{1-x}{1-x} = 1$. Assume the formula holds for n and let us show it for $n + 1$. By induction hypothesis, the first equality below holds.

$$1 + x + x^2 + \ldots + x^n + x^{n+1} = \frac{1 - x^{n+1}}{1 - x} + x^{n+1} = \frac{1 - x^{n+1}}{1 - x} + \frac{x^{n+1}(1 - x)}{1 - x} =$$

$$\frac{1 - x^{n+1} + x^{n+1} 1 - x^{n+2}}{1 - x} = \frac{1 - x^{n+2}}{1 - x}.$$

8. (a) If $n = 0$, then $n^3 + 2n = 0$ and 0 is divisible by 3. Assume that $n^3 + 2n$ is divisible by 3. Recall that this means that $n^3 + 2n = 3k$ for some natural number k. Let us show that $(n + 1)^3 + 2(n + 1)$ is also divisible by 3. Try to write this last expression as a sum of

n^3+2n, so that we can use the induction hypothesis, and another term which is a multiple of 3 and, hence, divisible by 3. Foil $(n+1)^3$ to get n^3+3n^2+3n+1 so that we have the following.

$$(n+1)^3+2(n+1) = n^3+3n^2+3n+1+2n+2 = n^3+2n+3n^2+3n+3 =$$

$$(n^3+2n)+3(n^2+n+1) = 3k+3(n^2+n+1) = 3(k+n^2+n+1)$$

The last expression is divisible by 3 since it is a multiple of 3.

(b) Since $n > 0$, we start the induction at $n = 1$. For $n = 1$, $6^n - 1 = 6 - 1 = 5$ and it is divisible by 5. Assume that $6^n - 1$ is divisible by 5 and write $6^n - 1 = 5k$ for some natural number k. Let us show that $6^{n+1} - 1$ is divisible by 5. Note that $6^{n+1} - 1 = 6 \cdot 6^n - 1$. From the induction hypothesis $6^n - 1 = 5k$, we have that $6^n = 5k+1$. Substituting $5k+1$ for 6^n in the inductive step, we have the following.

$$6^{n+1} - 1 = 6 \cdot 6^n - 1 = 6(5k+1) - 1 = 30k+6-1 = 30k+5 = 5(6k+1)$$

This last expression is divisible by 5 since it is a multiple of 5.

9. Use the limited induction starting with $n = 4$. The formula $n! > 2^n$ holds for $n = 4$ since it becomes $4! = 24 > 16 = 2^4$. Assume the formula to be true for n and let us show it for $n+1$. Note that $n+1 > 2$ for any $n \geq 4$ because $n+1$ is taking values $5, 6, 7, \ldots$ and they are all larger than 2. So, we have that

$$(n+1)! = n! \cdot (n+1) > 2^n \cdot (n+1) > 2^n \cdot 2 = 2^{n+1}$$

where the first relation holds by the recursive definition of the factorial, the second relation holds by the inductive hypothesis and the third relation holds by the observation that $n+1 > 2$ for $n \geq 4$.

10. When $m = n = 0$, the statement reduces to a true implication since the premise $0 \leq 0$ is true and the conclusion is true for $k = 0$.

Assuming that $P(0, n)$ holds, let us show that $P(0, n+1)$ holds. So, assume that the premise $0 \leq n+1$ of $P(0, n+1)$ holds. Taking $n+1$ for k, we have that $0 + k = k = n+1$. This concludes the proof of the first step.

To show the second step, assume that $P(m, n)$ holds and let us show $P(m+1, n)$. Assume that the assumption $m+1 \leq n$ of $P(m+1, n)$ holds. Hence $m < m+1 \leq n$ holds so the assumption $m \leq n$ of $P(m, n)$ also holds. So, there is unique l such that $m + l = n$. Since $m < n$, such l is strictly larger than zero (assuming otherwise $l = 0$ leads to a contradiction $m = m + 0 = n$). As $l > 0$, l is a successor of its predecessor, so $l = k+1$ for some natural number k. We have that $(m+1)+k = m+(k+1)$ by associativity and commutativity of $+$, so

$$(m+1)+k = m+(k+1) = m+l = n$$

where the last equality holds by the induction hypothesis.

11. (a) As the initial term is given with $n = 1$, use limited induction and show the claim for all $n \geq 1$. The closed form matches the recursive equation for $n = 1$ since $a_1 = 3$ and $5(1) - 2 = 3$. Assuming the closed form and the recursive formula to agree for n, let us show that they agree for $n+1$. On one hand, $a_{n+1} = a_n + 5 = 5n - 2 + 5 = 5n + 3$. On the other hand, $a_{n+1} = 5(n+1) - 2 = 5n + 5 - 2 = 5n + 3$. Thus, the two formulas match.

(b) The closed form matches the recursive equation for $n = 0$ since $a_0 = 2$ and $3(0) + 2 = 2$. Use complete induction, so assume the two formulas to match for all $k \leq n$ and show that $a_{n+1} = 3(n+1) + 2 = 3n + 5$ using the recursive formula. This holds by the argument below.

$$a_{n+1} = 2a_n - a_{n-1} = 2(3n+2) - (3(n-1)+2) = 6n + 4 - 3n + 3 - 2 = 3n + 5.$$

(c) The closed form matches the recursive equation for $n = 0$ since $a_0 = 0$ and $(0)2^0 = 0$. Use complete induction, so assume the two formulas to agree for all $k \leq n$ and show that $a_{n+1} = (n+1)2^{n+1}$.

$$a_{n+1} = 4a_n - 4a_{n-1} = 4n2^n - 4(n-1)2^{n-1} = 4 \cdot 2^{n-1}(n \cdot 2 - (n-1)) =$$

$$2^2 \cdot 2^{n-1}(2n - n + 1) = 2^{n+1}(n+1) = (n+1)2^{n+1}.$$