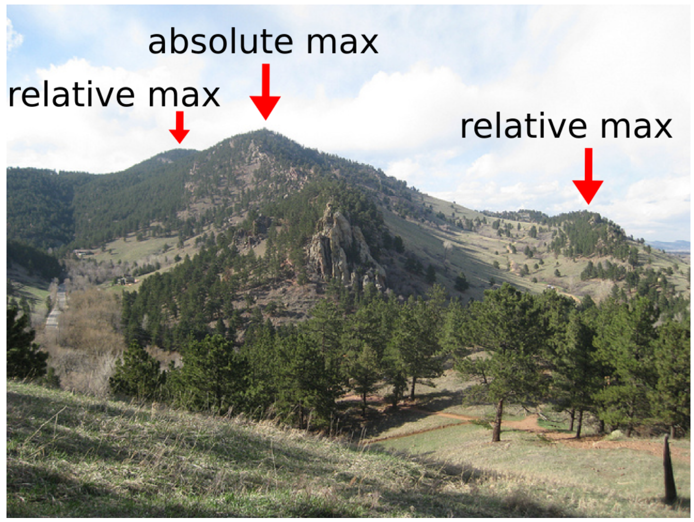
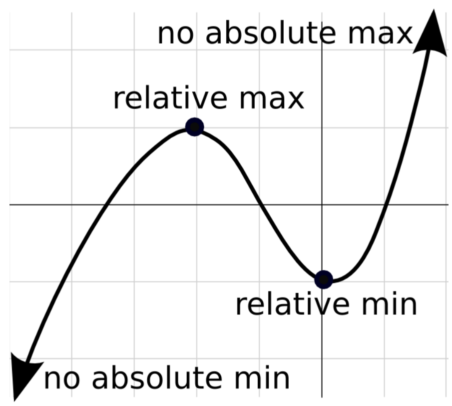
Fundamentals of Calculus
Lia Vas

Absolute Extrema and Constrained Optimization



Recall that a function $f(x)$ is said to have a **relative maximum** at $x = c$ if $f(c) \geq f(x)$ for all values of x in some open interval containing c. However, that does not mean that the value $f(c)$ is absolutely the largest value on entire domain of f. If $f(c) \geq f(x)$ for all the values x in the domain of f, then f is said to have an **absolute maximum** at $x = c$.

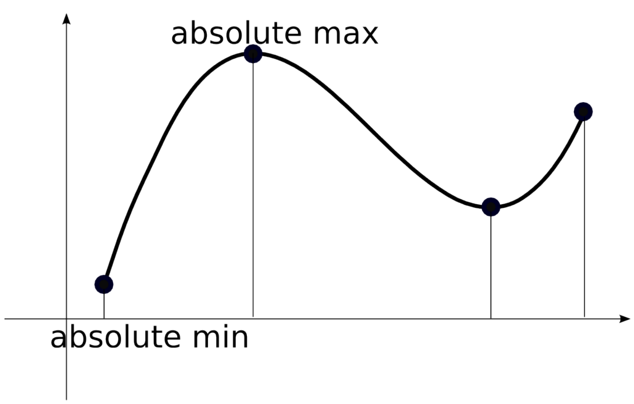
Similarly, $f(x)$ has a **relative minimum** at $x = c$ if $f(c) \leq f(x)$ for all values of x in some open interval containing c. If $f(c)$ is the absolutely smallest value on entire domain of f, that is if $f(c) \leq f(x)$ for all the values x in the domain of f, then f is said to have an **absolute minimum** at $x = c$.

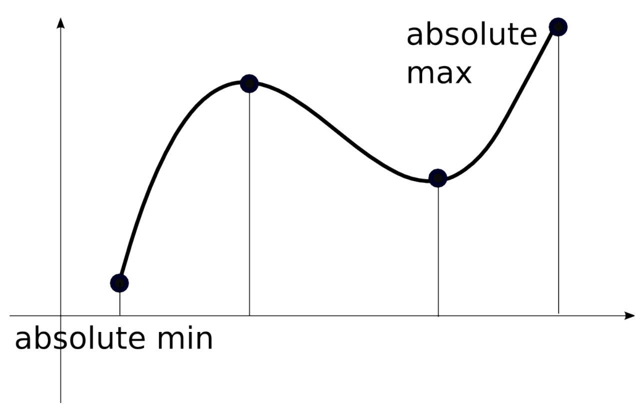


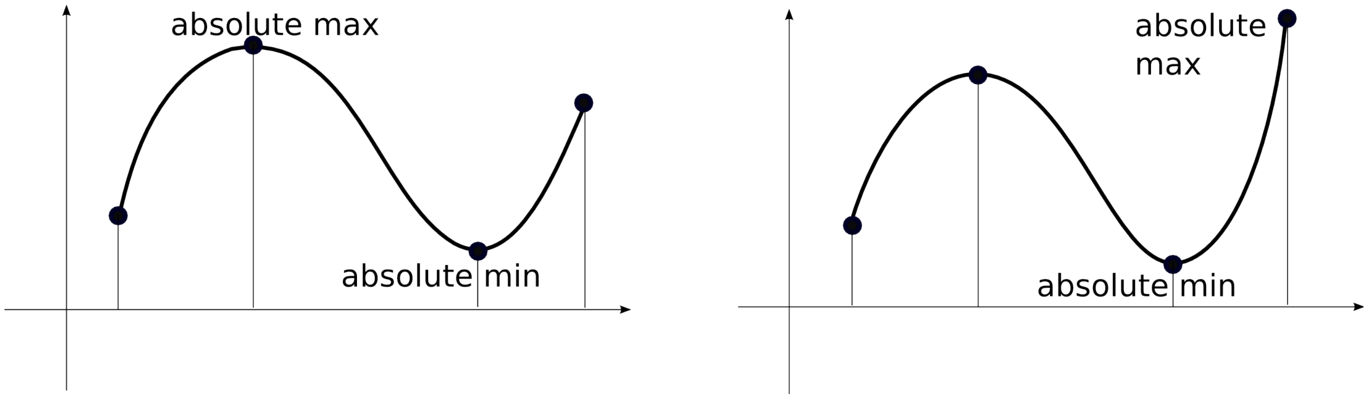
Even if having a relative extrema, a function does not have to have an absolute extrema. For example, the function on the figure on the right defined on $(-\infty, 2)$ has both relative minimum and a relative maximum but has neither an absolute minimum nor an absolute maximum.

However, if the domain of a continuous function $f(x)$ is a *closed* interval, then f *achieves both the absolute maximum and absolute minimum* on the interval.

This statement is known as the Extreme Value Theorem. We illustrate this theorem by the following figures.







In the figure above, we can see that the absolute extreme value is either at a critical point or at the end point of the interval. When one finds all the critical points and the endpoints and plugs them in the function, the largest value obtained is the absolute maximum and the lowest is the absolute minimum. Thus we have the following.

The Closed Interval Method. To find the absolute maximum and minimum values of a continuous function $f(x)$ on a closed interval $[a, b]$:

1. Find $f'(x)$ and the critical points in (a, b).
2. Evaluate $f(x)$ at the critical values in $[a, b]$ and the endpoints a and b. Then
   - the largest value you obtain is the absolute maximum and
   - the smallest value you obtain is the absolute minimum.

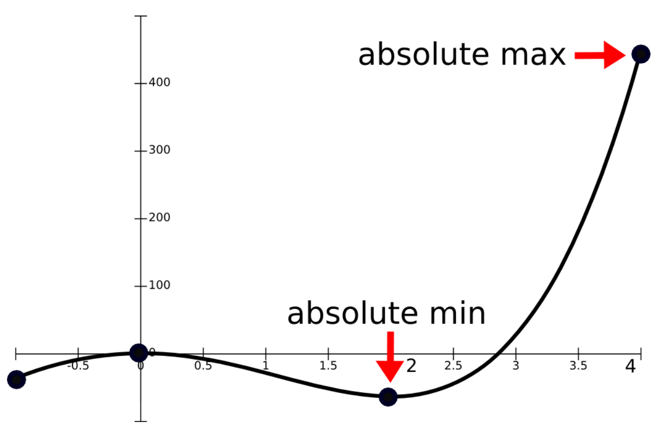
Example 1. Find the absolute minimum and maximum of $f(x) = 3x^4 + 4x^3 - 36x^2 + 1$ on the interval $[-1, 4]$.

Solutions. Find derivative $f'(x) = 12x^3 + 12x^2 - 72x = 12x(x^2 + x - 6) = 12x(x - 2)(x + 3)$. Thus the critical values are 0, 2 and -3. Note that -3 is not in the interval $[-1, 4]$, so it is not relevant for this problem.

Evaluate the function at the critical points 0 and 2 and at the endpoints -1 and 4. Obtain that

$$f(0) = 1 \qquad f(2) = -63$$
$$f(-1) = -36 \qquad f(4) = 449$$

As 449 is the largest of these four values, (4, 449) is the absolute maximum and as -63 is the smallest, $(2, -63)$ is the absolute minimum.



Example 2. Assume that the formula

$$C(t) = 2te^{-.4t} \text{ for } 0 \leq t \leq 4$$

computes the concentration C (in μg/cm^3) of a certain drug present in the body t hours after the drug was administered.

Find the minimal and the maximal concentration during the first four hours the drug is present in the body.

Solutions. Use the product rule for the derivative $C'(t) = 2e^{-.4t} + e^{-.4t}(-.4)2t = 2e^{-.4t}(1 - .4t)$. Find the critical points by setting the derivative to zero and solving for t.

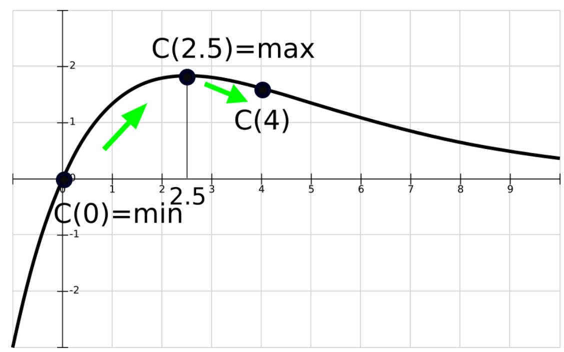
$$2e^{-.4t}(1 - .4t) = 0 \Rightarrow 2e^{-.4t} = 0 \text{ or } 1 - .4t = 0 \Rightarrow e^{-.4t} = 0 \text{ or } 1 = .4t \Rightarrow \text{ no sol. or } t = \frac{1}{.4} = \frac{5}{2} = 2.5.$$

Note that $e^{-.4t} = 0$ has no solutions since the exponential function is always positive (alternatively $e^{-.4t} = 0 \Leftrightarrow -.4t = \ln 0$ which is not defined). Hence $t = 2.5$ is the only critical point.

Plug the critical point 2.5 and the endpoints 0 and 4 into the function $C(t)$. As

$$C(2.5) = 1.84, \quad C(0) = 0, \quad \text{and} \quad C(4) = 1.615,$$

we conclude that the minimal concentration is 0 μg/cm^3 and it is reached at the very beginning, zero hours after the drug is administered. The maximal concentration is 1.84 μg/cm^3 and it is reached 2.5 hours after the drug is administered.



Constrained Optimization

Finding optimal conditions under which a certain event occurs is one of the most important applications of calculus. The term *optimization problem* refers to a problem of finding such optimal conditions. The quantity which needs to be optimized is referred to as the **objective**. The objective can depend on more than one variable. In this case, an equation that relates the variables is called the **constraint**.

To solve an applied optimization problem follow the steps below.

1. Read the problem carefully. Sketch a diagram if possible in order to visualize the relevant information.

2. List the relevant quantities in the problem and assign them appropriate variables.

3. Determine the quantity to be maximized or minimized and write down how it depends on the independent variables. This gives you the **objective**. Look for the *key words* in the problem (the largest, the smallest, the shortest, the quickest, the cheapest and so on) indicating the quantity that is to be optimized.

4. Determine how the independent variables are related. This gives you the **constraint equation**. The constraint often involves the *numerical value* given in the problem.

5. Using the constraint, express one independent variable in terms of the other. Using this, eliminate a variable from the objective equation making it a function of single variable.

6. Find the extreme values of the objective simplified by the previous step. If the domain of the objective is a closed interval, use *the Closed Interval Method*. If not, you need to use *either the First or the Second Derivative Test* to determine whether there is a minimum or a maximum at each of the relevant critical points.

   When you have found the needed value of the first independent variable, use the constraint to find the value of the other independent variable.

7. Interpret the solution. Write a sentence that answers the question posed in the problem.

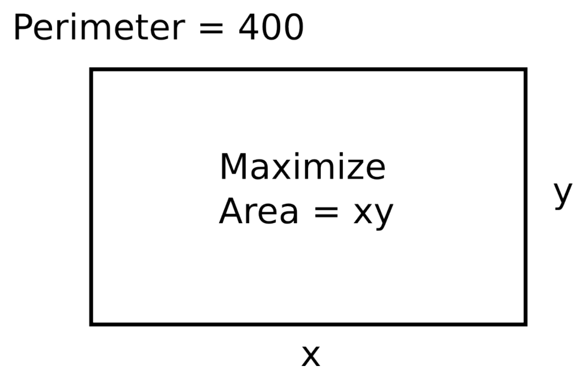
We illustrate this method with examples below.

Example 1. Find the dimensions of the rectangular garden of the greatest area that can be fenced off with 400 feet of fencing.

Solution. The problem is asking for optimal dimensions of the rectangular region so let us start by graphing a rectangular region and denoting the length and width by x and y.

Determine the objective. Note the words "the greatest area". This means that the area of the rectangular region is the objective. If we denote the area by A, the objective is $A = xy$.

Determine the constraint. The numerical reference "400 feet of fencing" indicates the constraint. The length of the fence corresponds to the perimeter of the rectangle $2x + 2y$.



Thus the perimeter being 400 is the constraint equation. So, $2x+2y = 400$, or simplified $x+y = 200$ is the constraint.

Eliminate a variable. Solve the constraint for x or y. For example with $y = 200 - x$ the objective becomes

$$A = xy = x(200 - x) = 200x - x^2.$$

Note that x and y are nonnegative numbers so the domain of $A(x)$ is bounded below by $x = 0$. When $y = 0$, x is the largest possible $x = 200$. So the domain of A is $[0, 200]$.

Find the maximum. The derivative of the area is $A'(x) = 200 - 2x$ and the only critical point is $200 - 2x = 0 \Rightarrow x = 100$. At this point we know only that this is a critical point − we cannot assume that it maximizes the function. So, to check whether it really maximizes A, you need to do *one of the following:* the First Derivative Test, the Second Derivative Test, or the Closed Interval Method.

1. *Choosing the First Derivative Test.* Perform the line test for A'. Obtain

A'	$+$		$-$
A	$\nearrow$	100	$\searrow$

   Conclude that there is a maximum at $x = 100$.

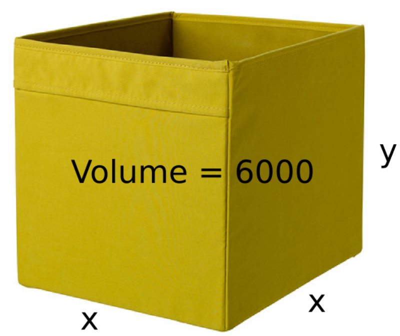
2. *Choosing the Second Derivative Test.* Find the second derivative $A''(x) = -2$. Since it is less than zero at any point, including the critical point $x = 100$, we conclude that the function is concave down at $x = 100$, so there is a maximum at 100.

3. *Choosing the Closed Interval Method.* Note that x and y are nonnegative numbers so the domain of $A(x)$ is bounded below by $x = 0$. When $y = 0$, x is the largest possible $x = 200$. So the domain of A is $[0, 200]$. Plug the endpoints 0, and 200, and the critical point 100 into the objective to determine the absolute extremes. $A(0) = A(200) = 0$ is the minimum and $A(100) = 10,000$ is the maximum.

When $x = 100, y = 200 - x = 200 - 100 = 100$.
Make a conclusion. The dimensions of 100 ft with 100 ft produce the largest area of $10,000$ ft^2.

Example 2. An open top box is made with a square base and should have a volume of 6000 cubic inches. If the material for the sides costs \$.20 per square inch and the material for the base costs \$.30 per square inch, determine the dimensions of the box that minimize the cost of the materials.

Solution. The problem is asking for the dimensions that minimize the cost. You can start by graphing a open top box with a square base and denoting the sides of the base by one variable and the height with the other. For example, x and y.



Determine the objective. With the requirement that the cost needs to be minimized, the cost of the material is the objective. The total cost is the sum of the cost for the bottom and the cost for the sides. We are given the prices in dollars per square inch so these prices need to be multiplied with corresponding areas in square inch to produce the cost in dollars. If we denote the cost by C, we have that

$$\begin{aligned} \text{Total cost } C &= \text{cost for the base} && + \text{ cost for the sides} \\ &= 0.3\text{ (area of the base)} && + 0.2\text{ (area of the four sides)} \\ &= 0.3\,(x^2) && + 0.2\,(4 \text{ times } xy) \\ &= 0.3\,x^2 && + 0.8\,xy. \end{aligned}$$

Determine the constraint. The numerical reference "6000 cubic inches" indicates the constraint. It refers to the volume of the box and so the volume being 6000 is the constraint equation. Since the volume is the product of the area of the base x^2 and the height y, we obtain the constraint

$$x^2 y = 6000.$$

Eliminate a variable. Note that it is easier to solve the constraint for y instead of x. So $y = \frac{6000}{x^2}$ and the objective becomes

$$C = 0.3x^2 + 0.8xy = 0.3x^2 + 0.8x\frac{6000}{x^2} = 0.3x^2 + \frac{4800}{x}$$

Find the minimum. Find the derivative $C'(x) = 0.6x - \frac{4800}{x^2}$ and the critical points 0 and the solution of $0.6x - \frac{4800}{x^2} = 0 \Rightarrow 0.6x^3 = 4800 \Rightarrow x^3 = 8000 \Rightarrow x = \sqrt[3]{8000} = 20$.

Using the First Derivative Test, obtain $\begin{array}{c|cc} C' & - & + \\ \hline C & \searrow \; 20 & \nearrow \end{array}$. Thus, there is an absolute minimum at 20. Alternatively, you can plug 20 in the second derivative $C''(x) = 0.6 + \frac{9600}{x^3}$ and, since $C''(20) = 1.8 > 0$, conclude that there is a minimum at 20 using the Second Derivative Test.

When $x = 20$, determine that the height is $y = \frac{6000}{x^2} = \frac{6000}{400} = 15$.

Make a conclusion. To obtain the minimal cost of 360 dollars for making the box, the base needs to have a side of 20 inches and the height should be 15 inches.

Practice Problems.

1. Find the absolute minimum and maximum of the function $f(x) = x^3 - 3x^2 - 12x + 24$ on the indicated interval. (a) $[0, 6]$ (b) $[-3, 6]$

2. The percent concentration of a certain medication during the first 20 hours after it has been administered is approximated by
$$p(t) = \frac{230t}{t^2 + 6t + 9} \qquad 0 \leq t \leq 20.$$
Determine the minimal and maximal concentration during the first 20 hours.

3. The function $B(t) = 5 - \frac{1}{9}\sqrt[3]{(8-3t)^5}$ models the biomass (total mass of the members of the population) in kilograms of a mice population after t months. Determine when the population is smallest and when it is the largest between 3 and 6 months after it started being monitored.

4. In a physics experiment, temperature T (in Fahrenheit) and pressure P (in kilo Pascals) have a constant product of 5000 and the function $F = T^2 + 50P$ is being monitored. Determine the temperature T and pressure P that minimize the function F.

5. A fence must be built in a large field to enclose a rectangular area of 400 square meters. One side of the area is bounded by existing fence; no fence is needed there. Material for the fence cost \$ 8 per meter for the two ends, and \$ 4 per meter for the side opposite the existing fence. Find the cost for the least expensive fence.

6. Consider a box with a square base. Find the dimensions of the box with the surface area 96 square inches, such that the volume is as large as possible.

7. A company wishes to manufacture a box with a volume of 36 cubic feet that is open on the top and is twice as long as it is wide. Find the dimensions of the box produced from the minimal amount of the material.

8. If p denotes the frequency of the dominant allele and q the frequency of recessive allele so that $p + q = 1$, the Hardy - Weinberg Law states that the proportion of individuals in a population who are heterozygous is $2pq$ and the proportion of individuals who are homozygous is $p^2 + q^2$.
   (a) Find the maximal and minimal percentage of people that are heterozygous.
   (b) Find the maximal and minimal percentage of people that are homozygous.

Solutions.

1. $f(x) = x^3 - 3x^2 - 12x + 24 \Rightarrow f'(x) = 3x^2 - 6x - 12 = 3(x^2 - 2x - 4) = 3(x-4)(x+2)$. Thus, the critical points are at $x = 4$ and at $x = -2$.

   For part (a) only $x = 4$ is relevant since -2 is outside of [0,6]. Evaluate the function $f(x)$ at the critical point 4 and at the endpoints 0 and 6. Obtain that $f(0) = 24$, $f(6) = 60$, and $f(4) = -8$. Hence, the maximum is 60 at $x = 6$ and the minimum is -8 at $x = 4$.

For part (b), both critical points are in the given interval $[-3, 6]$. Evaluate the function $f(x)$ at both critical points and at both endpoints. In addition to $f(6) = 60$ and $f(4) = -8$, obtain that $f(-2) = 28$ and $f(-3) = 6$. So, the absolute maximum is still 60 at $x = 6$ and the absolute minimum is still -8 at $x = 4$.

2. $p(t) = \frac{230t}{t^2+6t+9} \Rightarrow p'(t) = \frac{230(t^2+6t+9)-(2t+6)230t}{(t^2+6t+9)^2} = \frac{230(t^2+6t+9-2t^2-6t)}{(t^2+6t+9)^2} = \frac{230(9-t^2)}{(t^2+6t+9)^2} = \frac{230(3-t)(3+t)}{(t+3)^4} = \frac{230(3-t)}{(t+3)^3}$. Thus the critical points are ± 3. Since only 3 is in the interval $[0, 20]$, only 3 is relevant.

   Plug the critical point 3 and the endpoints 0 and 20 into the function $p(t)$ and have

   $$p(3) = \frac{115}{6} \approx 19.17, \quad p(0) = 0, \quad \text{and} \quad p(20) = \frac{4600}{529} \approx 8.696.$$

   Conclude that 0 % is the minimal and 19.17 % is the maximal percent concentration.

3. $B(t) = 5 - \frac{1}{9}\sqrt[3]{(8-3t)^5} \Rightarrow B'(t) = \frac{-5}{27}(8-3t)^{2/3}(-3) = \frac{5}{9}(8-3t)^{2/3}$. The only critical point is $8 - 3t = 0 \Rightarrow t = \frac{8}{3}$ and it is not in the interval. Evaluate function at the endpoints 3 and 6. Since $B(3) \approx 5.11$ and $B(6) \approx 10.16$, the absolute maximum is 10.16 kg at $t = 6$ months and the absolute minimum is 5.11 kg at $t = 3$ months.

4. The objective is $F = T^2 + 50P$ and the constraint is $PT = 5000$. Solving for P for example, we have that $P = \frac{5000}{T}$ and so $F = T^2 + \frac{250000}{T}$. Then $F' = 2T - \frac{250000}{T^2} = \frac{2T^3 - 250000}{T^2}$. The critical points are the solutions of $2T^3 - 250000 = 0 \Rightarrow 2T^3 = 250000 \Rightarrow T^3 = 125000 \Rightarrow T = 50$ and $T^2 = 0 \Rightarrow T = 0$. Use the First or the Second Derivative Test. With the latter, $F'' = 2 + \frac{500000}{T^3}$ and $F''(50) = 2 + 4 = 6 > 0$, so there is a minimum at $T = 50$. F is not defined at 0, so there is no extreme value at 0. When $T = 50$, $P = \frac{5000}{50} = 100$. Thus, the pressure of 100 kPa and the temperature of 50 degrees Fahrenheit minimize the function F.

5. Using x for the length of the side opposite to the existing fence and y for the other side, the objective, the cost function, is $C = 4x + 8 \cdot 2y = 4x + 16y$. The constraint is $xy = 400$. Solving for y, for example, you obtain that $y = \frac{400}{x}$ so that $C = 4x + 16\frac{400}{x} = 4x + \frac{6400}{x}$. Thus, $C' = 4 - \frac{6400}{x^2} = \frac{4x^2 - 6400}{x^2}$. The critical points are the solutions of $4x^2 - 6400 = 0 \Rightarrow x^2 = 1600 \Rightarrow x = \pm 40$ and $x^2 = 0 \Rightarrow x = 0$. Thus, the only relevant critical point is $x = 40$. To check that there is a minimum at $x = 40$, use the First or the Second Derivative Test. Using the latter, $C''(x) = \frac{12800}{x^3}$ and so $C''(40) = \frac{12800}{40^3} = 0.2 > 0$. Hence there is a minimum at $x = 40$. When $x = 40$, $y = \frac{400}{40} = 10$. Hence, 40 and 10 are dimensions that minimize the cost which becomes \$ 320 in that case.

6. The objective is the volume $V = x^2 y$. The constraint is the surface area being 96 in^2. Hence, $2x^2 + 4xy = 96 \Rightarrow x^2 + 2xy = 48$. Solving the constraint for y produces $2xy = 48 - x^2 \Rightarrow y = \frac{48-x^2}{2x}$. The volume becomes $V = x^2 \frac{48-x^2}{2x} = \frac{1}{2}x(48 - x^2) = \frac{1}{2}(48x - x^3)$. Hence, $V' = \frac{1}{2}(48 - 3x^2)$ and the critical points are $48 - 3x^2 = 0 \Rightarrow 48 = 3x^2 \Rightarrow 16 = x^2 \Rightarrow x = \pm 4$. Since negative values are not relevant, $x = 4$ is the only feasible critical point. To check that there is a minimum at $x = 4$, use the First or the Second Derivative Test. Using the latter, $V''(x) = \frac{1}{2}(-6x) = -3x$ so $V''(4) = -12 < 0$. Hence, the volume is maximal when $x = 4$ in which case $y = \frac{48-4^2}{2(4)} = \frac{32}{8} = 4$. Thus, the box with the maximal volume is a cube with the side of 4 inches.

7. Using x for the length of the shorter side of the base and y for the height, the dimensions of the box are $x, 2x$ and y. The objective is the surface area function $S = 2x^2+2xy+4xy = 2x^2+6xy$. The constraint is the volume being 36, so $2x^2y = 36$. Solve for y to get $y = \frac{18}{x^2}$ and substitute into the objective $S = 2x^2 + 6x\frac{18}{x^2} = 2x^2 + \frac{108}{x}$. As $S' = 4x - \frac{108}{x^2} = \frac{4x^3-108}{x^2}$, the critical points are $x = 0$ and $4x^3 - 108 = 0 \Rightarrow 4x^3 = 108 \Rightarrow x^3 = 27 \Rightarrow x = 3$. To check that there is a minimum at $x = 3$, use the First or the Second Derivative Test. Using the latter, $S'' = 4 + \frac{216}{x^3}$ and $S''(3) = 4 + \frac{216}{27} = 12 > 0$, so $x = 3$ minimizes S. When $x = 3$, $y = \frac{18}{3^2} = 2$. So, 3, 6 and 2 feet are the dimensions that minimize the amount of the material for the box.

8. (a) The objective is $F = 2pq$ and the constraint is $p + q = 1$. Thus $q = 1 - p$ and $F = 2p(1-p) = 2p-2p^2$. The only critical point is $F' = 2-4p = 0 \Rightarrow p = \frac{1}{2}$. Since p is the frequency (probability), we have that the domain of F is the closed interval [0,1], so that we can use the Closed Interval Method to find both minimum and maximum. Since $F(\frac{1}{2}) = \frac{1}{2} = 50\%$ and $F(0) = F(1) = 0\%$ the percent of heterozygous individuals in a population varies from 0 to 50.

   (b) The objective is $F = p^2 + q^2$ and the constraint is $p + q = 1$. Hence, $q = 1 - p$ and $F = p^2 + (1 - p)^2$. Thus, $F' = 2p + 2(1 - p)(-1) = 2p - 2 + 2p = 4p - 2$. The only critical point is $4p - 2 = 0 \Rightarrow p = \frac{1}{2}$. As in part (a), the extreme values are at the critical point or at the endpoints of $[0, 1]$. As $F(0) = 1, F(1) = 1$, and $F(\frac{1}{2}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, we can conclude that the minimal value is $\frac{1}{2}$ and the maximal value is 1. Hence, the percent of homozygous individuals in a population varies from 50 to 100 percent.