Fundamentals of Mathematics
Lia Vas

Review for Exam 4

1. (a) Show that the relation $\sim$ used to define $\mathbb{Z}$ and given on $\mathbb{N}\times\mathbb{N}$ by
$$(k,l)\sim(m,n) \quad\Leftrightarrow\quad k+n=l+m$$
is an equivalence relation.
   (b) Show that the integer $[(0,0)]$ is the identity for addition (i.e. show that $[(m,n)]+[(0,0)] = [(m,n)]$ and $[(0,0)]+[(m,n)] = [(m,n)]$).
   (c) For any integer $[(m,n)]$, show that its additive inverse is $[(n,m)]$ (i.e. show that $[(m,n)]+[(n,m)] = [(0,0)]$ and $[(n,m)]+[(m,n)] = [(0,0)]$).
   (d) Show that $[(n,m)] = [(0,m-n)]$ if $m \geq n$ and $[(n,m)] = [(n-m,0)]$ if $m < n$ for any natural numbers m and n.

2. The relation $\leq$ on $\mathbb{Z}$ matches the familiar order of integers when $[(m,n)]$ is shortened to $m-n$. Rearrange the integer numbers below, if needed, so that the elements in the new list are non-decreasing.
$$[(6,3)],\quad [(1000,1005)],\quad [(6,8)],\quad [(57,56)],\quad [(56,58)]$$

3. If $[(m,n)]\in\mathbb{Q}$ is such that $m\neq 0$, show that $[(m,n)]\cdot[(n,m)] = [(1,1)]$ and $[(n,m)]\cdot[(m,n)] = [(1,1)]$).).

4. For integers k,l,m,n such that $nl>0$, the relation $\leq$ on the set of rational numbers $\mathbb{Q}$ is given by
$$[(k,l)]\leq[(m,n)] \quad\text{if and only if}\quad kn\leq lm$$
where the relation $\leq$ on the right side of the equivalence above is the relation $\leq$ on $\mathbb{Z}$. Show that the relation $\leq$ on $\mathbb{Q}$ is reflexive and antisymmetric assuming that the relation $\leq$ on $\mathbb{Z}$ is reflexive and antisymmetric.

5. The relation $\leq$ on $\mathbb{Q}$ matches the familiar order of rationals when $[(m,n)]$ is shortened to $\frac{m}{n}$. Rearrange the rational numbers below, if needed, so that the elements in the new list are non-decreasing.
$$[(5,15)],\quad [(50,-100)],\quad [(15,10)],\quad [(20,-10)],\quad [(-10,20)]$$

6. Show that $\mathbb{Q}$ has no zero divisors, that is show that $ab=0\Rightarrow a=0$ or $b=0$ for all $a,b\in\mathbb{Q}$. Given the way how rational numbers are defined and that $0=[(0,1)]$, this translates to showing that $[(m,n)]\cdot[(k,l)] = [(0,1)] \Rightarrow [(m,n)] = [(0,1)]$ or $[(k,l)] = [(0,1)]$ for any $m,n,k,l\in\mathbb{Z}$ such that $n\neq 0$ and $l\neq 0$. You can assume that $\mathbb{Z}$ has no zero divisors (that is: $mn=0\Rightarrow m=0$ or $n=0$ for $m,n\in\mathbb{Z}$).

7. Find the limit of the following recursive sequences.

   (a) $a_{n+1}=\sqrt{2+a_n}, \quad a_0=0$ (b) $a_{n+1}=\frac{1}{1+a_n}, \quad a_0=1$.

8. Show that the following pairs of sets are in a bijective correspondence. You can assume the existence of any of the bijective correspondences from the formula sheet.

   (a) $(3,5)\cup[8,9)$ and $(7,\infty)$ (b) $(3,5]\cup[0,9)\cup[7,\infty)$ and $(-\infty,1]$

   (c) $\bigcup_{n\in\mathbb{N}-\{0\}}(-n,n)$ and $(0,1)$ (d) $\bigcap_{n\in\mathbb{N}}[0,n+1)$ and $\mathbb{R}$

   (e) $\bigcup_{n\in\mathbb{N}}(-\infty,-n)$ and $(1,\infty)$

9. Represent the following decimal numbers as quotients of two integer numbers.

   (a) $0.222222\ldots$ (b) $0.27272727\ldots$ (c) $1.2345454545\ldots$

10. Determine the moduli and the arguments given the following complex numbers in algebraic forms: $-3i$, $\sqrt{2}-\sqrt{2}i$, $-\sqrt{3}+i$, $-2-i$.

11. Determine the real and imaginary parts of the complex numbers given by their moduli and arguments: $\theta=\frac{-\pi}{2}, r=5$; $\theta=\frac{5\pi}{6}, r=2$; $\theta=\frac{-2\pi}{3}, r=3$.

12. Determine the n-th power of the given complex numbers and given n. Express your answers in algebraic form. (a) $z=-\sqrt{3}+i$, $n=4$; (b) $z=-2-i$, $n=6$.

13. Find all solutions of the following equations.

   (a) $z^5-32=0$ (b) $z^5+32=0$

   (c) $z^4=3+3i$ (d) $z^4=3-3i$

14. Using the definitions of the complex-valued trigonometric functions $\sin z=\frac{1}{2i}(e^{iz}-e^{-iz})$ and $\cos z=\frac{1}{2}(e^{iz}+e^{-iz})$, show the identities below.

   (a) $e^{iz}=\cos z+i\sin z$ (b) $\sin^2 z+\cos^2 z=1$

   (c) $\sin^2 z=\frac{1}{2}(1-\cos(2z))$ (d) $\cos^2 z=\frac{1}{2}(1+\cos(2z))$

Solutions

1. (a) *Reflexivity.* We need to show that $(k,l)\sim(k,l)$ for any nonnegative integers k and l.

$$\begin{aligned}(k,l)\sim(k,l)\ &\Leftrightarrow\ k+l=l+k && \text{(by the definition of }\sim)\\ &\Leftrightarrow\ k+l=k+l && \text{(by commutativity of }+)\\ &\Leftrightarrow\ \top && \text{(by the reflexivity of }=)\end{aligned}$$

Symmetry. Assume that $(k,l)\sim(m,n)$ for some $k,l,m,n\in\mathbb{N}$ and show that $(m,n)\sim(k,l)$.

$$\begin{aligned}(k,l)\sim(m,n)\ &\Leftrightarrow\ k+n=l+m && \text{(by the definition of }\sim)\\ &\Leftrightarrow\ m+l=n+k && \text{(by commutativity of + and symmetry of }=)\\ &\Leftrightarrow\ (m,n)\sim(k,l) && \text{(by the definition of }\sim)\end{aligned}$$

Transitivity. Assume that $(k,l) \sim (m,n)$ and that $(m,n) \sim (o,p)$ and show that $(k,l) \sim (o,p)$.

$$\begin{array}{rcll} (k,l) \sim (m,n) \ \wedge \ (m,n) \sim (o,p) & \Leftrightarrow & k+n = l+m \ \wedge \ m+p = n+o & \text{(by the definition of } \sim) \\ & \Rightarrow & k+n+m+p = l+m+n+o & \text{(by adding the equations)} \\ & \Leftrightarrow & k+p = l+o & \text{(by cancelling } n+m) \\ & \Leftrightarrow & (k,l) \sim (o,p) & \text{(by the definition of } \sim) \end{array}$$

(b) $[(m,n)] + [(0,0)] = [(m+0, n+0)] = [(m,n)]$. For the other relation, either argue that it holds by the first one and commutativity, or show directly that $[(0,0)] + [(m,n)]+ = [(0+m, 0+n)] = [(m,n)]$.

(c) $[(m,n)] + [(n,m)] = [(m+n, n+m)] = [(m+n, m+n)] = [(0,0)]$ where the last relation holds since $(m+n, m+n) \sim (0,0)$ as $m+n+0 = m+n+0$. The other relation holds since the first holds and addition is commutative.

(d) Let us consider the case $m \geq n$ first. In this case, $m-n$ is a natural number and the relation $[(n,m)] = [(0, m-n)]$ is equivalent with $(n,m) \sim (0, m-n)$ and this last relation is, by definition of $\sim$ equivalent with $n+m-n = m+0$. This last relation is true since both $m+0$ and $n+m-n$ are equal to m.

Let us consider the case $m < n$ now. In this case $n-m$ is a natural number and the relation $[(n,m)] = [(n-m, 0)]$ is equivalent with $(n,m) \sim (n-m, 0)$. This last relation is equivalent with $n+0 = m+n-m$ by the definition of $\sim$. The relation $n+0 = m+n-m$ is true since both sides are equal to n.

2. $[(6,3)]$ can be shortened to $6-3=3$, $[(1000,1005)]$ to $1000-1005=-5$, $[(6,8)]$ to $6-8=-2$, $[(57,56)]$ to $57-56=1$, and $[(56,58)]$ to $56-58=-2$. As $-5 < -2 = -2 < 1 < 3$, we have that

$$[(1000,1005)] < [(6,8)] = [(56,58)] < [(57,56)] < [(6,3)].$$

3. $[(m,n)] \cdot [(n,m)] = [(mn, nm)] = [(mn, mn)] = [(1,1)]$ where the last relation holds since $(mn, mn) \sim (1,1)$ as $mn \cdot 1 = mn \cdot 1$. Similarly, $[(n,m)] \cdot [(m,n)] = [(nm, mn)] = [(nm, nm)] = [(1,1)]$.

4. *Reflexivity:* we need to show that $[(k,l)] \leq [(k,l)]$ is true for any $k, l \in \mathbb{Z}, l \neq 0$.

$$\begin{array}{rcll} [(k,l)] \leq [(k,l)] & \Leftrightarrow & kl \leq lk & \text{(by the definition of } \leq) \\ & \Leftrightarrow & kl \leq kl & \text{(by commutativity of } \cdot \text{ for } \mathbb{Z}) \\ & \Leftrightarrow & \top & \text{(since } \leq \text{ on } \mathbb{Z} \text{ is reflexive)} \end{array}$$

Antisymmetry: we need to show that $[(k,l)] \leq [(m,n)] \wedge [(m,n)] \leq [(k,l)]$ implies $[(k,l)] = [(m,n)]$.

$$\begin{array}{rcll} [(k,l)] \leq [(m,n)] \wedge [(m,n)] \leq [(k,l)] & \Leftrightarrow & kn \leq lm \wedge ml \leq nk & \text{(by the definition of } \leq) \\ & \Leftrightarrow & kn \leq lm \wedge lm \leq kn & \text{(by commutativity of } \cdot \text{ for } \mathbb{Z}) \\ & \Rightarrow & kn = lm & (\leq \text{ is antisymmetric on } \mathbb{Z}) \\ & \Leftrightarrow & (k,l) \sim (m,n) & \text{(by the definition of } \sim) \\ & \Leftrightarrow & [(k,l)] = [(m,n)] & \text{(by the def. of an equiv. class)} \end{array}$$

5. $[(5,15)]$ can be shortened to $\frac{5}{15}=\frac{1}{3}$, $[(50,-100)]$ to $\frac{50}{-100}=\frac{-1}{2}$, $[(15,10)]$ to $\frac{15}{10}=\frac{3}{2}$, $[(20,-10)]$ to $\frac{20}{-10}=-2$, and $[(-10,20)]$ to $\frac{-10}{20}=\frac{-1}{2}$. As $-2<\frac{-1}{2}=\frac{-1}{2}<\frac{1}{3}<\frac{3}{2}$, we have that

$$[(20,-10)]<[(50,-100)]=[(-10,20)]<[(5,15)]<[(15,10)].$$

6.

$$\begin{array}{rcll}[(m,n)]\cdot[(k,l)]=[(0,1)] & \Leftrightarrow & [(mk,nl)]=[(0,1)] & \text{(by the definition of multiplication)}\\ & \Leftrightarrow & (mk,nl)\sim(0,1) & \text{(by the definition of equiv. class)}\\ & \Leftrightarrow & mk\cdot 1=nl\cdot 0 & \text{(by the definition of }\sim)\\ & \Leftrightarrow & mk=0 & \text{(simplifying)}\\ & \Rightarrow & m=0\ \vee\ k=0 & \text{(no zero divisors in }\mathbb{Z})\\ & \Leftrightarrow & [(m,n)]=[(0,1)]\vee[(k,l)]=[(0,1)] & \text{(by the definition of }\sim)\end{array}$$

7. (a) Let a stand for the limit of this sequence in case it exists. Note that then $a=\lim_{n\to\infty}a_n$ and $a=\lim_{n\to\infty}a_{n+1}$ as well. To find the value of a let $n\to\infty$ in the equation $a_{n+1}=\sqrt{2+a_n}$. The left side converges to a and the right side to $\sqrt{2+a}$. So, a can be found from the equation $a=\sqrt{2+a}\Rightarrow a^2=2+a\Rightarrow a^2-a-2=0\Rightarrow(a-2)(a+1)=0\Rightarrow a=2$ or $a=-1$. Since -1 is an extraneous root (it does not satisfy the equation $a=\sqrt{2+a}$), the limit of the sequence is $a=2$. Alternatively, you can also argue that starting with the nonnegative term $a_0=0$, all the terms of the sequence are nonnegative and so the solution $a=-1$ can be discarded.

   (b) Let a stand for the limit of this sequence in case it exists. Note that then $a=\lim_{n\to\infty}a_n$ and $a=\lim_{n\to\infty}a_{n+1}$ as well. To find the value of a let $n\to\infty$ in the equation $a_{n+1}=\frac{1}{1+a_n}$. The left side converges to a and the right side to $\frac{1}{1+a}$. So, a can be found from the equation $a=\frac{1}{1+a}\Rightarrow a(1+a)=1\Rightarrow a^2+a-1=0\Rightarrow a=\frac{-1+\sqrt{5}}{2}\approx 0.618$ or $a=\frac{-1-\sqrt{5}}{2}\approx-1.618$. Starting with the positive term $a_0=1$, all the terms of the sequence are positive, so the sequence converges towards the positive value $a=\frac{-1+\sqrt{5}}{2}\approx 0.618$.

8. (a) $|(3,5)\cup[8,9)|=|(3,5)|+|[8,9)|=|\mathbb{R}|+|\mathbb{R}|=|\mathbb{R}|$ and $|(7,\infty)|=|\mathbb{R}|$.

   (b) Note that $[0,9)\cup[7,\infty)=[0,\infty)$ and $(3,5]\cup[0,\infty)=[0,\infty)$. So, $|(3,5]\cup[0,9)\cup[7,\infty)|=|[0,\infty)|=|\mathbb{R}|$ and $|(-\infty,1]|=|\mathbb{R}|$.

   (c) Note that $\bigcup_{n\in\mathbb{N}}(-n,n)=(-1,1)\cup(-2,2)\cup(-3,3)\cup\ldots=(-\infty,\infty)=\mathbb{R}$. As $|(0,1)|=|\mathbb{R}|$, the two sets have the same cardinality.

   (d) $\bigcap_{n\in\mathbb{N}}[0,n+1)=[0,1)\cap[0,2)\cap[0,3)\cap\ldots=[0,1)$. As $|[0,1)|=|\mathbb{R}|$, the two sets have the same cardinality.

   (e) $\bigcup_{n\in\mathbb{N}}(-\infty,-n)=(-\infty,0)\cup(-\infty,-1)\cup(-\infty,-2)\cup(-\infty,-3)\cup\ldots=(-\infty,0)$. Since $|(-\infty,0)|=|\mathbb{R}|$ and $|(1,\infty)|=|\mathbb{R}|$, the two sets have the same cardinality.

9. (a) $0.222222\ldots=0.2+0.02+0.002+\ldots=\frac{2}{10}+\frac{2}{10^2}+\frac{2}{10^3}+\ldots=\sum_{n=1}^{\infty}2\left(\frac{1}{10}\right)^n$. Using the formula $\frac{ar^k}{1-r}$ with $a=2$, $r=\frac{1}{10}$ and $k=1$, we have that the sum is $\frac{\frac{2}{10}}{\frac{9}{10}}=\frac{2}{9}$.

   (b) $0.27272727\ldots=0.27+0.0027+0.000027+\ldots=\frac{27}{100}+\frac{27}{100^2}+\frac{27}{100^3}+\ldots=\sum_{n=1}^{\infty}27\left(\frac{1}{100}\right)^n$. Using the formula $\frac{ar^k}{1-r}$ with $a=27$, $r=\frac{1}{100}$ and $k=1$, we have that the sum is $\frac{\frac{27}{100}}{\frac{99}{100}}=\frac{27}{99}=\frac{3}{11}$.

(c) $1.2345454545\ldots = 1.23 + 0.0045 + 0.000045 + 0.00000045 + \ldots = 1.23 + \frac{45}{100^2} + \frac{45}{100^3} + \frac{45}{100^4} + \ldots = 1.23 + \sum_{n=2}^{\infty} 45\left(\frac{1}{100}\right)^n$. Using the formula $\frac{ar^k}{1-r}$ with $a = 45$, $r = \frac{1}{100}$ and $k = 2$, we have that the sum is $1.23 + \frac{\frac{45}{100^2}}{\frac{99}{100}} = \frac{123}{100} + \frac{45}{99(100)} = \frac{123(99)+45}{99(100)} = \frac{12222}{9900} = \frac{679}{550}$.

10. The complex number $-3i$ is on the negative part of y axis. Hence, $\theta = \frac{-\pi}{2}$. We have that $r = \sqrt{(-3)^2} = 3$.

The complex number $\sqrt{2} - \sqrt{2}i$ is on the $y = -x$ line and in the fourth quadrant. Hence, $\theta = \frac{-\pi}{4}$. We have that $r = \sqrt{\sqrt{2}^2 + (-\sqrt{2})^2} = \sqrt{2+2}\sqrt{4} = 2$.

The complex number $-\sqrt{3} + i$ is in the second quadrant. Hence, $\theta = \pi + \tan^{-1}\frac{1}{-\sqrt{3}} = \pi + \frac{-\pi}{6} = \frac{5\pi}{6}$. The modulus is $r = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$.

The complex number $-2 - i$ is in the third quadrant. Hence, $\theta = \pi + \tan^{-1}\frac{-1}{-2} = \pi + \tan^{-1}\frac{1}{2} \approx \pi + 0.4636 \approx 3.605$. The modulus is $r = \sqrt{(-2)^2 + (-1)^2} = \sqrt{5} \approx 2.24$.

11. If $\theta = \frac{-\pi}{2}$, the number is on the negative part of y-axis. As $r = 5$, $(x, y) = (0, -5)$. Alternatively, $x = 5\cos\frac{-\pi}{2} = 0$ and $y = 5\sin\frac{-\pi}{2} = -5$.

If $\theta = \frac{5\pi}{6}$ and $r = 2$, $x = r\cos\theta = 2\cos\frac{5\pi}{6} = 2\cdot\frac{-\sqrt{3}}{2} = -\sqrt{3}$ and $y = r\sin\theta = 2\sin\frac{5\pi}{6} = 2\cdot\frac{1}{2} = 1$. Thus, $(x, y) = (-\sqrt{3}, 1)$.

If $\theta = \frac{-2\pi}{3}$ and $r = 3$, $x = r\cos\theta = 3\cos\frac{-2\pi}{3} = 3\cdot\frac{-1}{2} = \frac{-3}{2}$ and $y = r\sin\theta = 3\sin\frac{-2\pi}{3} = 3\cdot\frac{-1}{\sqrt{2}} = \frac{-3}{\sqrt{2}}$. Thus, $(x, y) = (\frac{-3}{2}, \frac{-3}{\sqrt{2}})$.

12. (a) From problem (1), we have that $z = -\sqrt{3} + i = 2e^{5\pi/6i}$. Hence, $z^4 = 2^4 e^{4\cdot 5\pi/6i} = 16e^{10\pi/3i} = 16(\cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3}) = 16(\frac{-1}{2} - \frac{\sqrt{3}}{2}) = -8 - 8\sqrt{3}i$.

(b) From problem (1), we have that $z = -2 - i \approx \sqrt{5}e^{3.605i}$. Hence, $z^6 \approx (\sqrt{5})^6 e^{6\cdot 3.605i} = 125e^{21.63i} = 125(\cos 21.63 + i\sin 21.63) = 125(-0.936 + 0.352) = -117 + 44i$.

13. (a) We need to find all five solutions of $z^5 = 32$. Note that 32 corresponds to the complex number $(32, 0)$ which is on the positive side of the x-axis so $\theta = 0$. The distance from $(32, 0)$ to the origin is 32 so $r = 32$. Hence, the five solutions of the characteristic equation can be found by the formula

$$\sqrt[5]{32}e^{\frac{0+2k\pi}{5}i} = 2e^{\frac{2k\pi}{5}i} \quad \text{for } k = 0, 1, \ldots, 4.$$

These five solutions form a regular polygon with five sides on the circle of radius 2 centered at the origin.

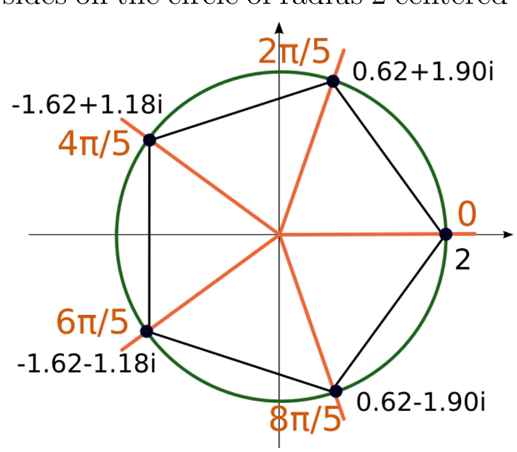
$k = 0 \Rightarrow \quad z_0 = 2e^{0i} = 2,$

$k = 1 \Rightarrow \quad z_1 = 2e^{\frac{2\pi}{5}i} = 2(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}) \approx 0.62 + 1.90i,$

$k = 2 \Rightarrow \quad z_2 = 2e^{\frac{4\pi}{5}i} = 2(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}) \approx -1.62 + 1.18i,$

$k = 3 \Rightarrow \quad z_3 = 2e^{\frac{6\pi}{5}i} = 2(\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}) \approx -1.62 - 1.18i,$

$k = 4 \Rightarrow \quad z_4 = 2e^{\frac{8\pi}{5}i} = 2(\cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}) \approx 0.62 - 1.90i.$



(b) $z^5 = -32 = 32e^{\pi i}$. Hence, $z_k = \sqrt[5]{32}e^{\frac{\pi+2k\pi}{5}i} = 2e^{\frac{(2k+1)\pi}{5}i}$ for $k = 0, 1, \ldots, 4$.

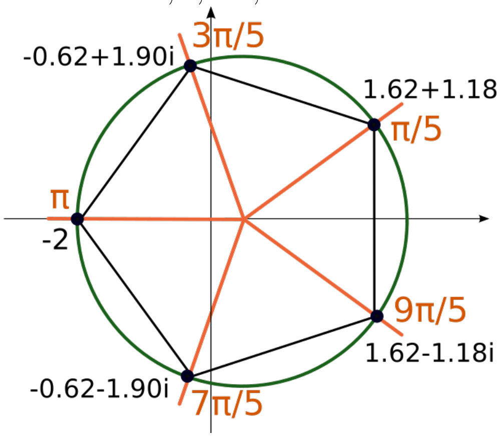
$k = 0 \Rightarrow$ $z_0 = 2e^{\frac{\pi}{5}i} = 2(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}) \approx 1.62 + 1.18i$,

$k = 1 \Rightarrow$ $z_1 = 2e^{\frac{3\pi}{5}i} = 2(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}) \approx -0.62 + 1.90i$,

$k = 2 \Rightarrow$ $z_2 = 2e^{\frac{5\pi}{5}i} = 2e^{\pi i} = 2(\cos\pi + i\sin\pi) = -2$,

$k = 3 \Rightarrow$ $z_3 = 2e^{\frac{7\pi}{5}i} = 2(\cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}) \approx -0.62 - 1.90i$,

$k = 4 \Rightarrow$ $z_4 = 2e^{\frac{7\pi}{5}i} = 2(\cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}) \approx 1.62 - 1.18i$.



(c) $r = \sqrt{3^2 + 3^2} = \sqrt{18}$ or $3\sqrt{2}$, $\theta = \tan^{-1}(\frac{3}{3}) = \tan^{-1}(1) = \frac{\pi}{4}$, so $z = \sqrt{18}e^{\pi/4i}$.

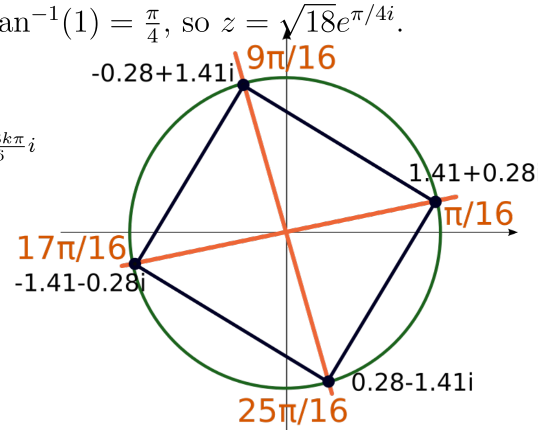
The four roots are obtained as

$z_k = \sqrt[4]{\sqrt{18}}e^{\frac{\pi/4+2k\pi}{4}i} = 18^{1/8}e^{\frac{\pi+8k\pi}{16}i} \approx 1.435e^{\frac{\pi+8k\pi}{16}i}$

for $k = 0, 1, 2, 3$. Thus,

$z_0 \approx 1.435e^{\pi/16i} = 1.435(\cos\frac{\pi}{16} + i\sin\frac{\pi}{16}) \approx 1.435(\ 0.98 + i0.195) = \ 1.41 + 0.28i$

$z_1 \approx 1.435e^{9\pi/16i} = 1.435(\cos\frac{9\pi}{16} + i\sin\frac{9\pi}{16}) \approx 1.435(-0.195 + 0.98i) = -0.28 + 1.41i$



$z_2 \approx 1.435e^{17\pi/16i} = 1.435(\cos\frac{17\pi}{16} + i\sin\frac{17\pi}{16}) = 1.435(-0.98 - 0.195i) = -1.41 - 0.28i$

$z_3 \approx 1.435e^{25\pi/16i} = 1.435(\cos\frac{25\pi}{16} + i\sin\frac{25\pi}{16}) = 1.435(\ 0.195 - 0.98i) = \ \ 0.28 - 1.41i$

(d) $r = \sqrt{3^2 + (-3)^2} = \sqrt{18}$ or $3\sqrt{2}$, $\theta = \tan^{-1}(\frac{-3}{3}) = \tan^{-1}(-1) = \frac{-\pi}{4}$, so $z = \sqrt{18}e^{-\pi/4i}$.

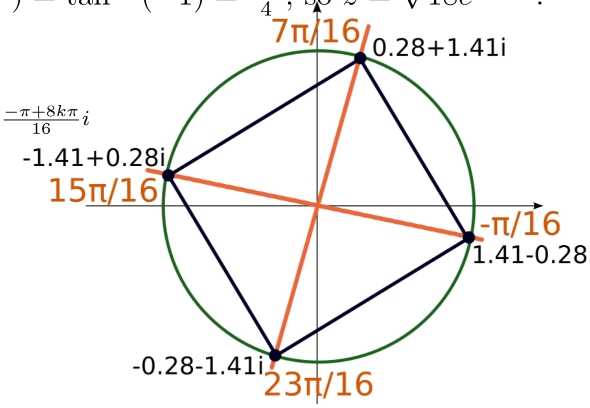
The four roots are obtained as

$z_k = \sqrt[4]{\sqrt{18}}e^{\frac{-\pi/4+2k\pi}{4}i} = 18^{1/8}e^{\frac{-\pi+8k\pi}{16}i} \approx 1.435e^{\frac{-\pi+8k\pi}{16}i}$

for $k = 0, 1, 2, 3$. Thus,

$z_0 \approx 1.435e^{-\pi/16i} = 1.435(\cos\frac{\pi}{16} + i\sin\frac{\pi}{16}) \approx 1.435(\ 0.98 - i0.195) = \ 1.41 - 0.28i$

$z_1 \approx 1.435e^{7\pi/16i} = 1.435(\cos\frac{7\pi}{16} + i\sin\frac{7\pi}{16}) \approx 1.435(0.195 + .98i) = 0.28 + 1.41i$



$z_2 \approx 1.435e^{15\pi/16i} = 1.435(\cos\frac{15\pi}{16} + i\sin\frac{15\pi}{16}) = 1.435(-0.98 + 0.195i) = -1.41 + 0.28i$

$z_3 \approx 1.435e^{23\pi/16i} = 1.435(\cos\frac{23\pi}{16} + i\sin\frac{23\pi}{16}) = 1.435(\ -0.195 - 0.98i) = \ \ -0.28 - 1.41i$

14. (a) $\cos z + i\sin z = \frac{1}{2}(e^{iz} + e^{-iz}) + i\frac{1}{2i}(e^{iz} - e^{-iz}) = \frac{1}{2}\left(e^{iz} + e^{-iz} + e^{iz} - e^{-iz}\right) = \frac{1}{2}\left(2e^{iz}\right) = e^{iz}$.

(b) $\sin^2 z + \cos^2 z = \frac{-1}{4}(e^{iz} - e^{-iz})^2 + \frac{1}{4}(e^{iz} + e^{-iz})^2 = \frac{-1}{4}(e^{2iz} - 2 + e^{-2iz}) + \frac{1}{4}(e^{2iz} + 2 + e^{-2iz}) = \frac{1}{4}\left(-e^{2iz} + 2 - e^{-2iz} + e^{2iz} + 2 + e^{-2iz}\right) = \frac{1}{4}(4) = 1.$

(c) $\sin^2 z = \left(\frac{1}{2i}(e^{iz} - e^{-iz})\right)^2 = \frac{1}{-4}(e^{2iz} - 2 + e^{-2iz}) = \frac{1}{4}\left(2 - (e^{2iz} + e^{-2iz})\right) =$
$\frac{1}{2}\left(1 - \frac{1}{2}(e^{2iz} + e^{-2iz})\right) = \frac{1}{2}(1 - \cos(2z))$

(d) $\cos^2 z = \left(\frac{1}{2}(e^{iz} + e^{-iz})\right)^2 = \frac{1}{4}(e^{2iz} + 2 + e^{-2iz}) = \frac{1}{4}(2 + e^{2iz} + e^{-2iz}) =$
$\frac{1}{2}\left(1 + \frac{1}{2}(e^{2iz} + e^{-2iz})\right) = \frac{1}{2}(1 + \cos(2z))$