Fundamentals of Mathematics
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Review for Exam 3

1. Show that the following pairs of sets have the same cardinality by explicitly producing a bijection between them.
   (a) The set of all positive integers and the set of even positive integers.
   (b) The interval $(5, 9)$ and the interval $(1, 7)$.
   (c) The interval $[5, 9]$ and the interval $[1, 7]$.

2. Consider the following sets.

$$\mathcal{P}(A),\ \ \mathcal{P}(B),\ \ A\times B,\ \ \mathcal{P}(A\times B),\ \ \mathcal{P}(A)\times B,\ \ A\times\mathcal{P}(B),\ \text{ and }\ \mathcal{P}(A)\times\mathcal{P}(B)$$

   Determine the cardinality of the above sets given the cardinalities of A and B. Express your answers in terms of the given cardinalities of A and B.
   (a) $|A| = 3$ and $|B| = 2$.
   (b) $|A| = \aleph_0$ and $|B| = 2$.

3. Determine the cardinality of the following sets.

$$A_n, \qquad \omega - A_n, \qquad \bigcap_{n\in\omega} A_n \qquad , \bigcup_{n\in\omega} A_n$$

   for each of the sets $A_n, n \in \omega$ given below.
   (a) $A_n = \{0, \ldots, n\}$. Determine also the cardinality of $A_{n+1} - A_n$.
   (b) $A_n = \omega - \{0, 1, 2, \ldots, n\}$. Determine also the cardinality of $A_n - A_{n+1}$.
   (c) $A_n = \{2n, 2n + 1, 2n + 2, \ldots, 3n\}$

4. Show that the relation $\approx$ (given by $A \approx B \Leftrightarrow (\exists f : A \to B) f$ is a bijection) is reflexive, symmetric and transitive.

5. If A, B, C, and D are sets such that $|A| = |C|$ and $|B| = |D|$, show that $|A| \cdot |B| = |C| \cdot |D|$.

6. Show the following properties of the cardinal multiplication.
   (a) $|A| \cdot 1 = 1 \cdot |A| = |A|$ (b) $|A| \cdot |B| = |B| \cdot |A|$

7. Use induction to show that the following formulas hold for every natural number.
   (a)

$$0 + 1 + 2 + \ldots + n = \frac{n(n + 1)}{2}$$

(b)
$$1+3+5+\ldots+(2n+1)=(n+1)^2$$

(c) For every real number $x \neq 1$,
$$1+x+x^2+\ldots+x^n=\frac{1-x^{n+1}}{1-x}$$

8. Show the following statements on divisibility using induction.

   (a) n^3+2n is divisible by 3 for any natural number n.

   (b) 6^n-1 is divisible by 5 for any natural number $n>0$.

9. Use induction to show that the given inequalities hold for the specified natural numbers.

   (a) $2n>n+2$ for all $n \geq 3$.

   (b) $n! > 2^n$ for all $n \geq 4$.

10. Show the inequality $(m+1)^n > mn$ for any natural numbers $m \geq 1$ and $n \geq 1$.

11. Show that the given formulas of the form $a_n = f(n)$ are closed forms of the given recursive sequences.

    (a) Recursive definition: $a_{n+1}=a_n+5, a_1=3$. Closed form: $a_n=5n-2$.

    (b) Recursive definition: $a_{n+1}=2a_n-a_{n-1}, a_0=2, a_1=5$. Closed form: $a_n=3n+2$.

    (c) Recursive definition: $a_{n+1}=4a_n-4a_{n-1}, a_0=0, a_1=2$. Closed form: $a_n=n2^n$.

Solutions

1. (a) Let $A=\{1,2,3,\ldots\}$ and $B=\{2,4,6,\ldots\}$ be the two sets as required. Then $f: A \to B$ mapping n onto $2n$ is a function. Its inverse $g: B \to A$ can be defined by mapping an even positive integer of the form $2n$ onto n. Then for any n, we have that $g(f(n))=g(2n)=n$ so $g \circ f$ is the identity on A and $f(g(2n))=f(n)=2n$, so $f \circ g$ is the identity on B. Thus, f is invertible and, hence, a bijection.

   (b) Any linear function mapping the endpoints of the interval onto the endpoints of the interval can be used. For example, we can take the linear function with the slope $\frac{7-1}{9-5}=\frac{6}{4}=\frac{3}{2}$ such that $y=1$ when $x=5$. Thus, $y-1=\frac{3}{2}(x-5) \Rightarrow y=\frac{3}{2}x-\frac{13}{2}$. Thus, let $f:(5,9)\to(7,1)$ be given by $f(x)=\frac{3}{2}x-\frac{13}{2}$. The formula for the inverse can be obtained by solving $y=\frac{3}{2}x-\frac{13}{2}$ for x : $y+\frac{13}{2}=\frac{3}{2}x \Rightarrow x=\frac{2}{3}y+\frac{13}{3}$, so let $g:(7,1)\to(5,9)$ be given by $g(x)=\frac{2}{3}x+\frac{13}{3}$. Both compositions $g \circ f$ and $f \circ g$ are identity maps:
$$g(f(x))=g\left(\frac{3}{2}x-\frac{13}{2}\right)=\frac{2}{3}\left(\frac{3}{2}x-\frac{13}{2}\right)+\frac{13}{3}=x-\frac{13}{3}+\frac{13}{3}=x \text{ and}$$

$$f(g(x)) = f\left(\frac{2}{3}x + \frac{13}{3}\right) = \frac{3}{2}\left(\frac{2}{3}x + \frac{13}{3}\right) - \frac{13}{2} = x + \frac{13}{2} - \frac{13}{2} = x$$

(c) Since f and g from the previous solution map the endpoints of the intervals onto the endpoints of the intervals, the same functions can be used.

2. (a) If $|A| = 3$ and $|B| = 2$, then $|\mathcal{P}(A)| = 2^3 = 8$, $|\mathcal{P}(B)| = 2^2 = 4$, $|A \times B| = 3 \cdot 2 = 6$, $|\mathcal{P}(A \times B)| = 2^6 = 64$, $|\mathcal{P}(A) \times B| = 8 \cdot 2 = 16$, $|A \times \mathcal{P}(B)| = 3 \cdot 4 = 12$, and $|\mathcal{P}(A) \times \mathcal{P}(B)| = 8 \cdot 4 = 32$.

   (b) If $|A| = \aleph_0$ and $|B| = 2$, then $|\mathcal{P}(A)| = 2^{\aleph_0}$, $|\mathcal{P}(B)| = 2^2 = 4$, $|A \times B| = \aleph_0 \cdot 2 = \aleph_0$, $|\mathcal{P}(A \times B)| = 2^{\aleph_0}$, $|\mathcal{P}(A) \times B| = 2^{\aleph_0} \cdot 2 = 2^{\aleph_0}$, $|A \times \mathcal{P}(B)| = \aleph_0 \cdot 4 = \aleph_0$, and $|\mathcal{P}(A) \times \mathcal{P}(B)| = 2^{\aleph_0} \cdot 4 = 2^{\aleph_0}$.

3. (a) $A_n = \{0, \ldots, n\}$ has $n + 1$ element. To see this, note that there are n elements from 1 to n and that 0 is the $n + 1$-st element. Alternatively, note that $A_0 = \{0\}$ has one element,$A_1 = \{0, 1, \}$ has 2 elements, $A_2 = \{0, 1, 2\}$ has 3 elements, so A_n has $n + 1$ element.

   $\omega - A_n = \omega - \{0, 1, \ldots, n\} = \{n + 1, n + 2, \ldots\}$ is a countably infinite set. Hence, $|\omega - A_n| = \aleph_0$.

   Since A_0 is contained in all $A_n, n \in \omega$, we have that $\bigcap_{n\in\omega} A_n = A_0 = \{0\}$ and $|\bigcap_{n\in\omega} A_n| = 1$.

   Every n is in A_n, so $\bigcup_{n\in\omega} A_n = \{0, 1, 2, \ldots\} = \omega$, and $|\bigcup_{n\in\omega} A_n| = \aleph_0$.

   $A_{n+1} - A_n = \{0, 1, \ldots, n + 1\} - \{0, 1, \ldots, n\} = \{n + 1\}$, so $|A_{n+1} - A_n| = 1$.

   (b) If $A_n = \omega - \{0, 1, 2, \ldots, n\} = \{n + 1, n + 2, \ldots\}$, then $|A_n| = \aleph_0$.

   $\omega - A_n = \omega - \{n + 1, n + 2, \ldots\} = \{0, 1, \ldots, n\}$, so $|\omega - A_n| = n + 1$.

   Note that $A_0 = \{1, 2, \ldots\}, A_1 = \{2, 3, \ldots\}, A_2 = \{3, 4, \ldots\} \ldots$, so $\bigcap_{n\in\omega} A_n = \emptyset$ and $|\bigcap_{n\in\omega} A_n| = 0$.

   We have that $\bigcup_{n\in\omega} A_n = \{1, 2, 3, \ldots\}$ so $|\bigcup_{n\in\omega} A_n| = |\omega| = \aleph_0$.

   $A_n - A_{n+1} = \{n + 1, n + 2, \ldots\} - \{n + 2, n + 3, \ldots\}) = \{n + 1\}$ so $|A_n - A_{n+1}| = 1$.

   (c) $A_n = \{2n, 2n + 1, 2n + 2, \ldots, 3n\} = \{2n + 0, 2n + 1, 2n + 2, \ldots 2n + n\}$. Note that we are adding values $0, 1, \ldots, n$ to n to obtain the elements of A_n. As there are $n+1$ elements in the list $0, 1, \ldots, n$, $|A_n| = n + 1$.

   Alternatively, write down A_n for the first few natural numbers and determine the pattern of their cardinalities.

$$A_0 = \{0\}, \quad A_1 = \{2, 3\}, \quad A_2 = \{4, 5, 6\}, \quad A_3 = \{6, 7, 8, 9\}, \quad A_4 = \{8, 9, \ldots 12\} \ldots$$

   Note that A_0 has 1 element, A_1 has 2 elements, A_2 has 3 elements. This argument continues so you can deduce that A_n has $n + 1$ element.

   $\omega - A_n = \omega - \{2n, 2n + 1, \ldots, 3n\} = \{0, 1, \ldots, 2n - 1\} \cup \{3n + 1, 3n + 2, \ldots\}$. Since the second term of this last union is countably infinite, $|\omega - A_n| = \aleph_0$.

   Since $A_0 \cap A_1 = \emptyset$, we have that $\bigcap_{n\in\omega} A_n = \emptyset$ and $|\bigcap_{n\in\omega} A_n| = 0$.

   If $n = 2m$, then n is in A_m. If $n = 2m+1$, then n is also in A_m, so $\bigcup_{n\in\omega} A_n = \{0, 1, 2, \ldots\} = \omega$, and $|\bigcup_{n\in\omega} A_n| = \aleph_0$.

4. Since id_A is a bijection $A \to A$, we have that $A \approx A$, so $\approx$ is reflexive.

   If $A \approx B$, then there is a bijection $f : A \to B$. As f is a bijection, there is the inverse $f^{-1} : B \to A$ which is also a bijection. This shows that $B \approx A$ and so $\approx$ is symmetric.

   If $A \approx B$ and $B \approx C$, then there are bijections $f : A \to B$ and $g : B \to C$. By a problem from the previous review sheet, $g \circ f : A \to C$ is a bijection, so $A \approx C$. Thus, $\approx$ is transitive.

5. As $|A| = |C|$ and $|B| = |D|$, there are bijections $f : A \to C$ and $g : B \to D$.

   Since $|A| \cdot |B|$ is defined as $|A \times B|$ and $|C| \cdot |D|$ is defined as $|C \times D|$, we need to show that $|A \times B| = |C \times D|$. This means that we need to define a function $F : A \times B \to C \times D$ which will turn out to be a bijection. Let us define such a function $F : A \times B \to C \times D$ by $(a, b) \mapsto (f(a), g(b))$. If f^{-1} and g^{-1} are the inverses of f and g respectively, then let us also define $G : C \times D \to A \times B$ by $(c, d) \mapsto (f^{-1}(c), g^{-1}(d))$. Check that both $G \circ F$ and $F \circ G$ are the identities.

   $$(G \circ F)(a, b) = G(F(a, b)) = G(f(a), g(b)) = (f^{-1}(f(a)), g^{-1}(g(b))) = (a, b) \text{ and}$$

   $$(F \circ G)(c, d) = F(G(c, d)) = F(f^{-1}(c), g^{-1}(d)) = (f(f^{-1}(c)), g(g^{-1}(d))) = (c, d).$$

   Thus, F and G are bijections, so $|A \times B| = |C \times D|$ holds.

6. (a) Let us use $\{0\}$ to represent 1. Checking that the function $f : A \to A \times \{0\}$ given by $a \mapsto (a, 0)$ is a bijection since $(a_1, 0) = (a_2, 0)$ implies $a_1 = a_2$ and a is the original of $(a, 0)$. This shows that $|A| \cdot 1 = |A|$. The relation $1 \cdot |A| = |A|$ can be shown analogously.

   (b) The function $f : A \times B \to B \times A$ given by $(a, b) \mapsto (b, a)$ is inverse to itself since

   $$f(f(a, b)) = f(b, a) = (a, b).$$

   So, it is invertible and, hence, a bijection. This shows that $|A \times B| = |B \times A|$ and so $|A| \cdot |B| = |B| \cdot |A|$.

7. (a) Let $P(n)$ denote the given identity. Start by writing what $P(0)$ is so that you can tell what the first induction step is.

   When $n = 0$, the left side consists of a single term 0 because the formula starts with zero and it ends with zero. The right side is $\frac{0(0+1)}{2} = 0$. So, $P(0)$ is the statement 0=0 which is true. This shows the induction base.

   To show the inductive step, assume that $P(n)$ holds and show that $P(n+1)$ holds. Start by writing down what $P(n+1)$ is so that you can tell which statement you are to show. So, $P(n+1)$ is obtained by replacing every instance of n in $P(n)$ by $n+1$, producing

   $$0 + 1 + 2 + \ldots + n + (n+1) = \frac{(n+1)(n+2)}{2}.$$

   Note that the part $0 + 1 + 2 + \ldots + n$ in the above formula is the left side of the identity $P(n)$, so this can be replaced by the right side of $P(n)$ since we assumed that $P(n)$ is true. Thus,

   $$\begin{aligned} 0 + 1 + 2 + \ldots + n + (n+1) &= \tfrac{n(n+1)}{2} + (n+1) && \text{(by the induction hypothesis)} \\ &= \tfrac{n(n+1)+2(n+1)}{2} && \text{(writing the terms with a common denominator} \\ &= \tfrac{(n+1)(n+2)}{2} && \text{(factoring } n+1\text{)} \end{aligned}$$

(b) Let $P(n)$ denote the given identity. Start by writing what $P(0)$ is.

When $n = 0$, the left side consists of a single term 1 because the formula starts with zero and it ends with $2(0) + 1 = 1$. The right side is $(0 + 1)^2 = 1$. So, $P(0)$ is the statement 1=1 which is true. This shows the induction base.

To show the inductive step, assume that $P(n)$ holds and show that $P(n+1)$ holds. Start by writing down what $P(n + 1)$ is so that you can tell which statement you are to show. So, $P(n + 1)$ is

$$1 + 3 + 5 + \ldots + (2n + 1) + (2n + 3) = (n + 2)^2.$$

Note that the last term of the left side is obtained by substituting n in the last term of the left side of $P(n)$ by $n + 1$ and $2(n + 1) + 1 = 2n + 2 + 1 = 2n + 3$. By the induction hypothesis, the part $1 + 3 + 5 + \ldots + (2n + 1)$ of the left side of $P(n + 1)$ can be replaced by $(n + 1)^2$. Hence,

$$\begin{aligned}
1 + 3 + 5 + \ldots + (2n + 1) + (2n + 3) &= (n + 1)^2 + (2n + 3) && \text{(by the induction hypothesis)}\\
&= n^2 + 2n + 1 + 2n + 3 && \text{(squaring)}\\
&= n^2 + 4n + 4 && \text{(simplifying)}\\
&= (n + 2)^2 && \text{(factoring)}
\end{aligned}$$

(c) Let $P(n)$ be the given formula for a real number $x \neq 1$. If $n = 0$, the left side consists of a single term 1 and the right side is $\frac{1-x^{0+1}}{1-x} = \frac{1-x}{1-x} = 1$. Hence, $P(0)$ holds. Assume that $P(n)$ holds and show that $P(n+1)$ holds. So, we need to show that $1 + x + x^2 + \ldots + x^n + x^{n+1}$ is equal to $\frac{1-x^{n+2}}{1-x}$.

$$\begin{aligned}
1 + x + x^2 + \ldots + x^n + x^{n+1} &= \tfrac{1-x^{n+1}}{1-x} + x^{n+1} && \text{(by the induction hypothesis)}\\
&= \tfrac{1-x^{n+1}}{1-x} + \tfrac{x^{n+1}(1-x)}{1-x} && \text{(finding the common denominator)}\\
&= \tfrac{1-x^{n+1}+x^{n+1}-x^{n+2}}{1-x} && \text{(algebra)}\\
&= \tfrac{1-x^{n+2}}{1-x} && \text{(cancelling } x^{n+1} \text{ in the numerator)}
\end{aligned}$$

8. (a) Let $P(n)$ denote the statement "$n^3 + 2n$ is divisible by 3". Hence, $P(n)$ is the sentence $(\exists k)\, n^3 + 2n = 3k$.

If $n = 0$, then $n^3 + 2n = 0$ and 0 is 3(0), so $P(0)$ holds by taking k to be zero. Assume that $P(n)$ holds. Hence, $n^3 + 2n = 3k$ for some natural number k. Let us show that $P(n + 1)$ holds, i.e. that $(n + 1)^3 + 2(n + 1) = 3l$ for some natural number l. To be able to use the induction hypothesis, try to write $(n + 1)^3 + 2(n + 1)$ as a sum of $n^3 + 2n$ and another term. Foil $(n + 1)^3$ to get $n^3 + 3n^2 + 3n + 1$ so that we have the following.

$$\begin{aligned}
(n + 1)^3 + 2(n + 1) &= n^3 + 3n^2 + 3n + 1 + 2n + 2 && \text{(foiling)}\\
&= (n^3 + 2n) + 3n^2 + 3n + 3 && \text{(simplifying and commuting)}\\
&= 3k + 3n^2 + 3n + 3 && \text{(by the induction hypothesis)}\\
&= 3(k + n^2 + n + 1) && \text{(by factoring 3)}
\end{aligned}$$

Hence, by taking $l = k + n^2 + n + 1$, we have that $(n + 1)^3 + 2(n + 1) = 3l$.

(b) Let $P(n)$ be the statement $(\exists k)6^n - 1 = 5k$. For $n = 0$, $6^0 - 1 = 1 - 1 = 0$ so $P(0)$ holds by taking k to be zero so that $6^0 - 1 = 0 = 5(0)$.

Assume that $P(n)$ holds so that $6^n - 1 = 5k$ for some natural number k and let us show that $P(n+1)$ holds, so that $6^{n+1} - 1 = 5l$ for some natural number l. Note that $6^{n+1} - 1 = 6 \cdot 6^n - 1$. From the induction hypothesis $6^n - 1 = 5k$, we have that $6^n = 5k + 1$. Substituting $5k + 1$ for 6^n in the inductive step, we have the following.

$$\begin{aligned} 6^{n+1} - 1 &= 6 \cdot 6^n - 1 && \text{(algebra)} \\ &= 6(5k+1) - 1 && \text{(by the induction hypothesis)} \\ &= 30k + 6 - 1 = 30k + 5 && \text{(algebra)} \\ &= 5(6k+1) && \text{(by factoring 5)} \end{aligned}$$

Thus, we can take l to be $6k + 1$ and have that $P(n+1)$ holds.

9. (a) Let $P(n)$ denote the inequality $2n > n+2$. The induction base is to show that $P(3)$ holds. When $n = 3$, the inequality becomes $2(3) > 3 + 2$ which is true since $6 > 5$.

   Assuming that $P(n)$ holds, let us show that $P(n+1)$ holds. Hence, we need to show the inequality $2(n+1) > n + 1 + 2$, that is the inequality $2n + 2 > n + 3$.

$$\begin{aligned} 2n + 2 &> n + 2 + 2 && \text{(by the induction hypothesis)} \\ &> n + 2 + 1 && \text{(since } 2 > 1\text{)} \\ &= n + 3 && \text{(algebra)} \end{aligned}$$

   (b) Use the limited induction starting with $n = 4$. The formula $n! > 2^n$ holds for $n = 4$ since it becomes $4! = 24 > 16 = 2^4$. Assume the formula to be true for n and let us show it for $n + 1$. Note that $n + 1 > 2$ for any $n \geq 4$ because $n + 1$ is taking values $5, 6, 7, \ldots$ and they are all larger than 2. So, we have that

$$(n+1)! = n! \cdot (n+1) > 2^n \cdot (n+1) > 2^n \cdot 2 = 2^{n+1}$$

   where the first relation holds by the recursive definition of the factorial, the second relation holds by the inductive hypothesis and the third relation holds by the observation that $n + 1 > 2$ for $n \geq 4$.

10. Let $P(m, n)$ denote the given inequality. Start by showing that $P(1, 1)$ holds. This is correct since $(1 + 1)^1 > (1)(1)$ is true as $2 > 1$. Then, assume that $P(m, 1)$ is true and show that $P(m + 1, 1)$ is true. Note that the left side of $P(m, 1)$ is $(m + 1)^1 = m + 1$ and the right side is $(m)(1)$, so we are assuming that $m + 1 > m$ (which is, in fact, true for every natural number m.) The statement we need to show, $P(m + 1, 1)$, is $(m + 1 + 1)^1 > (m + 1)(1)$, so we need to show that $m + 2 > m + 1$. This is also true for every natural number m even without using the induction hypothesis.

    Hence, it remains to show the implication $P(m, n) \Rightarrow P(m, n + 1)$. So, assume that $P(m, n)$ holds and let us show that $P(m, n + 1)$ holds which amounts to showing that $(m + 1)^{n+1} > m(n + 1)$.

$$\begin{aligned} (m+1)^{n+1} &= (m+1)^n \cdot (m+1) && \text{(algebra)} \\ &> mn \cdot (m+1) && \text{(by the induction hypothesis)} \\ &= m^2 n + mn && \text{(algebra)} \\ &\geq 1 \cdot mn + mn && \text{(since } m \geq 1\text{)} \\ &= mn + m \cdot 1 && \text{(since } n \geq 1\text{)} \\ &= m(n+1) && \text{(algebra)} \end{aligned}$$

11. (a) As the initial term is given with $n = 1$, use limited induction and show the claim for all $n \geq 1$.

Let $P(n)$ be $a_n = 5n - 2$. For $n = 1$, $a_1 = 3$ and $5(1) - 2 = 3$, so $P(1)$ is the true statement 3=3.

Assume that $P(n)$ holds so that $a_n = 5n - 2$ and show that $P(n+1)$ holds, i.e. that $a_{n+1} = 5(n+1) - 2 = 5n + 5 - 2 = 5n + 3$.

$$\begin{aligned} a_{n+1} &= a_n + 5 && \text{(by the recursive formula)} \\ &= 5n - 2 + 5 && \text{(by the induction hypothesis)} \\ &= 5n + 3 && \text{(algebra)} \end{aligned}$$

(b) Let $P(n)$ be the relation $a_n = 3n+2$. For $n = 0$, we have that $a_0 = 2$ and that $3(0)+2 = 2$, so $P(0)$ is the true statement $2 = 2$.

Use complete induction and assume that $P(k)$ holds for all $k \leq n$. In particular, assume that $P(n)$ and $P(n-1)$ hold, so that $a_n = 3n + 2$ and $a_{n-1} = 3(n-1) + 2$. Show that $P(n+1)$ holds, i.e. that a_{n+1} can be computed as $3(n+1) + 2 = 3n + 5$.

$$\begin{aligned} a_{n+1} &= 2a_n - a_{n-1} && \text{(by the given recursive formula)} \\ &= 2(3n+2) - (3(n-1)+2) && \text{(by the induction hypothesis)} \\ &= 6n + 4 - 3n + 3 - 2 = && \text{(algebra)} \\ &= 3n + 5 && \text{(algebra)} \end{aligned}$$

(c) Let $P(n)$ be $a_n = n2^n$. For $n = 0$, $a_0 = 0$ and $(0)2^0 = 0$, so $P(0)$ is the true statement 0=0.

Assume that $P(k)$ holds for all $k \leq n$. In particular, assume that $P(n)$ and $P(n-1)$ hold so that $a_n = n2^n$ and $a_{n-1} = (n-1)2^{n-1}$ and show that $P(n+1)$ holds, i.e. that $a_{n+1} = (n+1)2^{n+1}$.

$$\begin{aligned} a_{n+1} &= 4a_n - 4a_{n-1} && \text{(by the recursive formula)} \\ &= 4n2^n - 4(n-1)2^{n-1} && \text{(by the induction hypothesis)} \\ &= 4 \cdot 2^{n-1}(n \cdot 2 - (n-1)) && \text{(factor the common terms)} \\ &= 2^2 \cdot 2^{n-1}(2n - n + 1) && \text{(algebra)} \\ &= 2^{n+1}(n+1) = (n+1)2^{n+1} && \text{(algebra)} \end{aligned}$$