Genral Calculus
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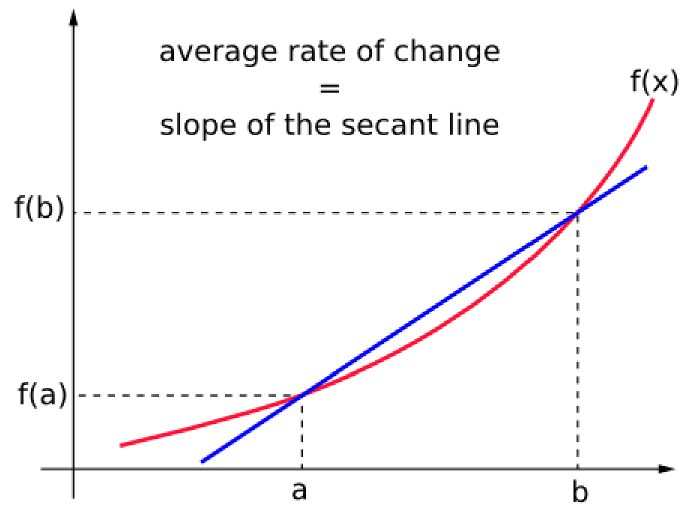
The Derivative

The rate of change

Knowing and understanding the concept of derivative will enable you to answer the following questions. Let us consider a quantity whose size is described by a function.

1. How fast is the quantity changing at a given moment?
2. Is the quantity increasing or decreasing in size?
3. When will the size be maximal and when will the size be minimal?
4. If the quantity is increasing, is the rate of the increase increasing or decreasing itself?
5. If another factor impacts the size of the quantity, how does it rate of change impacts the speed of the change of the initial quantity?

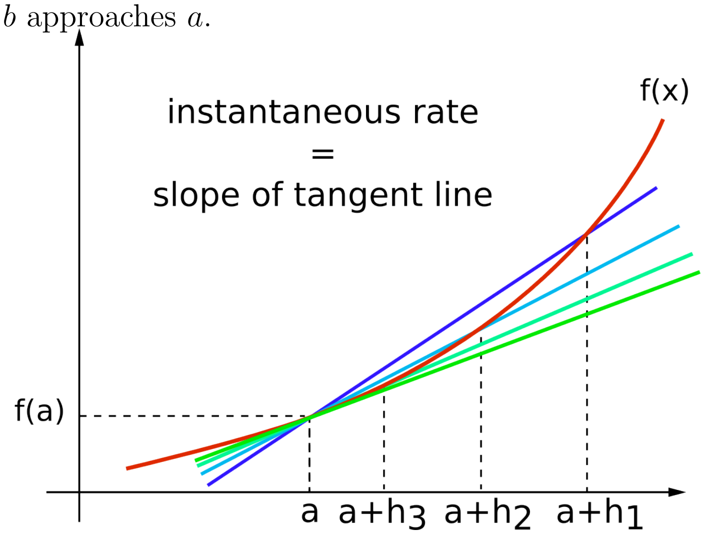
Rates of Change. To introduce the concept of derivative, let us recall the definition of the **average rate of change** of a function on an interval.



> The average rate of change of $f(x)$ over the interval $a \leq x \leq b$ is
>
> $$\frac{\text{rise}}{\text{run}} = \frac{f(b)-f(a)}{b-a}$$

Besides finding the rate of change over an interval, it may be relevant to find the rate of change at a specific point. This rate, called the **instantaneous rate of change or derivative of f at a** can be computed from the above formula $\frac{f(b)-f(a)}{b-a}$ when b approaches a.

If we denote the difference $b-a$ by h then $b = a+h$ and the condition $b \to a$ is equivalent to $h \to 0$, the formula $\frac{f(b)-f(a)}{b-a}$ becomes $\frac{f(a+h)-f(a)}{h}$. Hence, the can be computed as



> The instantaneous rate of change of $f(x)$ at $x = a$ is
>
> $$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Geometric Interpretation. We have seen that the average rate of change of $f(x)$ from a to b represents the slope of the secant line. In the limiting case when $b \to a$, the secant line becomes the tangent line as the previous figure illustrates. Thus,

The instantaneous rate of change of $f(x)$ at $x = a$ is the slope of the line **tangent** to the graph of $f(x)$ at $x = a$.

Thus, the formula $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ computes the slope m of the tangent line. Recall the point-slope equation of a line with slope m passing point (x_0, y_0).

$$\boxed{y - y_0 = m(x - x_0)}$$

This formula computes the equation of the tangent line to $f(x)$ at $x = a$ for $x_0 = a$, $y_0 = f(a)$ and $m = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$.

Example 1. Let $f(x) = x^2 + 4$.

(a) Find the average rate of change of $f(x)$ for $1 \le x \le 2$. Then find the equation of the secant line of $f(x)$ passing the graph of $f(x)$ at $x = 1$ and $x = 2$.

(b) Find the instantaneous rate of change of $f(x)$ for $x = 1$. Then find the equation of the tangent line to $f(x)$ at $x = 1$.

Solution. (a) The average rate of change of $f(x)$ for $1 \le x \le 2$ can be computed as

$$\frac{f(2) - f(1)}{2 - 1} = \frac{2^2 + 4 - (1^2 + 4)}{1} = 4 + 4 - 1 - 4 = 3.$$

This also computes the slope of the secant line. The equation of the secant line can be obtained using the point slope equation with $m = 3$ and using either $(1, f(1))$ or $(2, f(2))$ for point (x_0, y_0). For example, with $(1, f(1)) = (1, 5)$ one gets the equation

$$y - 5 = 3(x - 1) \quad \Rightarrow \quad y = 3x + 2.$$

(b) The instantaneous rate of change of $f(x)$ at $x = 1$ can be computed as $\lim_{h\to 0} \frac{f(1+h)-f(1)}{h} =$

$$\lim_{h\to 0} \frac{(1+h)^2 + 4 - (1^2 + 4)}{h} = \lim_{h\to 0} \frac{1 + 2h + h^2 + 4 - 1 - 4}{h} = \lim_{h\to 0} \frac{2h + h^2}{h} = \lim_{h\to 0} 2 + h = 2.$$

This also computes the slope of the tangent line. The equation of the tangent line can be obtained using the point-slope equation with $m = 2$ and using $(1, f(1)) = (1, 5)$ for point (x_0, y_0). So, the tangent line is

$$y - 5 = 2(x - 1) \quad \Rightarrow \quad y = 2x + 3.$$

An application. If $f(x)$ computes the position (in units of length) of an object at time x (in time units) after the object started moving, then the average rate of change from $x = a$ to $x = b$ computes the average velocity between times a and b. The instantaneous rate of change at a computes the **instantaneous velocity** at time $x = a$.

average velocity	from $x = a$ to $x = b$ is	$\frac{f(b)-f(a)}{b-a}$
velocity	at $x = a$ is	$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

If $[x]$ denotes the units of quantity x and $[y]$ denotes the units of $f(x)$, the units of both the average and the instantaneous rates of change are $\frac{[y]}{[x]}$ since

$$\frac{[f(b)-f(a)]}{[b-a]} = \frac{\text{units of } y}{\text{units of } x} \quad \text{and} \quad \lim_{h\to 0} \frac{[f(a+h)-f(a)]}{[h]} = \frac{\text{units of } y}{\text{units of } x}.$$

Example 2. Assume that the position of a moving object x second after the object started moving can be computed by $f(x) = x^2 + 4$ feet.

(a) Determine the average velocity at which the object was moving between the first and the second second.

(b) Determine the velocity of the object one second after it started moving.

Solution. Note that the function describing the position is the same function as in Example 1. Part (a) is asking for the average rate of change of $f(x)$ for $1 \leq x \leq 2$ which we computed to be 3. Thus, the average velocity is 3 feet per second.

Part (b) us asking for the instantaneous rate of change of $f(x)$ at $x = 1$ which we computed to be 2 in part (b) of Example 1. Thus, the object has (instantaneous) velocity of 2 feet per second 1 second after it started moving.

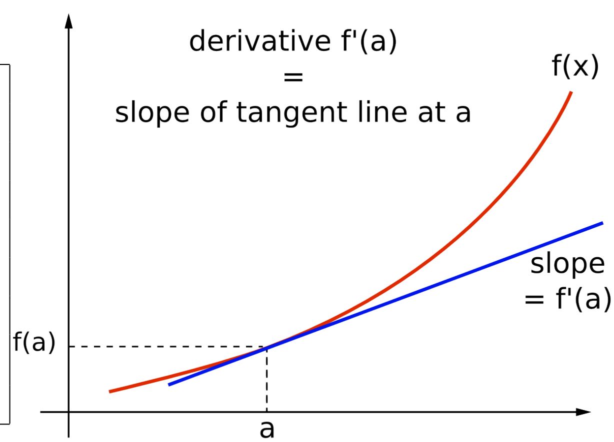
Derivative. The instantaneous rate of change of $f(x)$ at $x = a$ is the derivative of $f(x)$ at $x = a$. The notation $f'(a)$ is used to denote the derivative of $f(x)$ at $x = a$. Thus, the following concepts are all equivalent.

(1) The **derivative** $f'(a)$ of $f(x)$ at $x = a$;

(2) The **instantaneous rate of change** of $f(x)$ at $x = a$;

(3) The **slope of the tangent line** to $f(x)$ at $x = a$;

(4) $$\mathbf{f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}}$$



Alternative formula. If we denote the changing quantity $a + h$ by x so that $x - a = h$, the quotient $\frac{f(a+h)-f(a)}{h}$ can be written as $\frac{f(x)-f(a)}{x-a}$. When $h \to 0$, $x \to a$ so the derivative $f'(a)$ can also be found as follows.

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

Example 3. Find the derivative of $f(x) = x^2 + 4$ at $x = 1$.

Solution. Recall that we have found the instantaneous rate of change of this function at $x = 1$ to be 2 in Example 1. Thus $f'(1) = 2$.

Derivative as a function. The previous example illustrates that derivative at $x = a$ can be considered as a function of a. By using more familiar x instead of a to denote the independent variable, we obtain that the derivative $f'(x)$ can be considered to be a function of x since at every value of x it computes the slope of the tangent at the point $(x, f(x))$.

$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$$

Example 4. Find derivative of the line $f(x) = mx + b$.

Solution. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{m(x+h)+b-(mx+b)}{h} = \lim_{h\to 0} \frac{mx+mh+b-mx-b}{h} = \lim_{h\to 0} \frac{mh}{h} = m$. Thus, the derivative of $mx + b$ is m.

Alternatively, you can simply argue that, since the tangent line to a line is the same line, the slope of the tangent line is m at every point x.

Example 5. Find derivative of $f(x) = x^2$.

Solution. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0} 2x + h = 2x$. Thus, the derivative of x^2 is $2x$.

Notation for derivative. If the function $f(x)$ is denoted by y, sometimes y' is used to denote $f'(x)$. There are other notations for derivative besides $f'(x)$ and y'. Note that the quotient $\frac{f(x+h)-f(x)}{h}$ measures the quotient of the change in y over change of x. These two changes are denoted by Δy and Δx and the limit when $h = \Delta x \to 0$ is denoted by $\frac{dy}{dx}$. This notation is known as the **Leibniz notation**. In this notation, the formula computing the derivative can be written as follows.

$$\frac{dy}{dx} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}$$

In some cases, the notations $\frac{d}{dx}f(x)$ or $\frac{df}{dx}$ are also used. In this case, the term $\frac{d}{dx}$ is considered as a function that maps $y = f(x)$ to its derivative $y' = f'(x)$. In this case, the term $\frac{d}{dx}$ is refer to as **differentiation operator**. Keep in mind that the notations $\frac{d}{dx}f(x)$ or $\frac{df}{dx}$ are equivalent to $f'(x)$, y' and $\frac{dy}{dx}$. To summarize, the following denote the derivative of $y = f(x)$.

$$\boxed{f'(x), \quad y', \quad \frac{dy}{dx}, \quad \frac{df}{dx}, \quad \frac{d}{dx}f(x)}$$

If the derivative y' of a function $y = f(x)$ is evaluated at $x = a$, the following notation is also used besides $f'(a)$

$$\left.\frac{dy}{dx}\right|_{x=a}$$

Units of the derivative. If $[x]$ denotes the units of quantity x and $[f(x)]$ denotes the units of $f(x)$, the units of derivative are determined as follows.

$$\text{Units of derivative } f'(x) = [f'(x)] = \lim_{h\to 0} \frac{[f(x+h)-f(x)]}{[h]} = \frac{\text{units of } f(x)}{\text{units of } x} = \frac{[f(x)]}{[x]}.$$

In Example 2 we have seen that the the velocity is in feet per second if the position function is in feet and the time is in seconds.

Practice problems.

1. Find the average rate of change of the following functions over the given interval.

   (a) $y = \frac{1}{x+2}$, $[0, 2]$
   (b) $y = \sqrt{x+3}$, $[1, 5]$

2. Consider the function $f(x) = x^2 - 3x$.

   (a) Find the instantaneous rate of change of $f(x)$ at $x = 2$.
   (b) Find the derivative of $f(x)$ using the definition of derivative.

3. If we approximate the gravitational acceleration g by 9.8 meters per seconds squared, the displacement from the initial height of an object dropped from it to the ground can be described as $s(t) = \frac{g}{2}t^2 \approx 4.9t^2$.

   (a) Find the average velocity of the object in the first three seconds.
   (b) Find the velocity of the object three seconds into the fall.
   (c) Find the formula computing the velocity at any point t.

Solutions.

1. (a) $f(0) = \frac{1}{2}$ and $f(2) = \frac{1}{4}$ so the average rate is $\frac{f(2)-f(0)}{2-0} = \frac{\frac{1}{4}-\frac{1}{2}}{2} = \frac{-1}{8}$.

   (b) $f(1) = \sqrt{4} = 2$ and $f(5) = \sqrt{8}$ so the average rate is $\frac{f(5)-f(1)}{5-1} = \frac{\sqrt{8}-2}{4} \approx 0.207$.

2. (a) $f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{(2+h)^2-3(2+h)-(2^2-3(2))}{h} = \lim_{h\to 0} \frac{4+4h+h^2-6-3h-4+6}{h} = \lim_{h\to 0} \frac{h+h^2}{h} = \lim_{h\to 0} 1+h = 1$.

   (b) $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2-3(x+h)-(x^2-3x)}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2-3x-3h-x^2+3x}{h} = \lim_{h\to 0} \frac{2xh-3h+h^2}{h} = \lim_{h\to 0} 2x-3+h = 2x-3$.

3. (a) The average velocity in the first three seconds is $\frac{s(3)-s(0)}{3} = \frac{4.9(9)}{3} = 14.7$ meters per second.

   (b) $v(3) = s'(3) = \lim_{h\to 0} \frac{4.9(3+h)^2-4.9(3)^2}{h} = \lim_{h\to 0} \frac{4.9[9+6h+h^2-9]}{h} = \lim_{h\to 0} \frac{4.9[6h+h^2]}{h} = \lim_{h\to 0} 4.9(6+h) = 4.9(6) = 29.4$ meters per second.

   (c) The velocity at time t can be found as $v(t) = s'(t) = \lim_{h\to 0} \frac{4.9(t+h)^2-4.9t^2}{h} = \lim_{h\to 0} \frac{4.9[t^2+2th+h^2-t^2]}{h} = \lim_{h\to 0} \frac{4.9[2th+h^2]}{h} = \lim_{h\to 0} 4.9(2t+h) = 9.8t$ meters per second.

Finding and Using Derivative – the shortcuts

We have seen that the formula $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ is manageable for relatively simple functions like a linear or quadratic. For more complex functions, finding the derivative using this definition is not very effective. Because of this, many shortcuts to finding derivative have been introduced.

Derivative of a constant function. If c is a constant and $f(x) = c$ for every value of c, then $\frac{f(x+h)-f(x)}{h} = \frac{c-c}{h} = 0$. Thus, the derivative is zero. Alternatively, the same conclusion could be reached by noting that a horizontal line has the slope zero. Thus,

$$f(x) = c \Rightarrow f'(x) = 0$$

Derivative of the power function. In Examples 6 of previous section, we have seen that the derivative of the line $mx + b$ is m. Thus the derivative of x is 1. In Example 7 we have seen that the derivative of x^2 is $2x$. Let us demonstrate a more general formula which will compute the derivative of x^n for any positive integer n.

Let $f(x) = x^n$ and let us find the formula for $f'(a)$ using the formula $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ for the derivative at a. For this function the quotient $\frac{f(x)-f(a)}{x-a}$ becomes $\frac{x^n-a^n}{x-a}$. To determine the limit of this when $x \to a$, we want to factor the numerator. Note that $x^n - a^n$ factors as [1]

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \ldots + xa^{n-2} + a^{n-1})$$

Thus,

$$f'(a) = \lim_{x\to a} \frac{x^n - a^n}{x-a} = \lim_{x\to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + \ldots + xa^{n-2} + a^{n-1})}{x-a} =$$

$$\lim_{x\to a} x^{n-1} + x^{n-2}a + \ldots + xa^{n-2} + a^{n-1} = a^{n-1} + a^{n-1} + \ldots + a^{n-1} + a^{n-1} = na^{n-1}$$

Since $f'(a) = na^{n-1}$ we have

The Power Rule. $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$

Note that this formula confirms our calculation of derivative of x and x^2. Indeed when $n = 1$, it produces the derivative of $x = x^1$ as $1x^{1-1} = 1x^0 = 1$ and when $n = 2$, the derivative of x^2 as $2x^{2-1} = 2x^1 = 2x$.

Together with the following three rules, we shall be able to find derivative of any polynomial function without using the definition of derivative.

The Sum Rule.	$y = f(x) + g(x) \Rightarrow$	$y' = f'(x) + g'(x)$
The Difference Rule.	$y = f(x) - g(x) \Rightarrow$	$y' = f'(x) - g'(x)$
The Constant Multiple Rule.	$y = cf(x) \Rightarrow$	$y' = cf'(x)$

[1]To convince yourself of this formula, foil the right hand side and obtain $x^n + x^{n-1}a + \ldots + x^2a^{n-2} + xa^{n-1} - ax^{n-1} - x^{n-2}a^2 - \ldots - xa^{n-1} - a^n$ and note that all the terms cancel except the first x^n and the last one $-a^n$. Thus you have $x^n - a^n$.

Thus,

1. The derivative of the sum is the sum of the derivatives.
2. The derivative of the difference is the difference of the derivatives.
3. To find the derivative of a constant multiple of the function, carry the constant and find the derivative of the function.

These formulas hold basically because the same rules can be applied to limits: limits are additive and constant factors out of them. Thus, if $y = f(x) + g(x)$, the sum rule holds since

$$y' = \lim_{h\to 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h} = f'(x)+g'(x)$$

The difference rule and the constant multiple rules can be shown similarly.

These rules enables you to find derivative of any polynomial function by differentiating term by term. Each term is of the form ax^n which has derivative nax^{n-1} by power and constant multiple rules.

Example 1. Find derivative of $f(x) = 2x^3 + \frac{1}{4}x^2 - 5$.

Solution. By the Power Rule with $n = 3$, the derivative of x^3 is $3x^{3-1} = 3x^2$. So, the derivative of the first term $2x^3$ is $2 \cdot 3x^2 = 6x^2$.

By the Power Rule with $n = 2$, the derivative of x^2 is $2x^{2-1} = 2x^1 = 2x$. So, the derivative of the secod term $\frac{1}{4}x^2$ is $\frac{1}{4} \cdot 2x = \frac{1}{2}x$. The derivative of the constant term -5 is zero.

Hence, $f'(x) = 6x^2 + \frac{1}{2}x - 0 = 6x^2 + \frac{1}{2}x$.

General Power Rule. It can be shown that **the power rule holds for any real number** n**, not only when** n **is a positive integer.**

The General Power Rule.	$f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$ holds for every real n

This rule enables us to find derivatives of functions with negative or fractional powers. You need to make sure that your function is written in the form x^n before you apply the power rule. The following algebra rules may be useful when doing that.

$$\frac{1}{x^n} = x^{-n}$$

$$\sqrt[n]{x} = x^{1/n}$$

Example 2. Find derivatives of the following functions.

(a) $f(x) = \dfrac{x^4}{4} + \dfrac{4}{x^4}$ (b) $f(x) = \sqrt{x^3} - \dfrac{3}{\sqrt{x}}$

Solution. (a) Before finding derivative, write both terms of $f(x)$ in ax^n form: $f(x) = \frac{1}{4}x^4 + 4x^{-4}$. Then find derivative using the power rule for both terms. $f'(x) = \frac{d}{dx}\frac{1}{4}x^4 + \frac{d}{dx}4x^{-4} = \frac{1}{4}(4)x^{4-1} + 4(-4)x^{-4-1} = x^3 - 16x^{-5}$. If necessary, you can write your answer as $f'(x) = x^3 - \frac{16}{x^5}$.

(b) Write both terms of $f(x)$ in ax^n form first: $f(x) = (x^3)^{1/2} - 3x^{-1/2} = x^{3/2} - 3x^{-1/2}$. Then find derivative using the power rule for both terms. $f'(x) = \frac{d}{dx}x^{3/2} - \frac{d}{dx}3x^{-1/2} = \frac{3}{2}x^{3/2-1} - 3(\frac{-1}{2})x^{-1/2-1} = \frac{3}{2}x^{1/2} + \frac{3}{2}x^{-3/2}$. If necessary, you can write your answer as $f'(x) = \frac{3}{2}\sqrt{x} + \frac{3}{2\sqrt{x^3}}$.

The shortcuts. Having the differentiation formulas enables you to find

- the instantaneous rate of change,
- the slope of the tangent line at a point,
- velocity or any other rate of change in an applied problem

without using the definition of the derivative.

Example 3. $f(x) = x^2 + 4$. Find the instantaneous rate of change of $f(x)$ for $x = 1$. Then find the equation of the tangent line to $f(x)$ at $x = 1$.

Solution. Find the derivative to be $f'(x) = \frac{d}{dx}x^2 + \frac{d}{dx}4 = 2x^{1-1} + 0 = 2x$. Then plug $x = 1$ and obtain $f'(1) = 2(1) = 2$.

This computes the slope of the tangent line. The equation of the tangent line can be obtained using the point-slope equation with $m = 2$ and $(1, f(1)) = (1, 5)$. So, the tangent line is $y - 5 = 2(x - 1) \quad \Rightarrow \quad y = 2x + 3$.

More on velocity. Using the previous example, you can relate the formula computing the velocity of an object with position $s(t)$ at time t as $v = \frac{ds}{dt}$ with the pre-calculus formula you used for velocity $v = \frac{s}{t}$ now better expressed as $v = \frac{\Delta s}{\Delta t}$. The first formula computes the instantaneous velocity while the second formula computes the average velocity. As we have seen in the previous example, if the position function is a linear function (that is, if the velocity is constant), the two formulas amount to the same thing. One should keep in mind that the velocity cannot be computed by the pre-calculus formula $v = \frac{s}{t}$ when s is a nonlinear function.

More on other applications of derivative. Similar argument can be made for relation between several other physical quantities. For example, let us recall several other equivalent examples: (1) the force is the quotient of change in work and change in displacement caused by the force, (2) the current produced by a movement of electric charge is the quotient of the charge and time, (3) the density of a piece of wire is the quotient of mass and the length of the wire.

The average velocity	$v = \frac{\Delta s}{\Delta t}$,	**the velocity**	$v = \frac{ds}{dt}$
The average force	$F = \frac{\Delta W}{\Delta x}$,	**the force**	$F = \frac{dW}{dx}$
The average current	$I = \frac{\Delta Q}{\Delta t}$,	**the current**	$I = \frac{dQ}{dt}$
The average density	$\rho = \frac{\Delta m}{\Delta x}$,	**the density**	$\rho = \frac{dm}{dx}$

Example 4. The mass of a metal rod in kilograms depends on the length x measured in meters starting at the rod's end and can be computed by $m(x) = 6\sqrt[3]{x}$. Determine the density of the rod 1 meter from the rod's end.

Solution. The density ρ at $x = 1$ can be computed as the value of derivative $\frac{dm}{dx}\ |_{x=1}$ (recall that this is the same as $m'(1)$). Note that $m(x) = 6x^{1/3}$ so that the derivative is $m'(x) = 6\frac{1}{3}x^{1/3-1} = 2x^{-2/3} = \frac{2}{\sqrt[3]{x^2}}$. Thus $m'(1) = \frac{2}{\sqrt[3]{1^2}} = 2$ kilograms per meter.

Some further examples are given in practice problems below.

Practice problems.

1. Find the derivative of the given functions.

   (a) $y = 2x^5 - 3x^3 + 5x - 9$ (b) $y = x^{38} + 6$

   (c) $y = \frac{x^3}{2} + \sqrt{x^3}$ (d) $y = \frac{4}{x^2} - \frac{1}{3x^6}$

2. Find an equation of the line tangent to the curve at the indicated point.

   (a) $f(x) = \frac{2}{x} + \frac{x}{2}$ at $x = 2$. (b) $f(x) = \sqrt{x^3} + \sqrt[3]{x^2}$ at $x = 1$.

3. A company determines that its cost function is

$$C(x) = 1000 + 35x - .01x^2,$$

   $0 \leq x \leq 300$, where x is the number of items produced and $C(x)$ is the cost of producing x items in dollars. (a) Find the average rate of change in cost when x is changing from 100 to 150. (b) Find the instantaneous rate of change in cost when producing 200 items.

4. Assume that the mathematical model for the growth of a locust tree in its first century of life is given by $h(t) = 3\sqrt{t}$, $0 \leq t \leq 100$, where t is the age of the tree in years and $h(t)$ is the height of the tree in feet. Find $h(64)$ and $h'(64)$ and interpret the meaning of your answers in a full sentence.

5. The mass of bacteria culture at time t in hours, is approximated by $N(t) = 4t^{7/2}$, in milligrams. (a) Find $N(9)$ and $N'(9)$ and interpret the meaning of your answers in a full sentence. (b) Find how fast the mass of bacteria increases 4 hours after the experiment started.

6. The body mass index (BMI) is a number obtained as $BMI = \frac{703w}{h^2}$ where w is the weight in pounds and h is the height in inches. For a 125-lb female that is now 65 inches tall but growing, calculate how fast is BMI changing with each new inch. Explain the meaning of the answer.

7. A particle moves on a line away from its initial position so that after t hours it is $s(t) = 2t^2 - 1$ miles from its initial position. (a) Find the velocity of the particle 5 hours after it started moving. (b) Find the time when the velocity is 30 miles per hour.

8. The mass of a bacteria culture t hours after the start of experiment, is modeled by $N(t) = 3t^{5/2}$, in milligrams. (a) Determine the mass 16 hours after experiment started. (b) Determine how fast the mass of bacteria increases 9 hours after the experiment started. (c) Determine the time when the mass is 300 mg.

Solutions.

1. (a) $y' = 2(5)x^{5-1} - 3(3)x^{3-1} + 5(1)x^{1-1} - 0 = 10x^4 - 9x^2 + 5$

   (b) $y' = 38x^{38-1} + 0 = 38x^{37}$

   (c) $y' = \frac{x^3}{2} + \sqrt{x^3} = \frac{1}{2}x^3 + x^{3/2} \Rightarrow y' = \frac{1}{2}(3)x^{3-1} + \frac{3}{2}x^{3/2-1} = \frac{3}{2}x^2 + \frac{3}{2}x^{1/2}$ or $y' = \frac{3x^2}{2} + \frac{3\sqrt{x}}{2}$.

   (d) $y' = \frac{4}{x^2} - \frac{1}{3x^6} = 4x^{-2} - \frac{1}{3}x^{-6} \Rightarrow y' = 4(-2)x^{-2-1} - \frac{1}{3}(-6)x^{-6-1} = -8x^{-3} + 2x^{-7}$ or $y' = \frac{-8}{x^3} + \frac{2}{x^7}$

2. (a) To use the point-slope equation, you need to compute the y-value of point with $x = 2$ and the slope $f'(2)$.

   $f(2) = \frac{2}{2} + \frac{2}{2} = 1 + 1 = 2$. To find the derivative, note that $f(x) = 2x^{-1} + \frac{1}{2}x \Rightarrow f'(x) = 2(-1)x^{-1-1} + \frac{1}{2}x^{1-1} = \frac{-2}{x^2} + \frac{1}{2}$. Thus $f'(2) = \frac{-2}{2^2} + \frac{1}{2} = \frac{-1}{2} + \frac{1}{2} = 0$. So, the tangent line is $y - 2 = 0(x - 2) \Rightarrow y = 2$.

   (b) $f(1) = \sqrt{1^3} + \sqrt[3]{1^2} = 1 + 1 = 2$. So, the function passes (1,2). $f(x) = \sqrt{x^3} + \sqrt[3]{x^2} = x^{3/2} + x^{2/3} \Rightarrow f'(x) = \frac{3}{2}x^{1/2} + \frac{2}{3}x^{-1/3}$. At $x = 1$, $f'(1) = \frac{3}{2}1^{1/2} + \frac{2}{3}1^{-1/3} = \frac{3}{2} + \frac{2}{3} = \frac{9+4}{6} = \frac{13}{6}$. Thus, the tangent line is $y - 2 = \frac{13}{6}(x - 1) \Rightarrow y = \frac{13}{6}x - \frac{1}{6}$.

3. (a) When production changes from 100 to 150 items produced, the cost increased at an average rate of $\frac{C(150)-C(100)}{150-100} = \frac{6025-4400}{50} = 32.5$ dollars per item produced. (b) $C'(x) = 0 + 35x^{1-1} - 0.01(2)x^{2-1} = 35 - 0.02x$. When producing 200 items, the cost is increasing at a rate $C'(200) = 35 - 0.02(200) = 31$ dollars per item produced.

4. $h(t) = 3t^{1/2}$. $h(64) = 24$. 64 years after it starts growing, the tree is 24 feet tall. $h'(t) = 3\frac{1}{2}t^{1/2-1} = \frac{3}{2}t^{-1/2} = \frac{3}{2\sqrt{t}}$. $h'(64) = \frac{3}{16} = .1875 \approx 0.19$. 64 years after it starts growing, the tree is growing at the rate of .19 feet per year.

5. (a) $N(t) = 4t^{7/2}$, $N(9) = 8748$ mg = the mass of bacteria 9 hours after. $N'(t) = 4\frac{7}{2}t^{7/2-1} = 14t^{5/2}$. $N'(9) = 3402$ mg per hour. Thus, after 9 hours, the mass is increasing at the rate of 3402 mg per hour. (b) 4 hours after, the mass of bacteria is increasing at the rate of $N'(4) = 14(4)^{5/2} = 448$ mg per hour.

6. For $w = 125$, $BMI(h) = \frac{703(125)}{h^2} = 87875h^{-2} \Rightarrow BMI'(h) = 87875(-2)h^{-2-1} = \frac{-175750}{h^3}$. When $h = 65$, the value of the derivative is $\frac{-175750}{65^3} \approx -.64$. Thus, the BMI is decreasing by .64 per inch. The negative sign indicate that for a fixed weight, the BMI decreases when the height increases.

7. (a) $s(t) = 2t^2 - 1 \Rightarrow s'(t) = 4t \Rightarrow s'(5) = 20$ so that the velocity 5 hours after is 20 miles per hour. (b) $s'(t) = 4t = 30 \Rightarrow t = \frac{30}{4} = 7.5$. Thus, the velocity is 30 miles per hour 7.5 hours after.

8. (a) $N(t) = 3t^{5/2} \Rightarrow N(16) =$3072 mg. (b) $N'(t) = \frac{15}{2}t^{3/2} \Rightarrow N'(9) = \frac{15}{2}9^{3/2} = 202.5$ mg per hour. (c) $N(t) = 3t^{5/2} = 300 \Rightarrow t^{5/2} = 100 \Rightarrow t = 100^{2/5} \approx 6.31$ hours. Thus, the mass is 300 mg 6.31 hours after the experiment started.

Higher Derivatives. Differentiable Functions

The second derivative. The derivative itself can be considered as a function. The instantaneous rate of change of this function is the second derivative. Thus, the second derivative evaluated at a point computes the slope of the tangent line to the graph of the first derivative.

The second derivative $f''(x)$ is the derivative of the first derivative $f'(x)$

Notation: $f''(x)$, $\frac{d^2}{dx^2}f(x)$, y'', $\frac{d^2y}{dx^2}$

Evaluated at $x = a$: $f''(a)$, $\frac{d^2}{dx^2}f(x)|_{x=a}$, $\frac{d^2y}{dx^2}\ |_{x=a}$

Example 1. Find the second derivative of the following functions.

(a) $f(x) = 3x^2 + 5x - 6$

(b) $f(x) = \frac{7}{x^2}$

Solutions. (a) $f(x) = 3x^2 + 5x - 6 \Rightarrow f'(x) = 6x + 5 \Rightarrow f''(x) = 6x^{1-1} + 0 = 6.$

(b) $f(x) = \frac{7}{x^2} = 7x^{-2} \Rightarrow f'(x) = -14x^{-3} \Rightarrow f''(x) = -14(-3)x^{-3-1} = 42x^{-4} = \frac{42}{x^4}$

Applications. If $s(t)$ represents the position of an object at time t, we have seen that the derivative $s'(t)$ represents the velocity at time t. **The second derivative is the acceleration** since it calculates the rate at which the velocity is changing.

velocity	$v(t) = s'(t) = \frac{ds}{dt}$
acceleration	$a(t) = v'(t) = \frac{dv}{dt} = s''(t) = \frac{d^2s}{dt^2}$

Example 2. Consider the object whose position (in meters) is given as a function of time (in seconds) by the formula $s(t) = t^3 - 7t^2 + 13t$. Find the formulas for velocity and acceleration.

Solution. $s(t) = t^3 - 7t^2 + 13t \Rightarrow v(t) = s'(t) = 3t^2 - 14t + 13 \Rightarrow a(t) = s''(t) = 6t - 14.$

Higher Derivatives. Continuing differentiating the derivative, one obtains the higher derivatives: the second derivative as the derivative of the first, the third derivative as the derivative of the second and so on. The third derivative is usually denoted by $f'''(x)$. For the derivatives higher than three, $f^{(n)}$ is used to denote the n-th derivative. So, for example, the fourth derivative is written as $f^{(4)}(x)$ rather than $f''''(x)$.

Example 3. Find the fourth derivative of the function $f(x) = \sqrt{x} + x^2 + 5$.

Solution.

$$f(x) = \sqrt{x} + x^2 + 5 = x^{1/2} + x^2 + 5 \quad \Rightarrow \quad f'(x) = \frac{1}{2}x^{1/2-1} + 2x^1 + 0 = \frac{1}{2}x^{-1/2} + 2x \quad \Rightarrow$$
$$f''(x) = \frac{1}{2}\frac{-1}{2}x^{-1/2-1} + 2x^0 = \frac{-1}{4}x^{-3/2} + 2 \quad \Rightarrow \quad f'''(x) = \frac{-1}{4}\frac{-3}{2}x^{-3/2-1} + 0 = \frac{3}{8}x^{-5/2} \quad \Rightarrow$$
$$f^{(4)}(x) = \frac{3}{8}\frac{-5}{2}x^{-5/2-1} = \frac{-15}{16}x^{-7/2}.$$

Differentiable Functions. Recall that the derivative at $x = a$ is the limit $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ or, equivalently $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$. If either one (then necessarily both) of these limits exist, $f(x)$ is said to be **differentiable** at $x = a$. Thus, if you can find the derivative of a function (either by definition or using the differentiation formulas) and if $f'(x)$ is defined at $x = a$, then the function is differentiable at a.

Recall that a function is continuous at $x = a$ if limit of $f(x)$ when $x \to a$ exists and it is equal to $f(a)$. Thus, $f(a) = \lim_{x\to a} f(x)$ or, equivalently, $\lim_{x\to a} f(x) - f(a) = 0$. If a function $f(x)$ is differentiable at a, then

$$\lim_{x\to a} f(x) - f(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}(x - a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \lim_{x\to a} x - a = f'(a) \cdot 0 = 0$$

and so the function is continuous. Thus, **if a function is differentiable, it is continuous**.

Even when continuous, $f(x)$ may fail to be differentiable at a if the left and right limits of $\frac{f(a+h)-f(a)}{h}$ are different. In this case, the slope of the tangent on the left and the slope of the tangent on the right side of a are different and $f(x)$ is said to have a **corner** or a **sharp turn** at $x = a$.

The last scenario of $f(x)$ failing to be differentiable at a is when either left, right (or both) limits of $\frac{f(a+h)-f(a)}{h}$ exist but are not finite. In this case, $f(x)$ is said to have a **vertical tangent**.

Thus, $f(x)$ can fail to be differentiable at $x = a$ in any of the following cases.

1. $f(x)$ is not continuous at a.

2. $f(x)$ has a corner at a.

3. $f(x)$ has a vertical tangent at a.

Example 4. Discuss the differentiability of the following functions.

(a) $f(x) = x$ (b) $f(x) = \begin{cases} x & x \geq 0 \\ x + 1 & x < 0 \end{cases}$

(c) $f(x) = |x|$ (d) $f(x) = \sqrt[3]{x}$

Solution. (a) The derivative $f'(x)$ of $f(x)$ at any x is 1 (either by using the power rule or finding it using the definition as $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{x+h-x}{h} = \lim_{h\to 0} \frac{h}{h} = 1$. Since $f'(x) = 1$ is defined at every x, $f(x) = x$ is differentiable for every x.

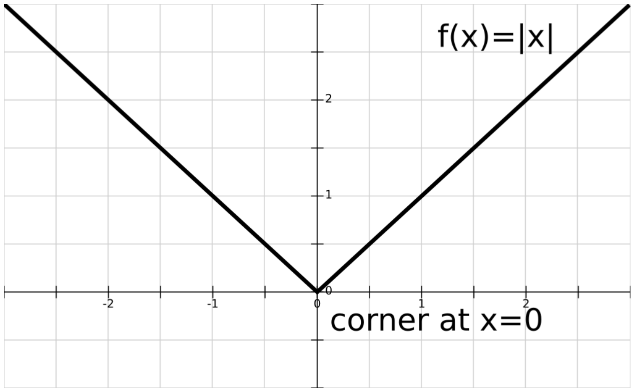
(b) For $x > 0$, $f(x) = x$ so $f'(x) = 1$ and thus $f(x)$ is differentiable for every $x > 0$. Similarly, for $x < 0$, $f(x) = x + 1$ so that $f'(x) = 1$ and thus $f(x)$ is differentiable for every $x < 0$ as well. At $x = 0$, the function is not continuous since the left limit when $x \to 0^-$ is 1 and the right limit when $x \to 0^+$ is 0. Since $f(x)$ is not continuous at 0, it is not differentiable at 0 as well.

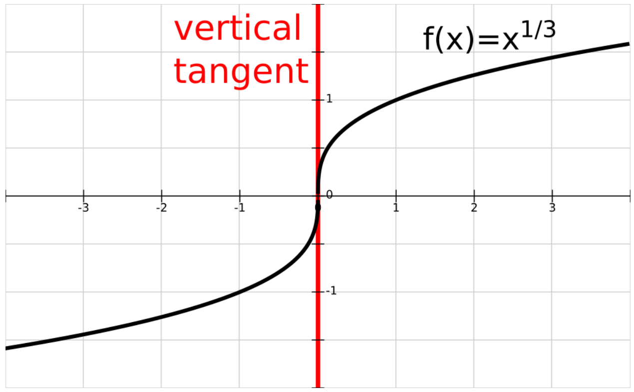
(c) Recall that $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$. For $x > 0$, we have seen that $f'(x) = 1$ so $f(x)$ is differentiable for every $x > 0$. Similarly, for $x < 0$, $f'(x) = -1$ so $f(x)$ is differentiable for every $x < 0$ as well. If it exists, the derivative $f'(0)$ at $x = 0$ is equal to $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|0+h|-|0|}{h} = \lim_{h\to 0} \frac{|h|}{h}$. Since the value of $|h|$ depends on the fact if h is positive or negative, we consider these cases separately:

1. when $h > 0$, $|h| = h$ and so $\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$.

2. when $h < 0$, $|h| = -h$ and so $\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$.

Thus, the left and the right limit of $\frac{f(0+h)-f(0)}{h}$ are not equal and so $f'(0)$ does not exist. Thus $f(x)$ is not differentiable at 0. Looking at the graph of $|x|$ one can notice it has a corner.

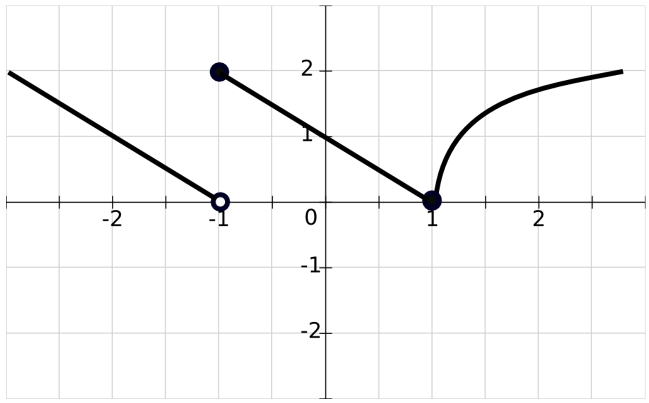




(d) $f(x) = \sqrt[3]{x} = x^{1/3} \Rightarrow f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$. This derivative is defined for every value of x except 0. So, $f(x)$ is differentiable for every $x \neq 0$. $f'(x)$ is not defined at 0 so $f(x)$ is not differentiable at 0. The graph of $f(x)$ reveals a vertical tangent at $x = 0$.

If a function is given by a graph, it is differentiable at a point if it has a (non-vertical) tangent at (both sides of) the point. If there is a corner, discontinuity or a vertical tangent, it is not differentiable.

Example 5. Discuss the differentiability of the function given by the graph on the right.



Solutions. The function is differentiable at every point different from -1 and 1 since there is a well defined tangent to the graph for all $x \neq \pm 1$. At $x = -1$ the function has a break so it is not continuous and thus also not differentiable. At $x = 1$ the function is not differentiable since there is a corner in the graph.

Practice problems.

1. Find the first and the second derivative of the following functions.

   (a) $f(x) = \frac{x^3}{2} + \frac{4}{x^2}$ (b) $f(x) = \sqrt{x^3} + \sqrt[3]{x^2}$

2. Find the first five derivatives of the function $f(x) = 2x^5 - 3x^3 + 5x - 9$.

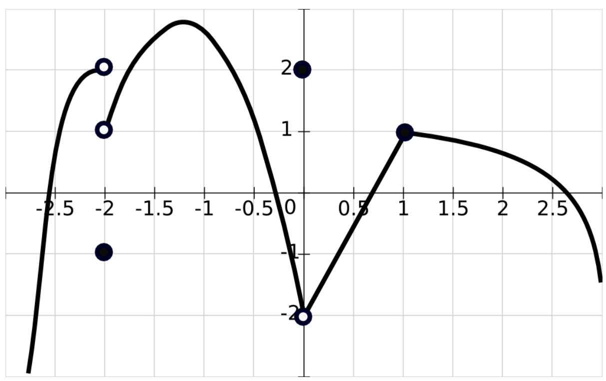
3. Discuss the differentiability of the following functions.

   (a) $f(x) = x^2 + 2$

   (b) $f(x) = 5\sqrt[3]{x^2}$

   (c) $f(x) = 2 - 3x^{1/5}$

   (d) The function given by the graph on the right.



Solutions.

1. (a) $f(x) = \frac{x^3}{2} + \frac{4}{x^2} = \frac{1}{2}x^3 + 4x^{-2} \Rightarrow f'(x) = \frac{3}{2}x^2 - 8x^{-3} \Rightarrow f''(x) = 3x + 24x^{-4} = 3x + \frac{24}{x^4}$

   (b) $f(x) = \sqrt{x^3} + \sqrt[3]{x^2} = x^{3/2} + x^{2/3} \Rightarrow f'(x) = \frac{3}{2}x^{1/2} + \frac{2}{3}x^{-1/3} \Rightarrow f''(x) = \frac{3}{4}x^{-1/2} - \frac{2}{9}x^{-4/3}$

2. $f(x) = 2x^5 - 3x^3 + 5x - 9 \Rightarrow f'(x) = 10x^4 - 9x^2 + 5 \Rightarrow f''(x) = 40x^3 - 18x \Rightarrow f'''(x) = 120x^2 - 18 \Rightarrow f^{(4)}(x) = 240x \Rightarrow f^{(5)}(x) = 240$.

3. (a) $f(x) = x^2 + 2 \Rightarrow f'(x) = 2x$. The derivative is defined at every x-value so $f(x)$ is differentiable for every x.

   (b) $f(x) = 5\sqrt[3]{x^2} = 5x^{2/3} \Rightarrow f'(x) = \frac{10}{3}x^{-1/3} = \frac{10}{3\sqrt[3]{x}}$. This function is defined for every value of x except $x = 0$. Graphing $f(x)$, you can notice that it has a corner (and a vertical tangent) at $x = 0$ so it is not differentiable at 0. Thus, $f(x)$ is differentiable for every $x \neq 0$.

   (c) $f(x) = 2 - 3x^{1/5} \Rightarrow f'(x) = \frac{-3}{5}x^{-4/5} = \frac{-3}{5\sqrt[5]{x^4}}$ This function is defined for every value of x except $x = 0$. Graphing $f(x)$, you can notice that it has a vertical tangent at $x = 0$ so it is not differentiable at 0. Thus, $f(x)$ is differentiable for every $x \neq 0$.

   (d) The function is differentiable at every point different from -2, 0 and 1 since there is a well defined tangent to the graph for all $x \neq -2, 0, 1$. At $x = -2$ and $x = 0$ the function is not differentiable since it is not continuous (a jump at -2 and a hole at 0). At $x = 1$ the function is not differentiable since there is a corner in the graph.