Fundamentals of Mathematics
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Review for Exam 4

1. (a) Show that the relation $\sim$ used to define $\mathbb{Z}$ and given on $\mathbb{N}\times\mathbb{N}$ by
$$(k,l)\sim(m,n) \quad\Leftrightarrow\quad k+n=l+m$$
is an equivalence relation.
   (b) Show that the integer $[(0,0)]$ is the identity for addition (i.e. show that $[(m,n)]+[(0,0)]=[(m,n)]$ and $[(0,0)]+[(m,n)]=[(m,n)]$).
   (c) For any integer $[(m,n)]$, show that its additive inverse is $[(n,m)]$ (i.e. show that $[(m,n)]+[(n,m)]=[(0,0)]$ and $[(n,m)]+[(m,n)]=[(0,0)]$).
   (d) Show that $[(n,m)]=[(0,m-n)]$ if $m\geq n$ and $[(n,m)]=[(n-m,0)]$ if $m<n$ for any natural numbers m and n.

2. The relation $\leq$ on $\mathbb{Z}$ matches the familiar order of integers when $[(m,n)]$ is shortened to $m-n$. Rearrange the integer numbers below, if needed, so that the elements in the new list are non-decreasing.
$$[(6,3)],\quad [(1000,1005)],\quad [(6,8)],\quad [(57,56)],\quad [(56,58)]$$

3. If $[(m,n)]\in\mathbb{Q}$ is such that $m\neq 0$, show that $[(m,n)]\cdot[(n,m)]=[(1,1)]$ and $[(n,m)]\cdot[(m,n)]=[(1,1)]$).).

4. The relation $\leq$ on $\mathbb{Q}$ matches the familiar order of integers when $[(m,n)]$ is shortened to $\frac{m}{n}$. Rearrange the rational numbers below, if needed, so that the elements in the new list are non-decreasing.
$$[(5,15)],\quad [(50,-100)],\quad [(15,10)],\quad [(20,-10)],\quad [(-10,20)]$$

5. Show that $\mathbb{Q}$ has no zero divisors, that is $ab=0\Rightarrow a=0$ or $b=0$ for any $a,b\in\mathbb{Q}$. You can assume that $\mathbb{Z}$ has no zero divisors (that is $mn=0\Rightarrow m=0$ or $n=0$ for $m,n\in\mathbb{Z}$).

6. Find the limit of the following recursive sequences.
   (a) $a_{n+1}=\sqrt{2+a_n},\quad a_0=0$
   (b) $a_{n+1}=\frac{1}{1+a_n},\quad a_0=1.$

7. Show that the following pairs of sets are in a bijective correspondence. You can assume the existence of any of the bijective correspondences from the formula sheet.
   (a) $(3,5)\cup[8,9)$ and $(7,\infty)$
   (b) $(3,5]\cup[0,9)\cup[7,\infty)$ and $(-\infty,1]$
   (c) $\bigcup_{n\in\mathbb{N}-\{0\}}(-n,n)$ and $(0,1)$
   (d) $\bigcap_{n\in\mathbb{N}}[0,n+1)$ and $\mathbb{R}$
   (e) $\bigcup_{n\in\mathbb{N}}(-\infty,-n)$ and $(1,\infty)$

8. Represent the following decimal numbers as quotients of two integer numbers.

(a) $0.222222\ldots$ (b) $0.27272727\ldots$ (c) $1.2345454545\ldots$

9. Determine the moduli and the arguments given the following complex numbers in algebraic forms: $-3i$, $\sqrt{2}-\sqrt{2}i$, $-\sqrt{3}+i$, $-2-i$.

10. Determine the real and imaginary parts of the complex numbers given by their moduli and arguments: $\theta=\frac{-\pi}{2}, r=5$; $\theta=\frac{5\pi}{6}, r=2$; $\theta=\frac{-2\pi}{3}, r=3$.

11. Determine the n-th power of the given complex numbers and given n. Express your answers in algebraic form. (a) $z=-\sqrt{3}+i$, $n=4$; (b) $z=-2-i$, $n=6$.

12. Find all solutions of the following equations.

(a) $z^5+32=0$ (b) $z^5-32=0$

13. (a) The exponential function e^z is defined by $e^z=e^{x+iy}=e^xe^{iy}=e^x(\cos y+i\sin y)$.
If n is a positive integer, show that $(e^z)^n=e^{nz}$.

(b) The complex-valued basic trigonometric functions are defined via exponential function by

$$\sin z=\frac{1}{2i}(e^{iz}-e^{-iz}) \qquad \cos z=\frac{1}{2}(e^{iz}+e^{-iz})$$

Verify the identities (i) $e^{iz}=\cos z+i\sin z$ (ii) $\sin^2 z+\cos^2 z=1$

Solutions

1. (a) *Reflexivity.* We need to show that $(k,l)\sim(k,l)$ for any nonnegative integers k and l.

$$\begin{aligned}(k,l)\sim(k,l) &\Leftrightarrow k+l=l+k \quad \text{(by the definition of } \sim)\\ &\Leftrightarrow k+l=k+l \quad \text{(by commutativity of } +)\end{aligned}$$

Since $k+l=k+l$ is true, we have that $(k,l)\sim(k,l)$ is also true.
Symmetry. Assume that $(k,l)\sim(m,n)$ for some $k,l,m,n\in\mathbb{N}$ and show that $(m,n)\sim(k,l)$.

$$\begin{aligned}(k,l)\sim(m,n) &\Leftrightarrow k+n=l+m \quad \text{(by the definition of } \sim)\\ &\Leftrightarrow m+l=n+k \quad \text{(by commutativity of } + \text{ and symmetry of } =)\\ &\Leftrightarrow (m,n)\sim(k,l) \quad \text{(by the definition of } \sim)\end{aligned}$$

Transitivity. Assume that $(k,l)\sim(m,n)$ and that $(m,n)\sim(o,p)$ and show that $(k,l)\sim(o,p)$.

$$\begin{aligned}(k,l)\sim(m,n)\ \wedge\ (m,n)\sim(o,p) &\Leftrightarrow k+n=l+m\ \wedge\ m+p=n+o && \text{(by the definition of } \sim)\\ &\Rightarrow k+n+m+p=l+m+n+o && \text{(by adding the equations)}\\ &\Leftrightarrow k+p=l+o && \text{(by cancelling } n+m)\\ &\Leftrightarrow (k,l)\sim(o,p) && \text{(by the definition of } \sim)\end{aligned}$$

(b) $[(m,n)] + [(0,0)] = [(m+0, n+0)] = [(m,n)]$. For the other relation, either argue that it holds by the first one and commutativity, or show directly that $[(0,0)] + [(m,n)]+ = [(0+m, 0+n)] = [(m,n)]$.

(c) $[(m,n)] + [(n,m)] = [(m+n, n+m)] = [(m+n, m+n)] = [(0,0)]$ where the last relation holds since $(m+n, m+n) \sim (0,0)$ as $m+n+0 = m+n+0$. The other relation holds since the first holds and addition is commutative.

(d) Let us consider the case $m \geq n$ first. In this case, $m-n$ is a natural number and the relation $[(n,m)] = [(0, m-n)]$ is equivalent with $(n,m) \sim (0, m-n)$ and this last relation is, by definition of $\sim$ equivalent with $n + m - n = m + 0$. This last relation is true since both $m+0$ and $n+m-n$ are equal to m.

Let us consider the case $m < n$ now. In this case $n-m$ is a natural number and the relation $[(n,m)] = [(n-m, 0)]$ is equivalent with $(n,m) \sim (n-m, 0)$. This last relation is equivalent with $n+0 = m+n-m$ by the definition of $\sim$. The relation $n+0 = m+n-m$ is true since both sides are equal to n.

2. $[(6,3)]$ can be shortened to $6-3 = 3$, $[(1000, 1005)]$ to $1000 - 1005 = -5$, $[(6,8)]$ to $6-8 = -2$, $[(57,56)]$ to $57 - 56 = 1$, and $[(56,58)]$ to $56 - 58 = -2$. As $-5 < -2 = -2 < 1 < 3$, we have that
$$[(1000, 1005)] < [(6,8)] = [(56,58)] < [(57,56)] < [(6,3)].$$

3. $[(m,n)] \cdot [(n,m)] = [(mn, nm)] = [(mn, mn)] = [(1,1)]$ where the last relation holds since $(mn, mn) \sim (1,1)$ as $mn \cdot 1 = mn \cdot 1$. Similarly, $[(n,m)] \cdot [(m,n)] = [(nm, mn)] = [(nm, nm)] = [(1,1)]$.

4. $[(5,15)]$ can be shortened to $\frac{5}{15} = \frac{1}{3}$, $[(50, -100)]$ to $\frac{50}{-100} = \frac{-1}{2}$, $[(15,10)]$ to $\frac{15}{10} = \frac{3}{2}$, $[(20, -10)]$ to $\frac{20}{-10} = -2$, and $[(-10, 20)]$ to $\frac{-10}{20} = \frac{-1}{2}$. As $-2 < \frac{-1}{2} = \frac{-1}{2} < \frac{1}{3} < \frac{3}{2}$, we have that
$$[(20, -10)] < [(50, -100)] = [(-10, 20)] < [(5,15)] < [(15,10)].$$

5. Let $a = [(m,n)]$ and $b = [(k,l)]$ and assume that $ab = 0$ so that $[(mk, nl)] = [(0,1)]$. This implies that $mk \cdot 1 = nl \cdot 0$ so that $mk = 0$. As $\mathbb{Z}$ has no zero divisors, this implies that $m = 0$ or $k = 0$. If $m = 0$, then $a = [(0,n)] = [(0,1)] = 0$. If $k = 0$, then $b = [(0,l)] = [(0,1)] = 0$.

6. (a) Let a stand for the limit of this sequence in case it exists. Note that then $a = \lim_{n\to\infty} a_n$ and $a = \lim_{n\to\infty} a_{n+1}$ as well. To find the value of a let $n \to \infty$ in the equation $a_{n+1} = \sqrt{2 + a_n}$. The left side converges to a and the right side to $\sqrt{2+a}$. So, a can be found from the equation $a = \sqrt{2+a} \Rightarrow a^2 = 2 + a \Rightarrow a^2 - a - 2 = 0 \Rightarrow (a-2)(a+1) = 0 \Rightarrow a = 2$ or $a = -1$. Since -1 is an extraneous root (it does not satisfy the equation $a = \sqrt{2+a}$), the limit of the sequence is $a = 2$. Alternatively, you can also argue that starting with the nonnegative term $a_0 = 0$, all the terms of the sequence are nonnegative and so the solution $a = -1$ can be discarded.

(b) Let a stand for the limit of this sequence in case it exists. Note that then $a = \lim_{n\to\infty} a_n$ and $a = \lim_{n\to\infty} a_{n+1}$ as well. To find the value of a let $n \to \infty$ in the equation $a_{n+1} = \frac{1}{1+a_n}$. The left side converges to a and the right side to $\frac{1}{1+a}$. So, a can be found from the equation $a = \frac{1}{1+a} \Rightarrow a(1+a) = 1 \Rightarrow a^2 + a - 1 = 0 \Rightarrow a = \frac{-1+\sqrt{5}}{2} \approx 0.618$ or $a = \frac{-1-\sqrt{5}}{2} \approx -1.618$. Starting with the positive term $a_0 = 1$, all the terms of the sequence are positive, so the sequence converges towards the positive value $a = \frac{-1+\sqrt{5}}{2} \approx 0.618$.

7. (a) $|(3,5)\cup[8,9)| = |(3,5)| + |[8,9)| = |\mathbb{R}| + |\mathbb{R}| = |\mathbb{R}|$ and $|(7,\infty)| = |\mathbb{R}|$.

   (b) Note that $[0,9)\cup[7,\infty) = [0,\infty)$ and $(3,5]\cup[0,\infty) = [0,\infty)$. So, $|(3,5]\cup[0,9)\cup[7,\infty)| = |[0,\infty)| = |\mathbb{R}|$ and $|(-\infty,1]| = |\mathbb{R}|$.

   (c) Note that $\bigcup_{n\in\mathbb{N}}(-n,n) = (-1,1)\cup(-2,2)\cup(-3,3)\cup\ldots = (-\infty,\infty) = \mathbb{R}$. As $|(0,1)| = |\mathbb{R}|$, the two sets have the same cardinality.

   (d) $\bigcap_{n\in\mathbb{N}}[0,n+1) = [0,1)\cap[0,2)\cap[0,3)\cap\ldots = [0,1)$. As $|[0,1)| = |\mathbb{R}|$, the two sets have the same cardinality.

   (e) $\bigcup_{n\in\mathbb{N}}(-\infty,-n) = (-\infty,0)\cup(-\infty,-1)\cup(-\infty,-2)\cup(-\infty,-3)\cup\ldots = (-\infty,0)$. Since $|(-\infty,0)| = |\mathbb{R}|$ and $|(1,\infty)| = |\mathbb{R}|$, the two sets have the same cardinality.

8. (a) $0.222222\ldots = 0.2+0.02+0.002+\ldots = \frac{2}{10}+\frac{2}{10^2}+\frac{2}{10^3}+\ldots = \sum_{n=1}^{\infty} 2\left(\frac{1}{10}\right)^n$. Using the formula $\frac{ar^k}{1-r}$ with $a=2$, $r=\frac{1}{10}$ and $k=1$, we have that the sum is $\frac{\frac{2}{10}}{\frac{9}{10}} = \frac{2}{9}$.

   (b) $0.27272727\ldots = 0.27+0.0027+0.000027+\ldots = \frac{27}{100}+\frac{27}{100^2}+\frac{27}{100^3}+\ldots = \sum_{n=1}^{\infty} 27\left(\frac{1}{100}\right)^n$. Using the formula $\frac{ar^k}{1-r}$ with $a=27$, $r=\frac{1}{100}$ and $k=1$, we have that the sum is $\frac{\frac{27}{100}}{\frac{99}{100}} = \frac{27}{99} = \frac{3}{11}$.

   (c) $1.2345454545\ldots = 1.23+0.0045+0.000045+0.00000045+\ldots = 1.23+\frac{45}{100^2}+\frac{45}{100^3}+\frac{45}{100^4}+\ldots = 1.23+\sum_{n=2}^{\infty} 45\left(\frac{1}{100}\right)^n$. Using the formula $\frac{ar^k}{1-r}$ with $a=45$, $r=\frac{1}{100}$ and $k=2$, we have that the sum is $1.23+\frac{\frac{45}{100^2}}{\frac{99}{100}} = \frac{123}{100}+\frac{45}{99(100)} = \frac{123(99)+45}{99(100)} = \frac{12222}{9900} = \frac{679}{550}$.

9. The complex number $-3i$ is on the negative part of y axis. Hence, $\theta = \frac{-\pi}{2}$. We have that $r = \sqrt{(-3)^2} = 3$.

   The complex number $\sqrt{2}-\sqrt{2}i$ is on the $y=-x$ line and in the fourth quadrant. Hence, $\theta = \frac{-\pi}{4}$. We have that $r = \sqrt{\sqrt{2}^2+(-\sqrt{2})^2} = \sqrt{2+2}\sqrt{4} = 2$.

   The complex number $-\sqrt{3}+i$ is in the second quadrant. Hence, $\theta = \pi+\tan^{-1}\frac{1}{-\sqrt{3}} = \pi+\frac{-\pi}{6} = \frac{5\pi}{6}$. The modulus is $r = \sqrt{(-\sqrt{3})^2+1^2} = \sqrt{4} = 2$.

   The complex number $-2-i$ is in the third quadrant. Hence, $\theta = \pi+\tan^{-1}\frac{-1}{-2} = \pi+\tan^{-1}\frac{1}{2} \approx \pi+0.4636 \approx 3.605$. The modulus is $r = \sqrt{(-2)^2+(-1)^2} = \sqrt{5} \approx 2.24$.

10. If $\theta = \frac{-\pi}{2}$, the number is on the negative part of y-axis. As $r=5$, $(x,y) = (0,-5)$. Alternatively, $x = 5\cos\frac{-\pi}{2} = 0$ and $y = 5\sin\frac{-\pi}{2} = -5$.

    If $\theta = \frac{5\pi}{6}$ and $r=2$, $x = r\cos\theta = 2\cos\frac{5\pi}{6} = 2\cdot\frac{-\sqrt{3}}{2} = -\sqrt{3}$ and $y = r\sin\theta = 2\sin\frac{5\pi}{6} = 2\cdot\frac{1}{2} = 1$. Thus, $(x,y) = (-\sqrt{3},1)$.

    If $\theta = \frac{-2\pi}{3}$ and $r=3$, $x = r\cos\theta = 3\cos\frac{-2\pi}{3} = 3\cdot\frac{-1}{2} = \frac{-3}{2}$ and $y = r\sin\theta = 3\sin\frac{-2\pi}{3} = 3\cdot\frac{-1}{\sqrt{2}} = \frac{-3}{\sqrt{2}}$. Thus, $(x,y) = (\frac{-3}{2},\frac{-3}{\sqrt{2}})$.

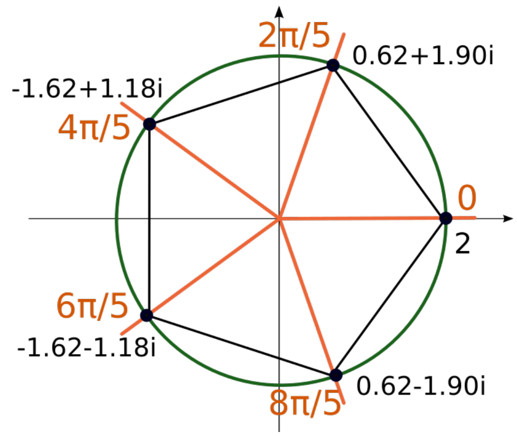
11. (a) From problem (1), we have that $z = -\sqrt{3}+i = 2e^{5\pi/6i}$. Hence, $z^4 = 2^4e^{4\cdot5\pi/6i} = 16e^{10\pi/3i} = 16(\cos\frac{10\pi}{3}+i\sin\frac{10\pi}{3}) = 16(\frac{-1}{2}-\frac{\sqrt{3}}{2}) = -8-8\sqrt{3}i$.

    (b) From problem (1), we have that $z = -2-i \approx \sqrt{5}e^{3.605i}$. Hence, $z^6 \approx (\sqrt{5})^6e^{6\cdot3.605i} = 125e^{12.63i} = 125(\cos 12.63+i\sin 12.63) = 125(-0.935+0.0636) = -116.88+7.95i$.

12. (a) We need to find all five solutions of $z^5 = 32$. Note that 32 corresponds to the complex number $(32, 0)$ which is on the positive side of the x-axis so $\theta = 0$. The distance from $(32, 0)$ to the origin is 32 so $r = 32$. Hence, the five solutions of the characteristic equation can be found by the formula

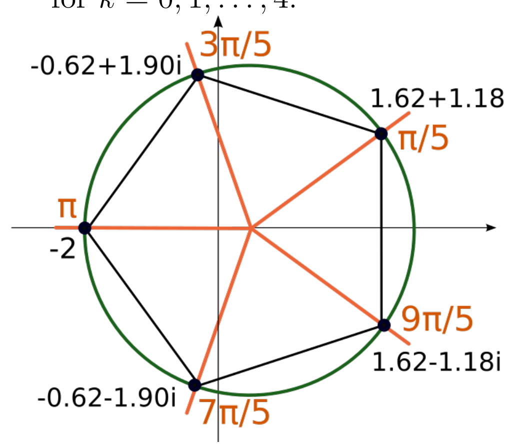
$$\sqrt[5]{32}e^{\frac{0+2k\pi}{5}i} = 2e^{\frac{2k\pi}{5}i} \quad \text{for } k = 0, 1, \ldots, 4.$$

These five solutions form a regular polygon with five sides on the circle of radius 2 centered at the origin.



$k = 0 \Rightarrow \quad z_0 = 2e^{0i} \quad = 2,$
$k = 1 \Rightarrow \quad z_1 = 2e^{\frac{2\pi}{5}i} = 2(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}) \approx 0.62 + 1.90i,$
$k = 2 \Rightarrow \quad z_2 = 2e^{\frac{4\pi}{5}i} = 2(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}) \approx -1.62 + 1.18i,$
$k = 3 \Rightarrow \quad z_3 = 2e^{\frac{6\pi}{5}i} = 2(\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}) \approx -1.62 - 1.18i,$
$k = 4 \Rightarrow \quad z_4 = 2e^{\frac{8\pi}{5}i} = 2(\cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}) \approx 0.62 - 1.90i.$

(b) $z^5 = -32 = 32e^{\pi i}$. Hence, $z_k = \sqrt[5]{32}e^{\frac{\pi+2k\pi}{5}i} = 2e^{\frac{(2k+1)\pi}{5}i}$ for $k = 0, 1, \ldots, 4$.



$k = 0 \Rightarrow \quad z_0 = 2e^{\frac{\pi}{5}i} = 2(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}) \approx 1.62 + 1.18i,$
$k = 1 \Rightarrow \quad z_1 = 2e^{\frac{3\pi}{5}i} = 2(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}) \approx -0.62 + 1.90i,$
$k = 2 \Rightarrow \quad z_2 = 2e^{\frac{5\pi}{5}i} = 2e^{\pi i} = 2(\cos\pi + i\sin\pi) = -2,$
$k = 3 \Rightarrow \quad z_3 = 2e^{\frac{7\pi}{5}i} = 2(\cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}) \approx -0.62 - 1.90i,$
$k = 4 \Rightarrow \quad z_4 = 2e^{\frac{7\pi}{5}i} = 2(\cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}) \approx 1.62 - 1.18i.$

13. (a) $(e^z)^n = (e^{x+iy})^n = (e^x e^{iy})^n = e^{nx}e^{iny} = e^{n(x+iy)} = e^{nz}$.

(b) (i) $\cos z + i\sin z = \frac{1}{2}(e^{iz} + e^{-iz}) + i\frac{1}{2i}(e^{iz} - e^{-iz}) = \frac{1}{2}\left(e^{iz} + e^{-iz} + e^{iz} - e^{-iz}\right) = \frac{1}{2}\left(2e^{iz}\right) = e^{iz}$.
(ii) $\sin^2 z + \cos^2 z = \frac{-1}{4}(e^{iz} - e^{-iz})^2 + \frac{1}{4}(e^{iz} + e^{-iz})^2 = \frac{-1}{4}(e^{2iz} - 2 + e^{-2iz}) + \frac{1}{4}(e^{2iz} + 2 + e^{-2iz}) = \frac{1}{4}\left(-e^{2iz} + 2 - e^{-2iz} + e^{2iz} + 2 + e^{-2iz}\right) = \frac{1}{4}(4) = 1.$