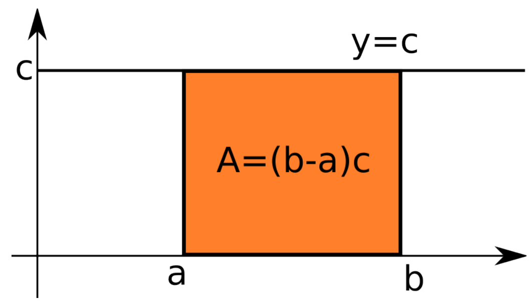
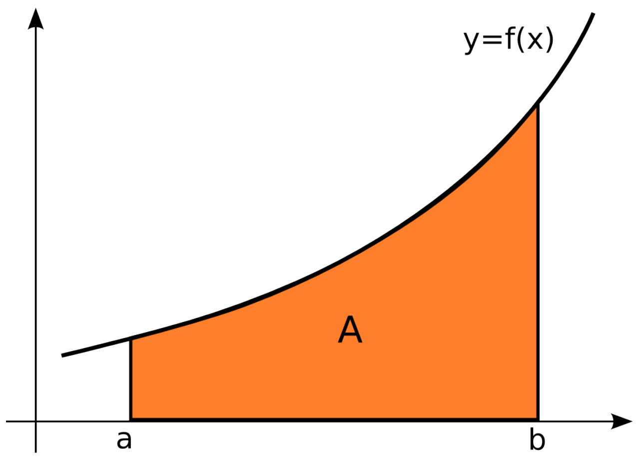
Calculus 1
Lia Vas

The Definite Integral

The Left and Right Sums. In this section we turn to the question of finding the area between a given curve and x-axis on an interval. At this time, this question seems unrelated to our consideration of indefinite integrals in the previous section. However, we relate the area under a curve with an antiderivative.

The area under a curve can be easily calculated if the curve is given by a simple formula. For example, if a function is a positive constant $f(x) = c$, the area under this curve on interval $[a, b]$ is the area of the rectangle with sides $b - a$ and c. Thus the area is $A = (b - a)c$.





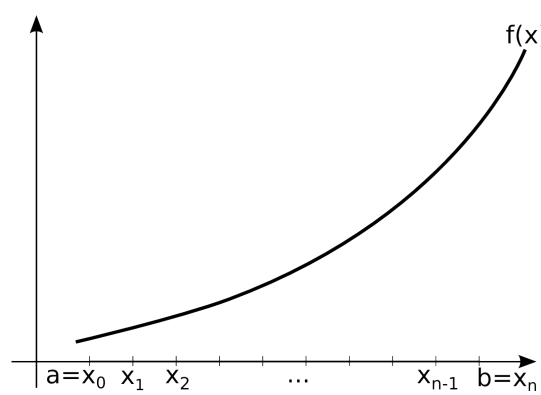
We are interested in calculating the area A under the curve f for $a \leq x \leq b$ for **any** non-negative, continuous curve $f(x)$. This, in general, rather complex problem can be approached the following idea: we divide the interval $[a, b]$ in small subintervals and approximate the area under $f(x)$ on each subinterval with the area of a rectangle with height $f(x_i)$ where x_i is a point in i-th subinterval of $[a, b]$.

Let us make this idea more precise using the sums of rectangles known as **the left and the right sums**.

Approximating the area.

Step 1. The partition. Divide $[a, b]$ into n-pieces. Get the points $a = x_0, x_1, \ldots, x_{n-1}, x_n = b$. This is called the **partition** of $[a, b]$. The distance between each two points is $h = \frac{b-a}{n}$.

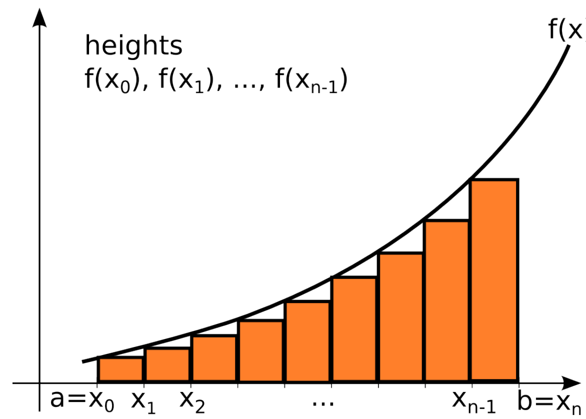
Step 2. The rectangles. The left and right sums differ in the way how we choose the point from the subinterval which will produce the height of the rectangle.



To approximate the area using the left sum, choose the **left** endpoint of each subinterval and evaluate the function at it to obtain the height of the rectangle. For example, x_0 is the left endpoint of the first subinterval $[x_0, x_1]$ and so the height of the first rectangle is $f(x_0)$.

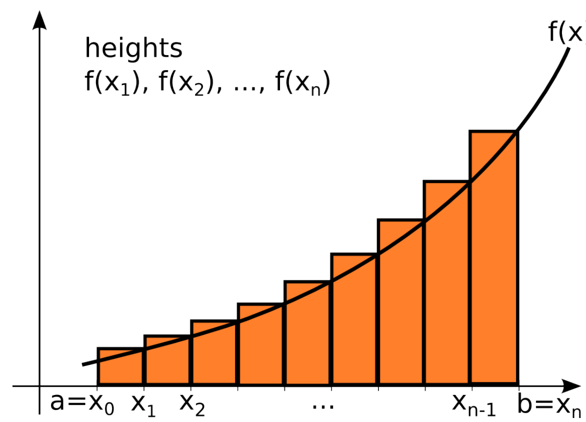
The area of the first rectangle is $A_0 = hf(x_0)$. The left endpoint of the second interval $[x_1, x_2]$ is x_1 and the second height is $f(x_1)$. The area of the second rectangle is $A_1 = hf(x_1)$. Continuing this process we obtain that the area of the $i+1$-st rectangle is $A_i = hf(x_i)$.

The process ends with the n-th rectangle of the area $A_{n-1} = hf(x_{n-1})$. The sum of all the rectangles is



$$\begin{aligned} \text{Left sum} &= hf(x_0) + hf(x_1) + \ldots + hf(x_{n-1}) = h\ \left(f(x_0) + f(x_1) + \ldots + f(x_{n-1})\right) \\ &= \tfrac{b-a}{n}\ \left(f(x_0) + f(x_1) + \ldots + f(x_{n-1})\right). \end{aligned}$$

Analogously, to approximate the area using the right sum, choose the **right** endpoint of each subinterval to form a rectangle. The point x_1 is the right endpoint of the first subinterval $[x_0, x_1]$ and so the height of the first rectangle is $f(x_1)$ and the area is $A_1 = hf(x_1)$. The right endpoint of the second interval $[x_1, x_2]$ is x_2, the second height is $f(x_2)$, and the area of the second rectangle is $A_2 = hf(x_2)$. Continuing this process we obtain that the area of the i-th rectangle is $A_i = hf(x_i)$.



The process ends with the n-th rectangle of the area $A_n = hf(x_n)$. The sum of all the rectangles is

$$\begin{aligned} \text{Right sum} &= hf(x_1) + hf(x_2) + \ldots + hf(x_n) = h\ \left(f(x_1) + f(x_2) + \ldots + f(x_n)\right) \\ &= \tfrac{b-a}{n}\ \left(f(x_1) + f(x_2) + \ldots + f(x_n)\right). \end{aligned}$$

Thus, the area under a curve can be approximated using both left and right sums.

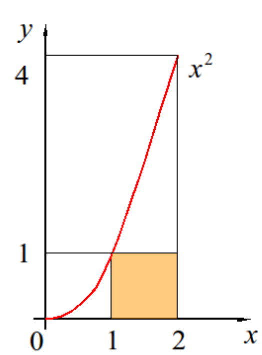
$$\text{Area } A \approx \text{ Left sum } = \frac{b-a}{n}\ \left(f(x_0) + f(x_1) + \ldots + f(x_{n-1})\right)$$

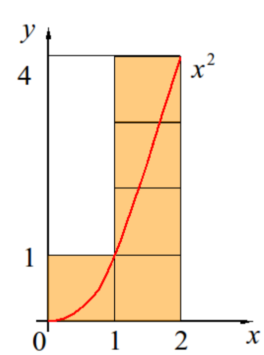
$$\text{Area } A \approx \text{ Right sum } = \frac{b-a}{n}\ \left(f(x_1) + f(x_2) + \ldots + f(x_n)\right)$$

Example 1. Approximate the area under $f(x) = x^2$ on [0,2] using the left and right sums with 2 subintervals.

Solution. Note that $a = 0, b = 2$ and $n = 2$. Thus $\frac{b-a}{n} = \frac{2-0}{2} = 1$. The partition consists of 3 points $a = x_0 = 0$, $x_1 = 1$ and $b = x_2 = 2$.

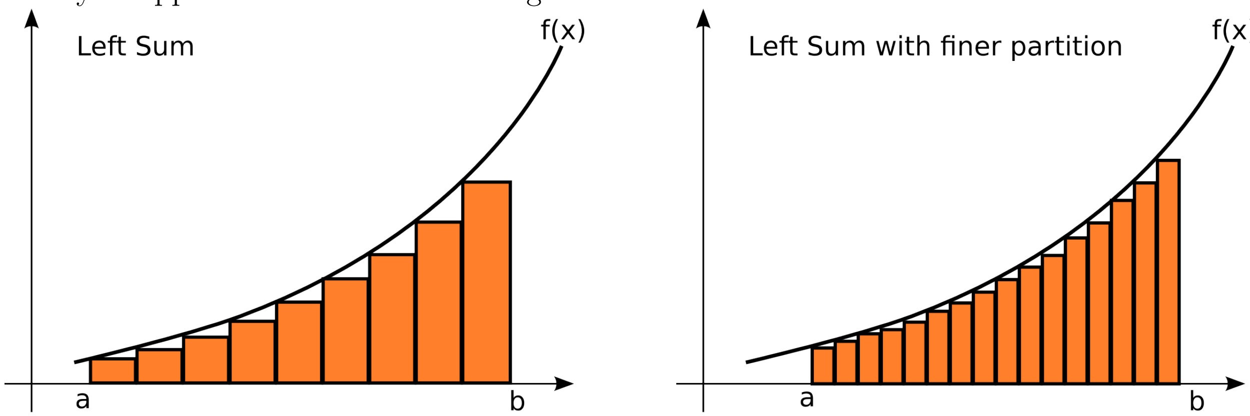
Compute the corresponding y-values to be $f(0) = 0^2 = 0$, $f(1) = 1^1 = 1$ and $f(2) = 2^2 = 4$. Thus the left sum is $L = \frac{2-0}{2}(0 + 1) = 1$ and the right sum is $L = \frac{2-0}{2}(1 + 4) = 5$.



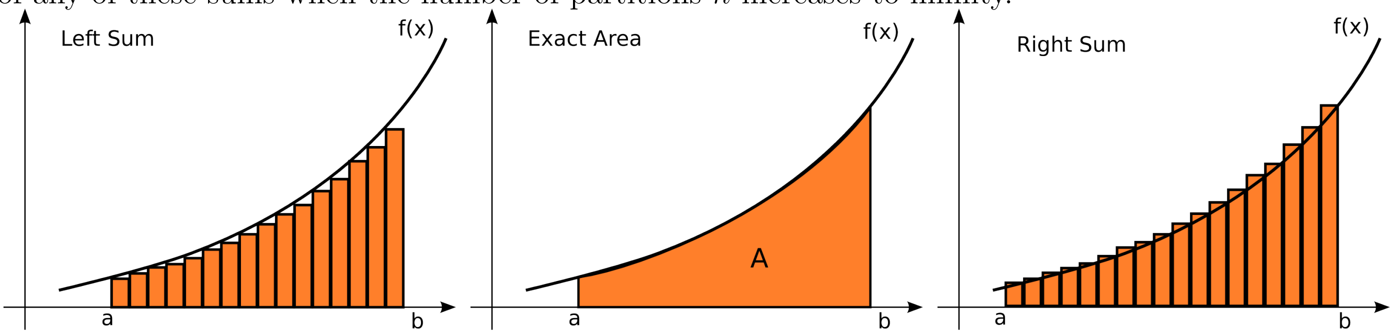


From the graph, we can conclude that the left sum 1 is an underestimate and the right sum 5 is an overestimate. Thus $1 \leq A \leq 5$.

Note that the approximation is not very accurate since we do not even have the first digit of the exact answer correct. Increasing the number of subintervals, that is using larger values of n, increase the accuracy of approximation as the next figure illustrates.



This fact suggests that finding the exact area under the curve can be obtained by taking the limit of any of these sums when the number of partitions n increases to infinity.



If f is increasing and nonnegative continuous function as on the last several figures, the exact area will be sandwiched between the left sum from below and the right sum from above (thus left and right sums are "two pieces of bread"). The limit of both left and right sum when $n \to \infty$ approaches the same value which is the exact value of the area under the curve ("ham") .

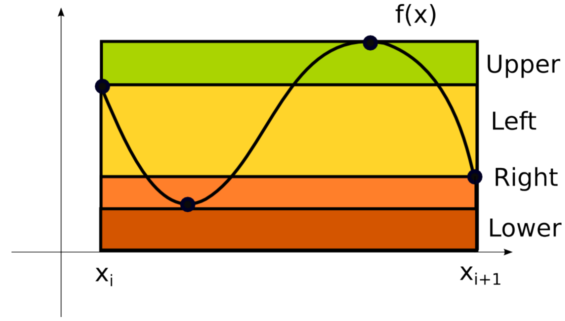
Area= ham, left and right sums = bread

If f is an *increasing*, nonnegative, continuous function, the left sum always produces an underestimate and the right sum an overestimate. For decreasing, nonnegative, continuous functions this is the opposite. For the functions with extreme values neither has to be the case.

When f is a nonnegative, continuous function, not necessary increasing, the area can still be sandwiched between the *lower sum* (obtained by forming rectangles using points with the smallest height on each subinterval) and the *upper sum* (obtained by forming rectangles using points with the highest height on each subinterval). Using the Closed Interval Method, such points can be found.

$$\text{Lower Sum} \leq \text{Area } A \leq \text{Upper Sum}$$

This also illustrates that the limit of the sum of rectangles formed by **any** point $\overline{x_i}$ from the i-th subinterval $[x_{i-1}, x_i]$, for $i = 1, 2, \ldots, n$, exists. Moreover this limit exists for any continuous function (not necessarily nonnegative) defined on a closed interval.



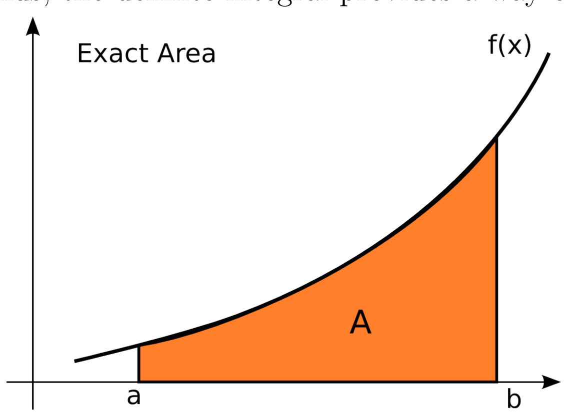
The sum of the rectangles is $(f(\overline{x_1}) + f(\overline{x_2}) + \ldots + f(\overline{x_n}))h$ and can be written shorter as $\sum_{i=1}^{n} f(\overline{x_i})h$. The length of each subinterval $h = \frac{b-a}{n}$ is also denoted as Δx. The limit $\lim_{n\to\infty} \sum_{i=1}^{n} f(\overline{x_i})\Delta x$ is defined to be the **definite integral** of a continuous function $f(x)$ on an interval $[a, b]$ and it is denoted as $\int_a^b f(x)dx$. Thus we have that

$$\textbf{The Definite Integral} \quad \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(\overline{x_i})\,\Delta x.$$

The following informal reasoning is the base for the notation $\int_a^b f(x)dx$: the small values of Δx correspond to dx, when the number of subintervals is very large and their number very small, picking values $\overline{x_i}$ from each subinterval amounts to picking all points x from the interval $[a, b]$. In the limiting case, the sum becomes the integral symbol indicating that we pick values x on continuous way from entire interval $[a, b]$.

$$\Delta x \rightsquigarrow dx, \quad f(\overline{x_i}) \rightsquigarrow f(x), \quad \sum_{i=1}^{n} \rightsquigarrow \int_a^b \quad \text{so that} \quad \sum_{i=1}^{n} f(\overline{x_i})\,\Delta x \to \int_a^b f(x)\,dx.$$

Thus, the definite integral provides a way of calculating the area under the curve.



If f nonnegative, continuous function on $[a, b]$, the area A under the curve f above the x-axis, for $a \leq x \leq b$ is

$$A = \int_a^b f(x)\ dx.$$

Before addressing the area under curves which are not nonnegative in the next section, we consider some applications of the left and right sums.

Example 2. A skydiver drops from an airplane. At the end of the first six seconds the diver's speed (in meters per second) is checked, and it reads as follows.

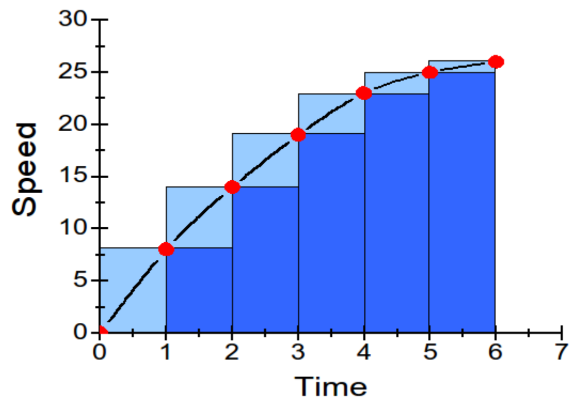
speed	8	14	19	23	25	26

You can assume that the diver's initial speed is zero. Use the appropriate left and right sums to approximate the distance the diver falls during the six-second period.

Solution. With the restrictions of the measurements above, we approximate the speed during each second with a constant. Thus the distance traveled in each second can be approximated with *the product of the speed and the time.* If the speed values are represented as dependent variable values and time values as the independent variable values, the product of velocity and time corresponds exactly to area of rectangles. Since the speed values are strictly increasing and positive, the left sum provides an underestimate and the right sum the overestimate of the total distance traveled.

Adding the time values to the given table, we obtain the following.

time	0	1	2	3	4	5	6
speed	0	8	14	19	23	25	26



The speed is measured on the time interval [0,6] so we can denote $a = 0$ and $b = 6$. The number of subintervals is $n = 6$. Note that n represents *the number of subintervals not the number of points* (7 points determines 6 subintervals). The left sum is $L = \frac{6-0}{6}(0+8+14+19+23+25) = 89$ and the right sum is $R = \frac{6-0}{6}(8 + 14 + 19 + 23 + 25 + 26) = 115$.

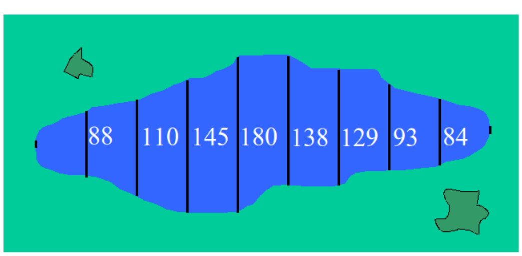
Thus the diver fell by an amount between 89 and 115 meters (note that the units of the answer are meters since the term $\frac{6-0}{6}$ is in seconds and the sums in parenthesis are in meters per seconds).

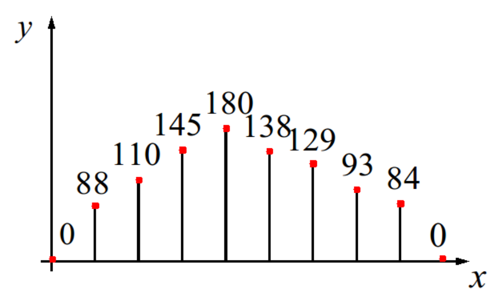
Example 3. Approximate the area of the lake using the shown measurements of its width which were taken 50 feet apart.

0	88	110	145	180	138	129	93	84	0

Solution. You can consider the measurement values as the dependent variable values at the values of the independent variable starting at 0 and being 50 feet apart.

x	0	50	100	150	200	250	300	350	400	450
y	0	88	110	145	180	138	129	93	84	0





From the table above we can see that $a = 0$ and $b = 50 \cdot 9 = 450$. There are 10 points, thus $n = 9$ subintervals. Alternatively, the measurements are taken 50 feet apart, thus $h = \frac{b-a}{n} = 50$.

The left sum is $L = \frac{450-0}{9}(0 + 88 + 110 + 145 + 180 + 138 + 129 + 93 + 84) = 48350$ and the right sum is $R = \frac{450-0}{9}(88 + 110 + 145 + 180 + 138 + 129 + 93 + 84 + 0) = 48350$. So, the area of the lake is estimated to be 48350 square feet. Try to generalize the conclusion of this problem: the left and the right sums will be equal whenever $f(x_0) = f(x_n)$.

Left and Right Sum Program for TI83–84

The program below calculates left and right sum of a given function f on a given interval $[a, b]$ with the given number of subintervals n. When executing this program in order to approximate $\int_a^b f(x)dx$ using left and right sums, the function f should be entered as $\mathbf{Y}_1$ and a, b and n should be entered when executing the program.

When entering the program in the calculator, editing or executing and existing program, start by using **PRGM** key.

```
PROGRAM: LFTRGT
Disp ''LOWER BOUND''          (to display Disp, choose PRGM then I/0 menu)
Input A                       (to display Input, choose PRGM then I/0 menu)
Disp ''UPPER BOUND''
Input B
Disp ''NUMBER OF SUBINTERVALS''
Input N
(B-A)/N → D
0 → L
For (I, 0, N-1)               (to display For, choose PRGM then CTL menu)
A+D*I → X
L+D*Y1 → L                    (to display Y1, choose VARS, Y-VARS, then 1: Function)
End                           (to display End, choose PRGM then CTL menu)
Disp ''LEFT SUM'', L
0 → R
For (J, 1, N)
A+D*J → X
R+D* Y1 → R
End
Disp ''RIGHT SUM'', R
```

You can exit the program editing by using **QUIT**.

Example 4. Use the calculator program to find the left and right sums of $f(x) = x^2$ on interval [0,2] with 50 subintervals.

Solution. Start by entering x^2 as your function $\mathbf{Y}_1$. Then execute the program by **PRGM** and choosing **LFTRGT**. Enter the bounds 0 and 2 and the number of subintervals 50. You should get 2.5872 for the left and 2.7472 for the right sum.

Example 5. Approximate the area under the curve of $f(x) = x^2$ on interval [0,2] to the first two nonzero digits.

Solution. Since x^2 is positive and increasing on [0,2], the left sum is an underestimate and the right sum is an overestimate of the exact area $\int_0^2 x^2 dx$.

We are given function and the interval but we are not given the number of subintervals. Thus, you can start by entering the given function, the endpoints of the interval and a relatively large n (e.g. 100 or 200) to monitor the left and right sum values L and R. Keep increasing n if necessary until L and R both round to the same value with desired number of significant digits. In that case, you obtain the value of the definite integral you are asked to find.

Note that you do not want to start with n that is too small (e.g. below 10 because it will be most likely that you will have to increase it anyway) nor n that is too large (e.g. 100,000 because it may take a long time for your calculator to produce the answers). Note also that you are not asked to find the smallest possible value of n but *any* value of n that will produce the desired accuracy.

In the previous example we obtained that with $n = 50$ the left sum rounds to 2.6 and the right sum to 2.7. Since we need to have the first two nonzero digits in the left and right sums equal, we need to increase the accuracy.

With $n = 200$, we obtain $2.6467 \approx 2.6$ for the left and $2.6867 \approx 2.7$ for the right which is still not accurate enough. With $n = 300$, we obtain $2.6533 \approx 2.7$ for the left and $2.6800 \approx 2.7$ Thus, we conclude that the area is 2.7 up to the first two nonzero digits.

Practice Problems.

1. Approximate the area under the following curves on the given intervals using left and right sums using the partition with n subintervals.
   (a) $f(x) = x, \ [0,2], \ n = 2.$
   (b) $f(x) = x^2, \ [0,2], \ n = 4.$
   (c) $f(x) = 8\sqrt{x}, \ [0,3], \ n = 100.$ Use the calculator program.

2. Approximate the integral
$$\int_1^4 x^{-2}\,dx$$
   using the Left-Right Sums calculator program with $n = 150$ subintervals. Note that the function $\frac{1}{x^2}$ is decreasing on [1,4] so that the left sums gives you the overestimate and the right sum the underestimate.

3. Approximate the following definite integrals using your calculator program to the first two nonzero digits.

   (a) $\displaystyle\int_0^2 \ln(x^2+1)\,dx$ (b) $\displaystyle\int_1^3 \frac{e^{2x}}{x}\,dx$

4. The speed of a runner increases during the first three seconds of a race. His speed (in meters per second) at half-second intervals is given in the table bellow. Find lower and upper estimates for the distance he traveled during the first three seconds.

time	0	0.5	1	1.5	2	2.5	3
speed	0	3.1	5.4	7.4	8.9	9.2	10.1

5. A chemical reaction produces a compound X with a rate of 23, 19, 12, 11, 9, 5, 2 liters per second at time intervals spaced by 1 second. Approximate the total volume of the compound X produced in the 6 seconds for which the rate is given using the left and the right sums.

Solutions.

1. (a) $n = 2$ and

x	0	1	2
y	0	1	2

thus $L = \frac{2-0}{2}(0+1) = 1$ and $R = \frac{2-0}{2}(1+2) = 3$.

(b) $n = 4$, and

x	0	0.5	1	1.5	2
y	0	0.25	1	2.25	4

thus $L = \frac{2-0}{4}(0 + 0.25 + 1 + 2.25) = \frac{3.5}{2} = 1.75$ and $R = \frac{2-0}{4}(0.25 + 1 + 2.25 + 4) = \frac{7.5}{2} = 3.75$.

(c) $f(x) = 8\sqrt{x}$, $a = 0$, $b = 3$ and $n = 100$. Obtain that $L = 27.496$ and $R = 27.912$.

2. Use the program with $f(x) = x^{-2}$, $a = 1$, $b = 4$ and $n = 150$. Obtain that $L = .759$ and $R = .741$. Thus $.741 \leq \int_1^4 \frac{1}{x^2}\, dx \leq .759$.

3. (a) $f(x) = \ln(x^2 + 1)$, $a = 0$ and $b = 2$. With $n = 100$ for example, both the left sum and the right sum sound to 1.4. Thus, $\int_0^2 \ln(x^2 + 1)\, dx \approx 1.4$.

(b) $f(x) = \frac{e^{2x}}{x}$, $a = 1$, and $b = 3$. With $n = 300$ for example, both the left sum and the right sum round to 81. Thus, $\int_1^3 \frac{e^{2x}}{x}\, dx \approx 81$.

4. Similarly as in Example 2, the left and the right sum provide the estimates for distance traveled. In this problem, $a = 0$, $b = 3$, $n = 6$ (careful: n is not 7). The left sum is $L = \frac{3-0}{6}(0+3.1+5.4+7.4+8.9+9.2) = \frac{34}{2} = 17$ and the right sum is $L = \frac{3-0}{6}(3.1+5.4+7.4+8.9+9.2+10.1) = \frac{44.1}{2} = 22.05$. Thus, 22.05 meters is the upper and 17 meters is the lower estimate for the distance.

5. With the restrictions of the measurements above, we approximate the rate during each second with a constant. Thus the volume added in each second can be approximated with *the product of the rate of increase and time.* If the rate values are represented as dependent variable values and time values as the independent variable values, the product of the rate and time corresponds exactly to area of rectangles. Since the rate values are strictly decreasing and positive, the left sum provides an overestimate and the right sum the underestimate of the total volume produced. Adding the time values to the given table, we obtain the following.

time (sec.)	0	1	2	3	4	5	6
rate (l/sec.)	23	19	12	11	9	5	2

Thus, $a = 0$, $b = 6$ and $n = 6$. $L = \frac{6-0}{6}(23 + 19 + 12 + 11 + 9 + 5) = 79$ liters, $R = \frac{6-0}{6}(19 + 12 + 11 + 9 + 5 + 2) = 58$ liters.

The Fundamental Theorem of Calculus and the Area Under a Curve

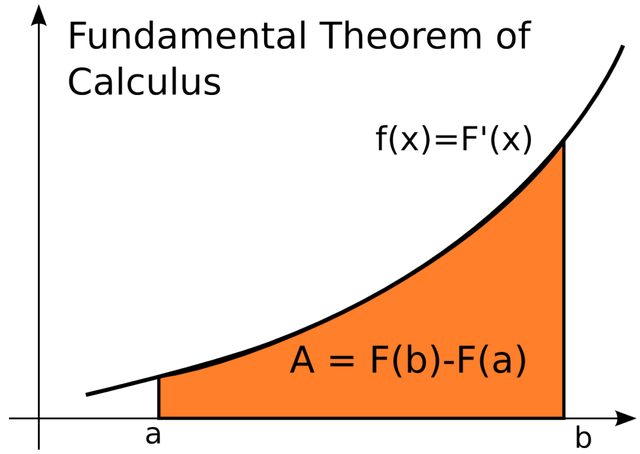
In Example 2 of the previous section, we used the left and right sums to approximate the distance traveled during some time interval given the speed values at certain points. Since the left and right sum approximate the definite integral, this suggests that the exact distance traveled can be obtained as the definite integral of the speed.

More generally, if a continuous function $f(x)$ represents the rate of change of $F(x)$ (so that $F(x)$ is an antiderivative of $f(x)$) on interval $[a, b]$, then

> the **total change** $F(b) - F(a)$ in $F(x)$ as x changes from a to b is given by $\int_a^b f(x)\, dx$.

The statement that $\int_a^b f(x)\ dx = F(b) - F(a)$ is known as the

Fundamental Theorem of Calculus.



$$\int_a^b f(x)\ dx = F(b) - F(a)$$

This statement relates the definite integral $\int_a^b f(x)dx$ with an antiderivative $F(x)$ so it relates definite and indefinite integral. Moreover, it provides us with the way of calculating a definite integral *without* computing the sums of any rectangles and finding their limits.

When evaluating the antiderivative $F(x)$ at b and a and subtracting the answers, the notation

$$F(x)|_a^b \text{ is used for } F(b) - F(a).$$

Thus,

to evaluate $\int_a^b f(x)\ dx$	1. find an antiderivative $F(x)$ of $f(x)$, 2. evaluate $F(x)\|_a^b$, 3. obtain the numerical answer $F(b) - F(a)$.

Example 1. Evaluate the integral $\int_0^2 x^2 dx$ computing the area under x^2 from 0 to 2.

Solution. The function x^2 has an antiderivative $\frac{x^3}{3}$. Thus

$$\int_0^2 x^2 dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3} = 2.666\ldots$$

Keep in mind the difference between the definite and indefinite integral. If $f(x)$ is a continuous function with antiderivative $F(x)$, then

the indefinite integral $\int f(x)\,dx$	the definite integral $\int_a^b f(x)\,dx$
is the family of **functions** $F(x) + c$,	is the **number** $F(b) - F(a)$.

The fact that the Fundamental Theorem of Calculus enables you to compute the total change in antiderivative of $f(x)$ when x changes from a to b is referred also as the **Total Change Theorem**. Thus, the definite integral can be used to find the *total change in a quantity on an interval given its rate.*

Example 2. Suppose that the speed of an object is given by the function $v(t) = 0.3t$ where t is the time in seconds and v is the speed in feet per second. Determine the distance traveled between 10 and 20 seconds.

Solution. The distance traveled is the total change $s(20) - s(10)$ in position $s(t)$ from 10 to 20 seconds. This can be found as the definite integral from 10 to 20 of the rate $v(t) = s'(t)$. Thus,

$$s(20) - s(10) = \int_{10}^{20} v(t)\, dt = \int_{10}^{20} 0.3t\, dt = 0.3\, \frac{1}{2} t^2 \Big|_{10}^{20} = 0.15(20^2 - 10^2) = 0.15(300) = 45 \text{ feet.}$$

Units of the total change. If $[x]$ denotes the units of quantity x and $[f(x)]$ denotes the units of $f(x)$, then

the definite integral $\int_a^b f(x)dx$ is in units $[f(x)] \cdot [x]$.

This is because $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^n f(\overline{x_i})\Delta x$ for some partition of $[a, b]$ and points $\overline{x_i}$ from the subinterval $[x_{i-1}x_i]$ and the units of the sum on the right are the units $[f(x)]$ of $f(\overline{x_i})$ multiplied by the units $[x]$ of Δx.

For instance, in Example 2 above, the units of the definite integral are the units of speed multiplied by the units of time. Thus, the answer is in meters since the product $\frac{\text{meters}}{\text{second}} \cdot$ second produces meters.

In some cases you may be instructed to use the calculator if a function is such that finding an antiderivative is out of the scope of the Calculus 1 course.

Example 3. The size of a certain bacteria culture grows at a rate of $f(t) = te^{t/2}$ milligrams per hour. Use the Left-Right Sums calculator program to approximate the total change in the bacteria size during the first 3 hours to the first two nonzero digits.

Solutions. The change in the bacteria size during the first 3 hours can be found as $\int_0^3 te^{t/2}dt$. Enter the function in your calculator as $xe^{x/2}$, start the program and enter that $a = 0$, $b = 3$. With $n = 300$ for example, both the left and the right sums round to 13. Thus, the size increased by 13 mg during the first three hours.

Properties of the Definite Integral

Assume that $f(x)$ and $g(x)$ are continuous functions on the interval $[a, b]$ with antiderivatives $F(x)$ and $G(x)$ respectively. The Fundamental Theorem of Calculus can be used to show the following properties of definite integrals.

1. Since $\int_a^a f(x)\, dx = F(a) - F(a) = 0$, we have that

$$\int_a^a f(x)\, dx = 0.$$

2. $\int_a^b f(x)\, dx = F(b) - F(a) = -(F(a) - F(b)) = -\int_b^a f(x)\, dx$ thus we have that

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx.$$

3. If c is any number from $[a, b]$, then $\int_a^b f(x)\, dx = F(b) - F(a) = F(b) - F(c) + F(c) - F(a) = (F(c) - F(a)) + (F(b) - F(c)) = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$ so that

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

This property can be shown to hold for any number c, not necessarily between a and b.

4. Since $F(x)+G(x)$ is an antiderivative of $f(x)+g(x)$ we have that $\int_a^b (f(x)+g(x))\ dx = F(b)+G(b)-(F(a)+G(a)) = F(b)-F(a)+G(b)-G(a) = \int_a^b f(x)\ dx + \int_a^b g(x)dx$. Thus

$$\int_a^b (f(x)+g(x))\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)dx.$$

5. If c is any constant, $cF(x)$ is an antiderivative of $cf(x)$ so that $\int_a^b cf(x)\ dx = cF(b)-cF(a) = c(F(b)-F(a)) = c\int_a^b f(x)\ dx$. Thus

$$\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx.$$

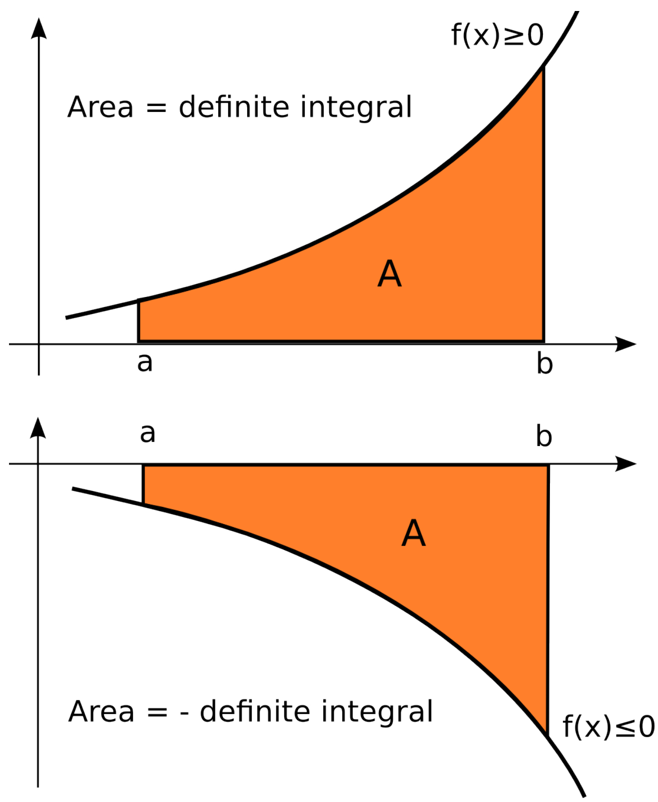
6. Since the definite integral of a continuous function is not a function but a constant, the variable name used in the definite integral does not matter. In other words,

$$\int_a^b f(x)\ dx = \int_a^b f(t)\ dt = \int_a^b f(u)\ du = \dots$$

Finding the area between $f(x)$ and x-axis on a given interval.

So far we related the definite integral and area under the graph of a continuous function just in the case the function is nonnegative. Recall that in this case the definite integral is exactly *equal* to the area.

Let us consider now the other computing the area if the function is not nonnegative on entire interval.



First, let us now consider the case when a given function is less or equal to zero on an interval $[a,b]$. If $f(x)\le 0$ then $|f(x)| = -f(x) \ge 0$ and the size of the area A between x-axis and $f(x)$ on $[a,b]$ is the same as the size of the area under the curve of $|f(x)| = -f(x)$ on $[a,b]$. Thus

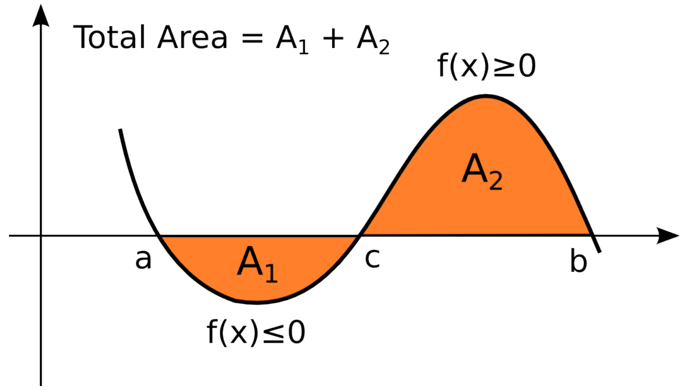
$$A = \int_a^b -f(x)dx = -\int_a^b f(x)dx$$

by property 5. above. We can unify the cases $f(x)\ge 0$ or $f(x)\le 0$ by writing that in either case $\int_a^b |f(x)|\ dx = \left|\int_a^b f(x)\ dx\right|$ but note that this <u>does not have to hold</u> if the function $f(x)$ changes sign on interval $[a,b]$.

Let us assume that $f(x)$ is changing the sign just once at c in $[a,b]$. Say that f is changing from negative to positive at c as in the figure below.

On interval $[a, c]$, $f(x) \leq 0$ so the area A_1 between $f(x)$ and x-axis can be found as $A_1 = -\int_a^c f(x)\,dx$. On interval $[c, b]$, $f(x) \geq 0$ so the area A_2 between $f(x)$ and x-axis can be found as $A_2 = \int_c^b f(x)\,dx$. The total area A can be obtained as the sum $A_1 + A_2$. Thus

$$A = A_1 + A_2 = \int_a^c -f(x)dx + \int_c^b f(x)dx.$$



Note that the total area cannot be evaluated using a single definite integral. The property 3. does not apply to the sums of two integrals in the last formula since the integrand in the first integral is $-f(x)$ and the integrand in the second integral is $f(x)$.

Similarly, if f is changing from positive to negative at c in $[a, b]$ we would have that $A = A_1 + A_2 = \int_a^c f(x)dx + \int_c^b -f(x)dx$.

If $f(x)$ has more than one x-intercept in $[a, b]$, one would need to divide the total area in more than two regions.

This brings us to the following procedure for finding the total area under the curve.

Finding Area. To find the total area A between the graph of a continuous function $f(x)$ and x-axis on $[a, b]$,

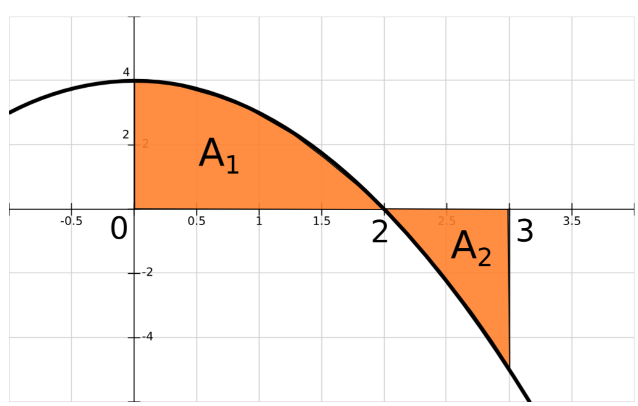
1. Graph the function on [a,b] and check if f is positive, negative or it changes the sign.

2. Consider the following cases.
   Case 1. If f is **positive** on $[a, b]$, then $A = \int_a^b f(x)\ dx$.
   Case 2. If f is **negative** on $[a, b]$, then $A = \int_a^b -f(x)\ dx$.
   Case 3. If f is **changing sign** on $[a, b]$ proceed with the following steps.

   3. Find all x-intercepts $c_1, c_2 \ldots c_k$ of $f(x)$ which are in $[a, b]$ and divide the interval into subintervals such that f does not change sign on each subinterval.
   4. Finding the area between f and x-axis on each subinterval falls under either case 1 or 2.
   5. The total area A is the sum of the areas on each subinterval.

Example 4. Find the area between the graph of $f(x) = 4 - x^2$ and x-axis on $[0, 3]$.

Solution. Consider the graph of the function on the interval [0,3]. Note that function changes the sign from positive to negative at a point between 0 and 3. Set the function to zero to find the x-intercepts. $4 - x^2 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$. Since -2 is not in [0,3], just 2 is relevant. Let A_1 denote the area on [0,2] and A_2 the area on [2,3].



The function is positive on [0,2) and negative on (2,3] so

$$A_1 = \int_0^2 (4-x^2)dx \quad \text{and} \quad A_2 = \int_2^3 -(4-x^2)dx = -\int_2^3 (4-x^2)dx.$$

Find the antiderivative $4x - \frac{1}{3}x^3$ of $4 - x^2$ and evaluate the two areas as follows.

$$A_1 = \int_0^2 (4-x^2)dx = (4x - \frac{1}{3}x^3)\Big|_0^2 = (4(2) - \frac{1}{3}2^3) - (4(0) - \frac{1}{3}0^3) = 8 - \frac{8}{3} = \frac{16}{3}.$$

$$A_2 = -\int_2^3 (4-x^2)dx = -(4x - \frac{1}{3}x^3)\Big|_2^3 = -\left((4(3) - \frac{1}{3}3^3) - (4(2) - \frac{1}{3}2^3)\right) = -(12-9-8+\frac{8}{3}) = \frac{7}{3}.$$

Thus the total area A is $A = A_1 + A_2 = \frac{16}{3} + \frac{7}{3} = \frac{23}{3} \approx 7.67$.

Practice Problems.

1. Evaluate the following. (a) $\int_0^2 2x \ dx$ (b) $\int_0^1 x^2 + 2 \ dx$ (c) $\int_1^4 x^{-2} \ dx$
2. Find the area between the graph of $f(x)$ and x-axis on $[a, b]$.
   (a) $f(x) = x^2; \quad a = 0, \quad b = 2.$
   (b) $f(x) = x - 3; \quad a = 1, \quad b = 2$
   (c) $f(x) = x^2 - 9; \quad a = -1, \quad b = 4$
   (d) $f(x) = x^2 - 2x \quad a = 1, \quad b = 3$
   (e) $f(x) = 2\sqrt{x} - 4; \quad a = 0, \quad b = 9$
3. From past records, a botanist knows that a certain species of tree has a rate of growth that can be modeled by $f(t) = \frac{2}{\sqrt{t}}$, $1 \leq t \leq 4$, where t is the age of the tree in years and $f(t)$ is the growth rate in feet per year. Determine how much did the tree grow from the time when it was a year old to the time it was four years old.
4. Suppose that the velocity of an object is given by the function $v(t) = \frac{t}{\sqrt{t^2+9}}$ where t is the time in seconds and v is the velocity in feet per second. Determine the distance traveled between 3 and 5 seconds.
5. The rate of change in the U.S. population can be modeled by $g(x) = 1.03e^{0.013t}$, $0 \leq t \leq 100$ where t represents the number of years since 1900 and g represents the rate of change in population measured in millions per year. Determine the total increase in the U.S. population from 1900 to 1950.
6. Geologists estimate that an oil field will produce oil at a rate given by $f(t) = 600e^{-0.1t}$ thousand barrels per month, t months into production. Estimate the total production during the first year of operation. Round to the nearest whole number.
7. Breathing is cyclic and a full respiratory cycle takes about 5 seconds. The function $f(t) = \frac{1}{2}\sin\frac{2\pi t}{5}$ in liters per second has often been used to model the rate of air flow into the lungs at time t. Find the volume of inhaled air in the lungs in one respiratory cycle.

Solutions.

1. (a) $\int_0^2 2x\,dx = 2\frac{x^2}{2}\Big|_0^2 = x^2|_0^2 = 4 - 0 = 4.$

   (b) $\int_0^1 x^2 + 2\,dx = (\frac{x^3}{3} + 2x)\Big|_0^1 = \frac{1^3}{3} + 2(1) - 0 = \frac{7}{3}.$

   (c) $\int_1^4 x^{-2}\,dx = \frac{x^{-1}}{-1}\Big|_1^4 = \frac{-1}{x}\Big|_1^4 = \frac{-1}{4} - \frac{-1}{1} = \frac{3}{4}.$

2. (a) Since $f(x)$ is positive on the given interval, the area is $A = \int_0^2 x^2 dx = \frac{x^3}{3}\Big|_0^2 = \frac{8}{3}.$

   (b) Graph the function $f(x) = x - 3$ and note that it is negative on [1,2]. Thus, the area is $A = -\int_1^2 (x-3)dx = -(\frac{x^2}{2} - 3x)\Big|_1^2 = -2 + 6 + \frac{1}{2} - 3 = 1 + \frac{1}{2} = \frac{3}{2}.$

   (c) Graph the function and check the sign of $f(x)$ on [-1,4]. Notice that $f(x)$ changes the sign. Find x-intercepts: $f(x) = x^2 - 9 = 0$ when $x^2 = 9 \Rightarrow x = \pm 3$. The relevant zero is 3 since -3 is not in the interval [-1,4]. From the graph, you can see that $f(x)$ is negative on [-1, 3) and positive on (3, 4]. Thus, the total area can be obtained as the sum of $A_1 = -\int_{-1}^3 f(x)dx$ and $A_2 = \int_3^4 f(x)dx.$

   $A_1 = -\int_{-1}^3 (x^2 - 9)dx = (-\frac{x^3}{3} + 9x)\Big|_{-1}^3 = -9 + 27 - \frac{1}{3} + 9 = \frac{80}{3} = 26.67.$ $A_2 = \int_3^4 (x^2 - 9)dx = (\frac{x^3}{3} - 9x)\Big|_3^4 = \frac{64}{3} - 36 - 9 + 27 = \frac{10}{3} = 3.33$ The total area is $A = A_1 + A_2 = \frac{80}{3} + \frac{10}{3} = 30.$

   Careful: the total area is *not* $\int_{-1}^4 f(x)dx = -23.33.$

   (d) Check the sign of $f(x)$ on [1,3] and note that $f(x)$ changes the sign. Find x-intercepts: $f(x) = x^2 - 2x = 0 \Rightarrow x(x-2) = 0 \Rightarrow x = 0, x = 2$. The relevant zero is 2 since 0 is not in the interval [1,3]. From the graph, you can see that $f(x)$ is negative on [1, 2) and positive on (2, 3]. Thus, the total area can be obtained as the sum of $A_1 = -\int_1^2 f(x)dx$ and $A_2 = \int_2^3 f(x)dx.$

   $A_1 = -\int_1^2 (x^2 - 2x)dx = (-\frac{x^3}{3} + x^2)\Big|_1^2 = -\frac{8}{3} + 4 + \frac{1}{3} - 1 = \frac{2}{3} = 0.67.$ $A_2 = \int_2^3 (x^2 - 2x)dx = (\frac{x^3}{3} - x^2)\Big|_2^3 = 9 - 9 - \frac{8}{3} + 4 = \frac{4}{3} = 1.33$ The total area is $A = A_1 + A_2 = \frac{2}{3} + \frac{4}{3} = 2.$

   Careful: the total area is *not* $\int_1^3 f(x)dx = 0.67.$

   (e) Check the sign of $f(x)$ on [0,9] and note that $f(x)$ changes the sign. Find x-intercepts $f(x) = 2\sqrt{x} - 4 = 0$ when $2\sqrt{x} = 4 \Rightarrow \sqrt{x} = 2 \Rightarrow x = 4$. From the graph, you can see that $f(x)$ is negative on [0, 4) and positive on (4, 9]. Thus, the total area can be obtained as the sum of the area $A_1 = -\int_0^4 f(x)dx$ and $A_2 = \int_4^9 f(x)dx$. Similarly as in previous problems, you can calculate that $A_1 = \frac{16}{3} = 5.33$ and $A_2 = \frac{16}{3} = 5.33$. The total area is $A = A_1 + A_2 = \frac{32}{3} = 10.67.$

   Careful: the total area is *not* $\int_0^9 f(x)dx = 0.$

3. The total growth can be found by integrating the growth rate from 1 to 4. $\int_1^4 \frac{2}{\sqrt{t}}dt = 2\frac{t^{1/2}}{1/2}\Big|_1^4 = 4\sqrt{t}\Big|_1^4 = 8 - 4 = 4$ feet.

4. If $s(t)$ denotes the position, the problem is asking for $s(5) - s(3) = \int_3^5 v(t)dt$. Using the substitution with $u = t^2 + 9$ find the antiderivative of the velocity function to be $\sqrt{t^2 + 9}$. Thus $s(5) - s(3) = \sqrt{t^2 + 9}\Big|_3^5 = \sqrt{34} - \sqrt{18} \approx 1.588$ ft.

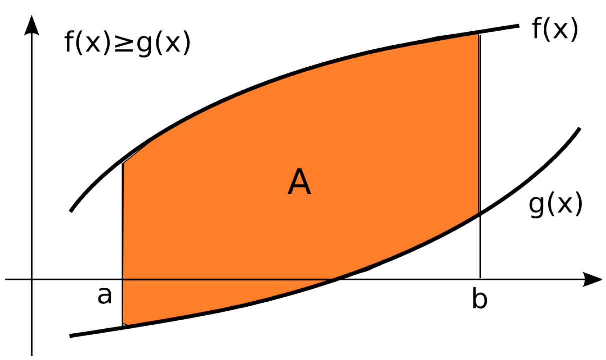
5. The total increase in the number of people from $t = 0$ to $t = 5-$ can be found as $\int_0^{50} 1.03e^{0.013t}dt$. Use the substitution $u = 0.013t$ to find the antiderivative and get $1.03\frac{1}{0.013}e^{0.013t}\big|_0^{50} = \frac{1.03}{0.013}(e^{0.013(50)} - 1) \approx 79.23(0.916) = 72.54$ millions.

6. Work out the details similarly to the previous problem. The total production for the first year of operation can be estimated to be $4192.834 \approx 4193$ thousands of barrels.

7. Graph the function to note that the inhaling occurs in the first 2.5 seconds of the cycle and exhaling in the second 2.5 seconds. Thus, the volume of inhaled air can be found as $\int_0^{5/2} \frac{1}{2}\sin\frac{2\pi t}{5}dt$. Using the substitution $u = \frac{2\pi t}{5}$, you obtain $\frac{1}{2}\,\frac{-5}{2\pi}\,\cos\frac{2\pi t}{5}\big|_0^{5/2} = \frac{-5}{4\pi}(-1-1) = \frac{5}{2\pi} \approx 0.796$ liters.

Areas between Curves

In this section we consider the area between two curves. Let $f(x)$ and $g(x)$ are two continuous functions defined on the interval $[a, b]$ such that $f(x) \geq g(x)$ for all x in $[a, b]$. If we consider a partition of $[a, b]$ with length of each subinterval $\Delta x = \frac{b-a}{n}$, choose the points $\overline{x_i}$ from each subinterval $[x_{i-1}, x_i]$, for $i = 1, \ldots, n$, then the sum of the areas of rectangles with height $f(\overline{x_i}) - g(\overline{x_i})$ and width Δx approaches the area between the curves when $n \to \infty$. Since this limit represents the definite integral



$$\int_a^b (f(x) - g(x))\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}(f(\overline{x_i}) - g(\overline{x_i})\,\Delta x,$$

this integral computes the area between the curves. Thus the area between the curves f and g on $[a, b]$ is

$$A = \int_a^b (f(x) - g(x))\;\; dx.$$

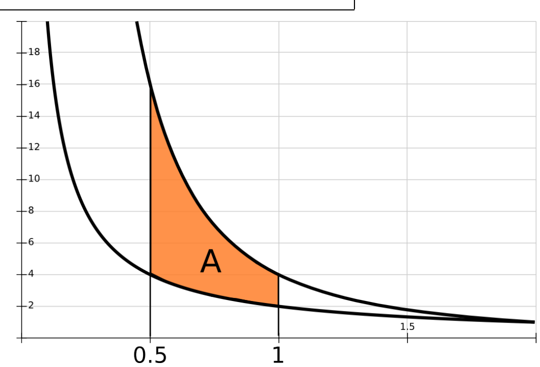
Note that in this consideration the position of f and g with respect to x-axis is not relevant. The only relevant factor is the position of f and g with respect to each other.

Analogously to the above consideration, if $g(x) \geq f(x)$ on $[a, b]$ the area can be computed as $A = \int_a^b (g(x) - f(x))\;\; dx$. So, you may remember the formula computing the area between the two curves which **do not intersect on interval** $[a, b]$ as

Area between two curves $= \int_a^b$ (upper curve - lower curve) dx

Example 1. Find the area between $f(x) = \frac{2}{x}$ and $g(x) = \frac{4}{x^2}$ on interval $[\frac{1}{2}, 1]$.

Solution. Graph the functions first. Zoom to see the relevant region. Note that the functions do not intersect on $[\frac{1}{2}, 1]$ and that $g(x) \geq f(x)$ on $[\frac{1}{2}, 1]$.



So, the area can be calculated as

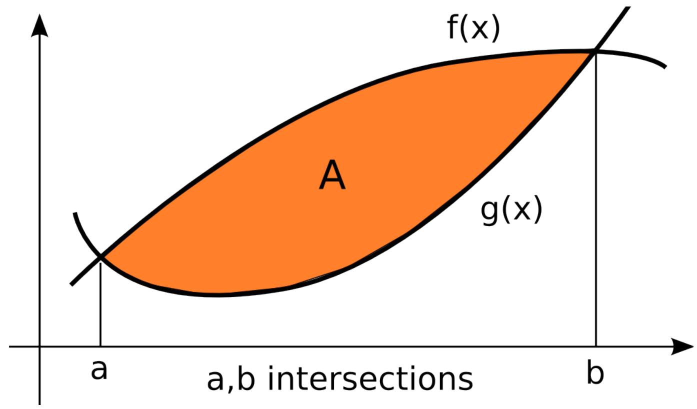
$$A = \int_{1/2}^{1} \left(\frac{4}{x^2} - \frac{2}{x} \right) \, dx$$

Find antiderivatives $2 \ln x$ of $f(x)$ and $4\frac{1}{-1}x^{-1} = \frac{-4}{x}$ of $g(x)$ and evaluate the integral.

$$A = \left(\frac{-4}{x} - 2 \ln x \right)\Big|_{1/2}^{1} = \left(\frac{-4}{1} - 2 \ln 1 \right) - \left(\frac{-4}{\frac{1}{2}} - 2 \ln \frac{1}{2} \right) = -4 + 8 + 2 \ln \frac{1}{2} \approx 2.61$$

Finding the area enclosed by two curves *without a specific interval given.*

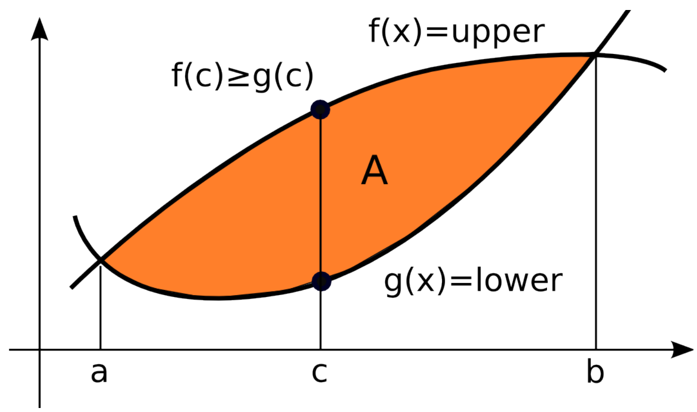
When finding the total area enclosed by two curves, the bounds of the integration are the intersections of the curves. For the time being, let us consider the case when the functions intersect just twice.



1. The bounds of integration are the intersections of the two curves and can be obtained by solving $f(x) = g(x)$ for x. The intersections $x = a$ and $x = b$ are the **bounds of the integration**.

2. Determine which function is larger on (a, b).

   When you graph two given curves and are still unsure of which curve is upper and which lower, you can take any point $x = c$ between a and b and plug it in both functions. Comparing $f(c)$ and $g(c)$ determines which function is greater on (a, b). Be careful to pick a point within (a, b) i.e. *between* a and b and remember that this method works just if f and g *do not intersect* on (a, b).



3. If $f(x) \geq g(x)$ (as on the figure above), then the area is $A = \int_a^b (f(x) - g(x)) \, dx$.
   If $f(x) \leq g(x)$, then the area is $A = \int_a^b (g(x) - f(x)) \, dx$. Both cases again follow the pattern:

Area between two curves $= \int_a^b$ (upper curve - lower curve) dx

Example 2. Find the area enclosed by the curves $f(x) = 4 - x^2$ and $g(x) = 2 - x$.

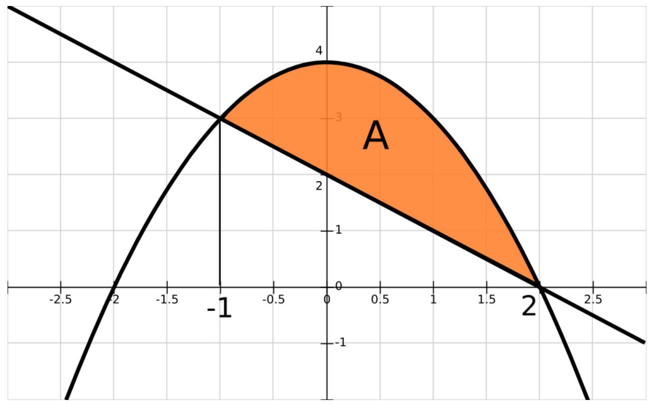
Solution. Graph both curves first and note that they intersect two times. These intersections are the bounds of the integration.

Find the intersections by solving

$$4 - x^2 = 2 - x \Rightarrow x^2 - x - 2 = 0 \Rightarrow$$

$$(x-2)(x+1) = 0 \Rightarrow x = 2 \text{ and } x = 1.$$

The graph indicates that the curve $4-x^2$ is upper and $2-x$ is lower. You can double check that by plugging a number between -1 and 2 into both. For example, for $x = 0$, $f(0) = 4 > g(0) = 2$.



Having established that $f(x)$ is upper and $g(x)$ lower, you can set up the integral computing the area and evaluate it using the Fundamental Theorem of Calculus. Simplify the integrand before integrating – add the similar terms to reduce the number of terms to integrate.

$$A = \int_{-1}^{2} \left(4 - x^2 - (2-x)\right)\, dx = \int_{-1}^{2} \left(2 - x^2 + x\right)\, dx = \left(2x - \frac{x^3}{3} + \frac{x^2}{2}\right)\Big|_{-1}^{2} =$$

$$\left(2(2) - \frac{2^3}{3} + \frac{2^2}{2}\right) - \left(2(-1) - \frac{(-1)^3}{3} + \frac{(-1)^2}{2}\right) = 5 - \frac{1}{2} = \frac{9}{2} = 4.5.$$

Practice Problems.

1. Find the area of the region between the given curves.

   (a) $y = x^2 + 3$, $y = x$, for x in $[-1, 1]$ (b) $y = 4x^2$, $y = x^2 + 3$

   (c) $y = x^2$, $y = x$ (d) $y = \sqrt{x+3}$, $y = \frac{x+3}{2}$

2. Pollution enters a lake at the rate $f(t) = 150 - 0.2e^{t/2}$ g/hour. Meanwhile, the pollution filter removes the pollution at the rate of $g(t) = 0.3e^{t/2}$ g/hour.

   (a) Find the time when the rate of pollution entering is the same as the rate pollution leaving the lake and the amount of pollution at that time.

   (b) If the initial amount of pollution is 500 g, determine the function computing the total amount of pollution at time t. Then find the time when the pollution is completely removed from the lake using your calculator.

3. A botanist knows that a certain species of oak tree grows at a rate of $\frac{4x^2+16x+9}{2x+4}$ feet per year, where x is the age of the tree in years. When restricting the light, the oak tree grows at a rate $\frac{2x^2+12x+9}{2x+4}$ feet per year in x years. Determine the difference in growth which results from restricting the amount of light that tree receives when the tree is between 3 and 8 years old. (Hint: simplify the difference of functions before integrating).

Solutions.

1. (a) The bounds of integration are given to be -1 and 1. Using either the graph or plugging a point from $(-1,1)$ into both curves (for example 0), you can see that $y = x^2 + 3$ is greater than $y = x$ on $(-1,1)$ $(3 = 0^2 + 3 > 0)$. Thus, $A = \int_{-1}^{1}(x^2 + 3 - x)dx = \frac{x^3}{3} - 3x - \frac{x^2}{2} = \frac{20}{3}$.

   (b) Find the intersections first. $4x^2 = x^2 + 3 \Rightarrow 3x^2 = 3 \Rightarrow x^2 = 1 \Rightarrow x = 1$ and $x = -1$. On interval $(-1,1)$, $y = x^2 + 3$ is greater than $y = 4x^2$ (for example, using $x = 0$, $3 = 0^2 + 3 > 4(0)^2 = 0$). So, $A = \int_{-1}^{1}(x^2 + 3 - 4x^2)dx = \int_{-1}^{1}(3 - 3x^2)dx = 3x - \frac{3x^3}{3}|_{-1}^{1} = 3 - 1 + 3 - 1 = 4$.

   (c) Intersections: $x^2 = x \Rightarrow x^2 - x = x(x-1) = 0 \Rightarrow x = 0$ and $x = 1$. On interval $(0,1)$, the curve $y = x$ is greater than $y = x^2$. The area is $A = \int_0^1(x - x^2)dx = \frac{x^2}{2} - \frac{x^3}{3}|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.

   (d) Intersections: $\sqrt{x+3} = \frac{x+3}{2} \Rightarrow x + 3 = \frac{(x+3)^2}{4} \Rightarrow 4(x+3) = (x+3)^2 \Rightarrow 0 = (x+3)^2 - 4(x+3)0 = (x+3)[x+3-4] \Rightarrow 0 = (x+3)(x-1) \Rightarrow x = -3$ and $x = 1$. On interval $(-3,1)$, the curve $y = \sqrt{x+3}$ is greater than $y = \frac{x+3}{2}$. The area is $A = \int_{-3}^{1}(\sqrt{x+3} - \frac{x+3}{2})dx$. Use substitution $u = x+3$ to obtain that this integral is $= \frac{2(x+3)^{3/2}}{3} - \frac{(x+3)^2}{4}|_{-3}^{1} = \frac{16}{3} - 4 = \frac{4}{3} = 1.33$.

2. (a) The rates are the same when $150 - 0.2e^{t/2} = 0.3e^{t/2} \Rightarrow 150 = 0.5e^{t/2} \Rightarrow 300 = e^{t/2} \Rightarrow \frac{t}{2} = \ln 300 \Rightarrow t = 2\ln 300 \approx 11.41$ hours. The amount of pollution at a time t_0 can be found as the integral from 0 to t_0 from the difference of rate in and rate out. Thus the amount of pollution at $t \approx 11.41$ is $\int_0^{11.41}\left(150 - 0.2e^{t/2} - 0.3e^{t/2}\right)dt = \int_0^{11.41}\left(150 - 0.5e^{t/2}\right)dt =$
   $\left(150t - 0.5(2)e^{t/2}\right)\Big|_0^{11.41} = \left(150t - e^{t/2}\right)\Big|_0^{11.41} = 150(11.41) - e^{11.41/2} + 1 = 1412.13$ grams.

   (b) The amount of pollution $A(t)$ at time t is the antiderivative $150t - e^{t/2} + c$. Since $A(0) = 500$, $500 = 150(0) - e^{0/2} + c \Rightarrow 500 = -1 + c \Rightarrow c = 501$. So, $A(t) = 150t - e^{t/2} + 501$. Set to zero and solve using your calculator. The lake becomes pollutant free in $t \approx 15.94$ hours.

3. The difference in the growth of the oak thee in the two different set up from 3 to 8 years can be found as the difference of the two definite integrals computing the total growth in two cases: $\int_3^8 \frac{4x^2+16x+9}{2x+4}\,dx - \int_3^8 \frac{2x^2+12x+9}{2x+4}\,dx$. Following the hint, combine the two integrals and then evaluate the resulting integral $\int_3^8\left(\frac{4x^2+16x+9}{2x+4} - \frac{2x^2+12x+9}{2x+4}\right)dx = \int_3^8 \frac{4x^2+16x+9-2x^2-12x-9}{2x+4}\,dx = \int_3^8 \frac{2x^2+4x}{2x+4}\,dx = \int_3^8 \frac{x(2x+4)}{2x+4}\,dx = \int_3^8 x\,dx = \frac{x^2}{2}|_3^8 = \frac{55}{2} = 27.5$ feet.