Fundamentals of Calculus
Lia Vas

Extreme Values and The Derivative Tests. Concavity and Inflection Points.

Increasing/Decreasing Test. Recall that a function $f(x)$ is **increasing** on an interval if the increase in x-values implies an increase in y-values for all x-values from that interval.

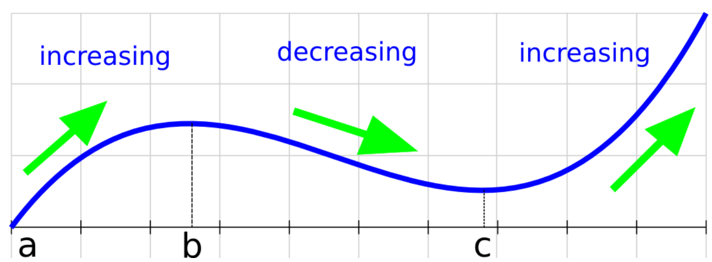
$$x_1 > x_2 \quad \Rightarrow \quad f(x_1) < f(x_2).$$

Analogously, $f(x)$ is **decreasing** if

$$x_1 < x_2 \quad \Rightarrow \quad f(x_1) > f(x_2).$$

These concepts completely correspond to the intuitive "going up" and "going down" when looking at the graph. One just needs to keep in mind the positive direction of the x-axis when considering the "direction" of the curve.

Example 1. Determine the intervals on which the function with the graph on the right defined on interval (a, ∞) is increasing/decreasing.

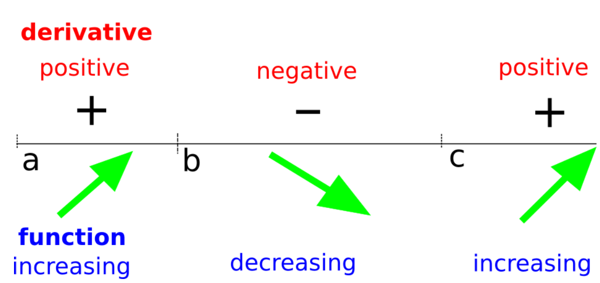


Solution. The function is increasing on intervals (a, b) and (c, ∞) and decreasing on (b, c).

The slope of the line tangent to an increasing function at a point is positive. Thus, increasing differentiable functions have positive derivatives. The converse is also true: if a derivative of a differential function is positive for all x-values from an interval, then the function is increasing on that interval. These two statements combine in the following equivalence and the analogous equivalence holds for decreasing functions as well.

> **Increasing/Decreasing Test.** A differentiable function $f(x)$ is:
>
> - increasing on an interval if and only if the derivative $f'(x)$ is positive on that interval,
> - decreasing on an interval if and only if the derivative $f'(x)$ is negative on that interval.

Thus, determining the intervals of increase/decrease can be done by analyzing the **sign of the derivative**. Recall the number line test method from Precalculus for finding the intervals on which a function is positive/negative. For example, the number line test applied to the derivative of the function from Example 1, would be as follows.

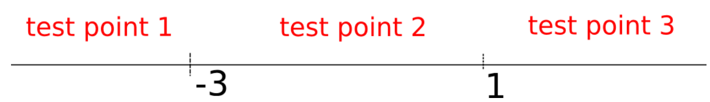


Example 2. Determine the intervals on which the following functions are increasing/decreasing.

(a) $f(x) = x^3 + 3x^2 - 9x - 8$ (b) $f(x) = \dfrac{x^2+4}{2x}$

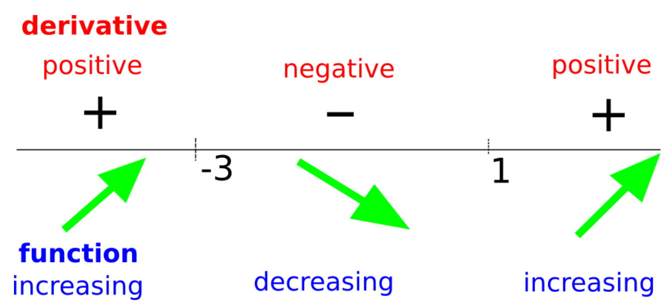
Solutions. (a) Find the derivative $f'(x) = 3x^2 + 6x - 9$ and factor it as $f'(x) = 3(x^2 + 2x - 3) = 3(x-1)(x+3)$. This tells you that the derivative can change sign when $x = 1$ and $x = -3$ so there are the relevant points to consider on the number line.

Recall the number line method from the pre-calculus course: In this case, put the points -3 and 1 on a number line and note that they divide the number line in three parts. Take a test point from each part and calculate the value of f' at it.



For example, with -4, 0, and 2 as the test points, you obtain that $f'(-4) = 15 > 0$, $f'(0) = -9 < 0$ and $f'(2) = 15 > 0$. *Careful:* you want to plug the test points in the derivative f' not in the function f since you are determining the sign of f', not f.

Thus we have the number line on the right.

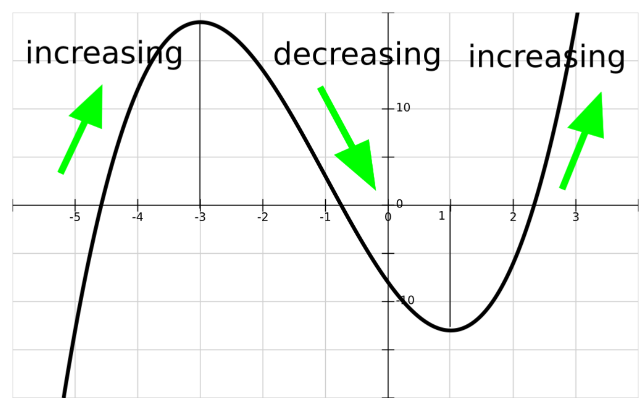


From the number line, we conclude that $f(x)$ is increasing for $x < -3$ and $x > 1$ and decreasing for $-3 < x < 1$. Alternatively, you can write your answer using interval notation as follows.

$f(x)$ is increasing on $(-\infty, -3)$ and $(1, \infty)$,

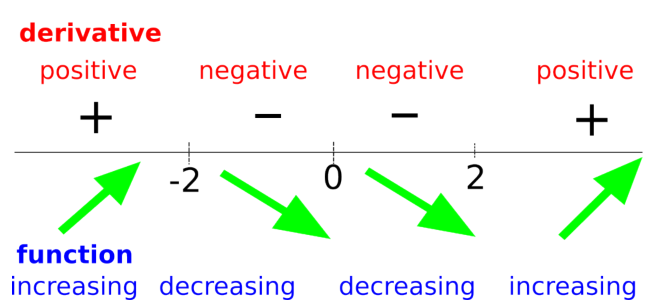
$f(x)$ is decreasing on $(-3, 1)$.

You can graph the function to check that the graph agrees with your findings.



(b) Use the quotient rule to find the derivative $f'(x) = \frac{2x(2x)-2(x^2+4)}{(2x)^2} = \frac{4x^2-2x^2-8}{4x^2} = \frac{2x^2-8}{4x^2} = \frac{x^2-4}{2x^2} = \frac{(x-2)(x+2)}{2x^2}$.

Three terms impact the sign of $f'(x)$: $x - 2, x + 2$ and $2x^2$. These terms change the sign for $x = 2, -2$ and 0 so those are the numbers relevant for the number line. Since these three values divide the number line to four pieces, you need four test points. Test the points and obtain the number line on the right.

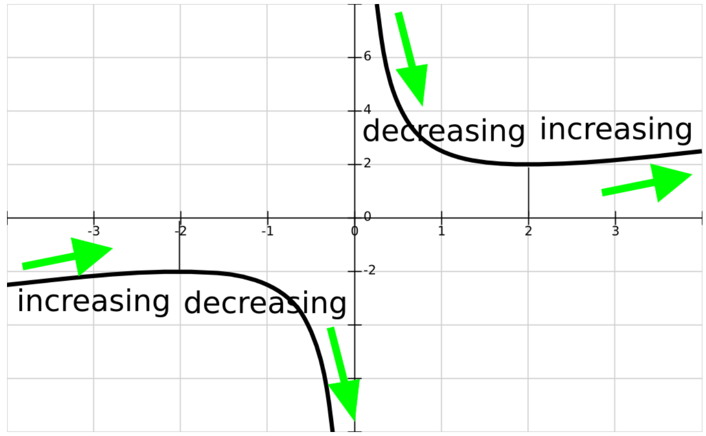


From the number line, we conclude that $f(x)$ is increasing for $x < -2$ and $x > 2$ and decreasing for $-2 < x < 0$ and $0 < x < 2$. Alternatively, using interval notation you have that.

$f(x)$ is increasing on $(-\infty, -2)$ and $(2, \infty)$,

$f(x)$ is decreasing on $(-2, 0)$ and $(0, 2)$.

You can graph the function to check that the graph agrees with your findings.

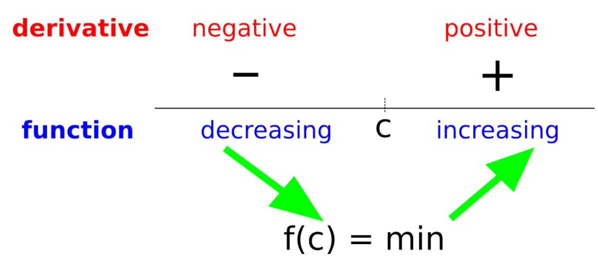


We have seen that the positive/negative sign of the first derivative corresponds exactly to function increasing/decreasing. Next we consider what happens at the points at which the derivative is neither positive, nor negative. At such points the derivative is either zero or undefined and are called the critical points.

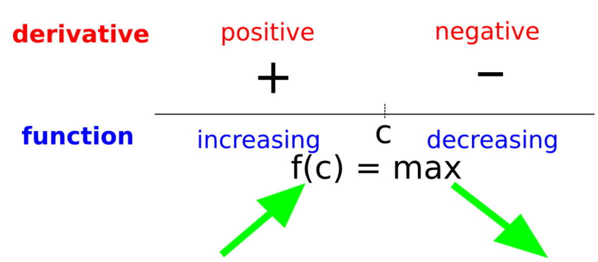
> The point $x = c$ is a **critical point** of a function $f(x)$ if
>
> $f'(c) = 0 \qquad$ or $\qquad f'(c)$ does not exist.

If f is a continuous function, and c is a critical point of f, the following three scenarios are possible.

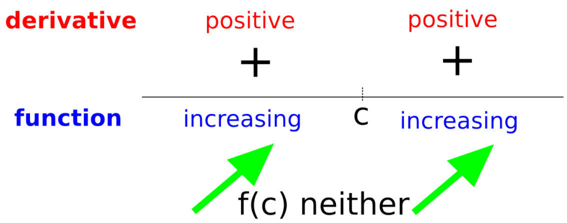
Case 1 The sign of f' is changing from negative to positive. This means that the function is decreasing before c, then reaches the bottom at $x = c$, and then increases after c. So, if defined, $f(c)$ is smaller than all values $f(x)$ when x is in some interval around c. In this case, the function f is said to have a **relative minimum** at $x = c$ and the y-value $f(c)$ is called the **minimal value.**

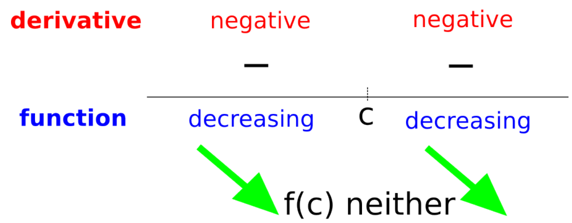


Case 2 The sign of f' is changing from positive to negative. This means that the function is increasing before c, then reaches the top at $x = c$, and then decreases after c. So, if defined, $f(c)$ is larger than all values $f(x)$ when x is in some interval around c. In this case, the function f is said to have a **relative maximum** at $x = c$ and the y-value $f(c)$ is called the **maximal value.**



Case 3 The sign of f' is not changing at $x = c$ (it is either positive both before or after c or negative both before or after c). In this case, f has neither minimum nor maximum at $x = c$.





The minimum and maximum values are collectively referred to as the **extreme values**.

The existence of the third case demonstrates that *a function does not necessarily have a minimum or maximum value at a critical value.*

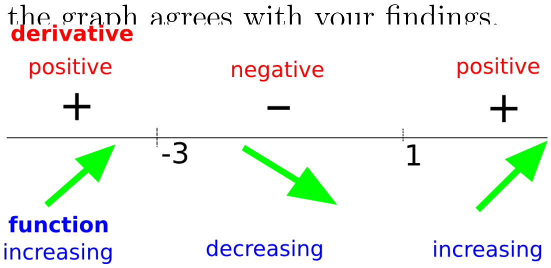
Using the number line test just as when determining increasing/decreasing intervals, one can readily classify the critical points into three categories matching the three cases above and determine the points at which a function has extreme values. This procedure is known as the **First Derivative Test.** Let us summarize.

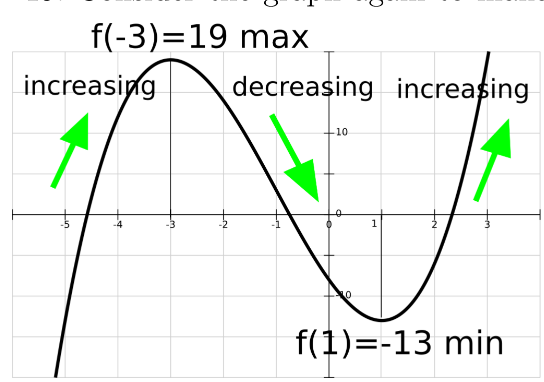
The First Derivative Test. To determine the extreme values of a continuous function $f(x)$:
1. Find $f'(x)$.
2. Set it to zero and find all the critical points.
3. Use the number line to classify the critical points into the three cases.
   - if $f'(x)$ changes from negative to positive at c and $f(c)$ is defined, then f has a minimum at c,
   - if $f'(x)$ changes from positive to negative at c and $f(c)$ is defined, then f has a maximum at c,
   - if $f'(x)$ does not change the sign at c, f has no extreme value c.

Example 2 revisited. Determine the extreme values of the following functions.

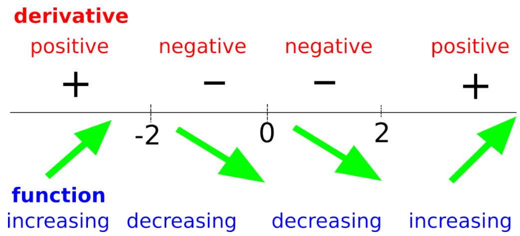
(a) $f(x) = x^3 + 3x^2 - 9x - 8$ (b) $f(x) = \dfrac{x^2+4}{2x}$

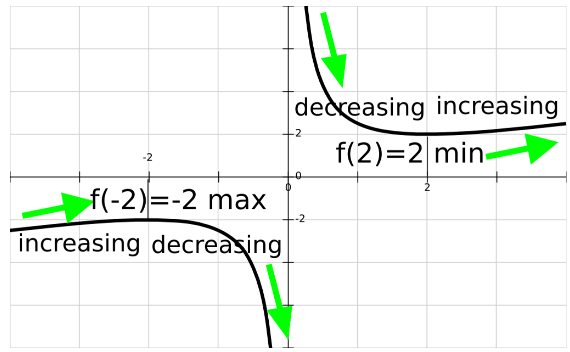
Solutions. (a) Recall that the derivative is $f'(x) = 3(x^2 + 2x - 3) = 3(x - 1)(x + 3)$ so that $x = 1$ and $x = -3$ are the critical points. In Example 2, we have performed the number line test for the derivative and obtained the number line below. Thus, both critical values are extreme values and there is a minimum at $x = 1$ and a maximum at $x = -3$. Compute the y-values to determine the minimal value $f(1) = -13$ and the maximal value $f(-3) = 19$. Consider the graph again to make sure that the graph agrees with your findings.





(b) Recall that the derivative is $f'(x) = \frac{(x-2)(x+2)}{2x^2}$. Thus the critical points are $x = 2, x = -2$ (at which f' is zero) and $x = 0$ at which f' is not defined. In Example 2, we have performed the number line test for the derivative and obtained the number line below. Thus, both 2 and -2 are extreme values and there is a minimum at $x = 2$ and a maximum at $x = -2$. The derivative is not changing sign at 0 so it is not an extreme value. f is also undefined at 0 so it cannot have an extreme value at 0 for that reason too. Compute the y-values to determine the minimal value $f(2) = 2$ and the maximal value $f(-2) = -2$. Consider the graph again to make sure that the graph agrees with your findings.





Example 3. A **drug concentration function** is any function describing the concentration C (in μg/cm^3) of a drug in the body at time t hours after the drug was administered. Consider the drug concentration function

$$C(t) = 2te^{-.4t} \quad \text{for} \quad t \geq 0.$$

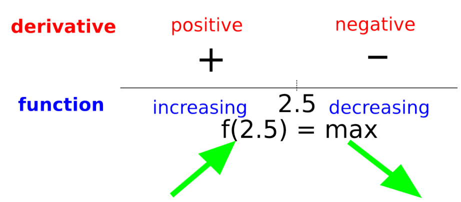
Determine the time intervals when the drug concentration is increasing/decreasing. Determine also the maximal concentration and the time when it is reached.

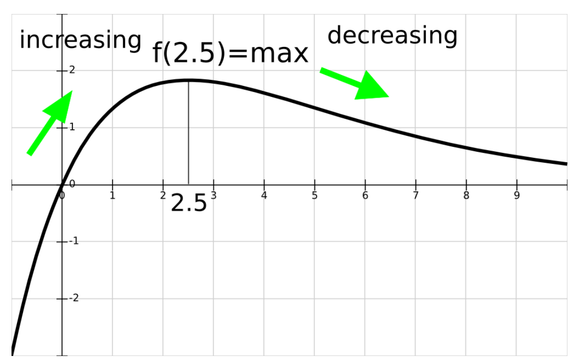
Solution. Use the product rule for the derivative $C'(t) = 2e^{-.4t} + e^{-.4t}(-.4)2t = 2e^{-.4t}(1 - .4t)$. Find the critical points by setting the derivative to zero and solving for t.

$$2e^{-.4t}(1 - .4t) = 0 \Rightarrow 2e^{-.4t} = 0 \text{ or } 1-.4t = 0 \Rightarrow e^{-.4t} = 0 \text{ or } 1 = .4t \Rightarrow \text{ no sol. or } t = \frac{1}{.4} = \frac{5}{2} = 2.5.$$

Note that $e^{-.4t} = 0$ has no solutions since the exponential function is always positive (alternatively $e^{-.4t} = 0 \Leftrightarrow -.4t = \ln 0$ which is not defined). Hence $t = 2.5$ is the only critical point.

Perform the number line test and obtain the findings on the figure. From the number line, we conclude that the function is increasing on $(-\infty, 2.5)$ and decreasing on $(2.5, \infty)$. At the critical point 2.5 the function reaches its maximum of $f(2.5) = 5e^{-1} \approx 1.84$. Graph the function to check if the graph agrees with your findings.

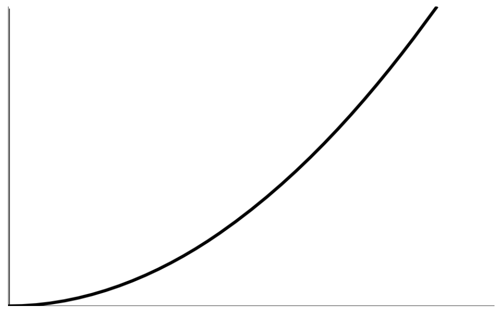


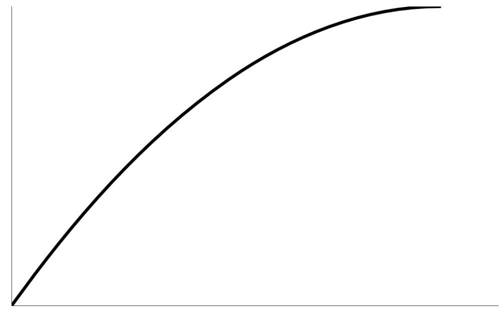


Interpret the conclusions in context of the problem. Since the negative time values are not relevant, we can consider the interval $(0, 2.5)$ instead of $(-\infty, 2.5)$ and conclude that the drug concentration is increasing in the first 2.5 hours and decreasing after 2.5 hours. At 2.5 hours, it reaches a max value of 1.84 μg/cm^3.

Concavity and Inflection Points. Extreme Values and The Second Derivative Test.

Consider the following two increasing functions. While they are both increasing, their concavity distinguishes them.





The first function is said to be concave up and the second to be concave down. More generally, a function is said to be **concave up** on an interval if the graph of the function is above the tangent at each point of the interval. A function is said to be **concave down** on an interval if the graph of the function is below the tangent at each point of the interval.

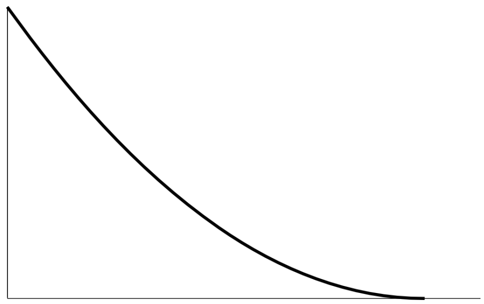
Concave up

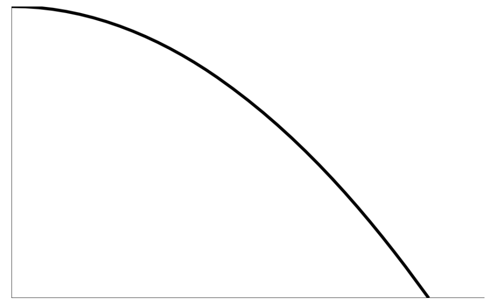
Concave down

In case of the two functions above, their concavity relates to the **rate of the increase**. While the first derivative of both functions is positive since both are increasing, the rate of the increase distinguishes them. The first function *increases at an increasing rate* because the slope of the tangent line becomes steeper and steeper as the x-values increase. So, the *first derivative of the first function is increasing*. Thus, the derivative of the first derivative, the **second derivative is positive**. Note that this function is concave up.

The second function *increases at an decreasing rate* (see how it flattens towards the right end of the graph) so that the *first derivative of the second function is decreasing* because the slope of the tangent line becomes less and less steep as the x-values increase. So, the derivative of the first derivative, **second derivative is negative**. Note that the second function is concave down.

Similarly, consider the following two decreasing functions. We consider again how the concavity relates both to the rate of the decrease and to the sign of the second derivative.

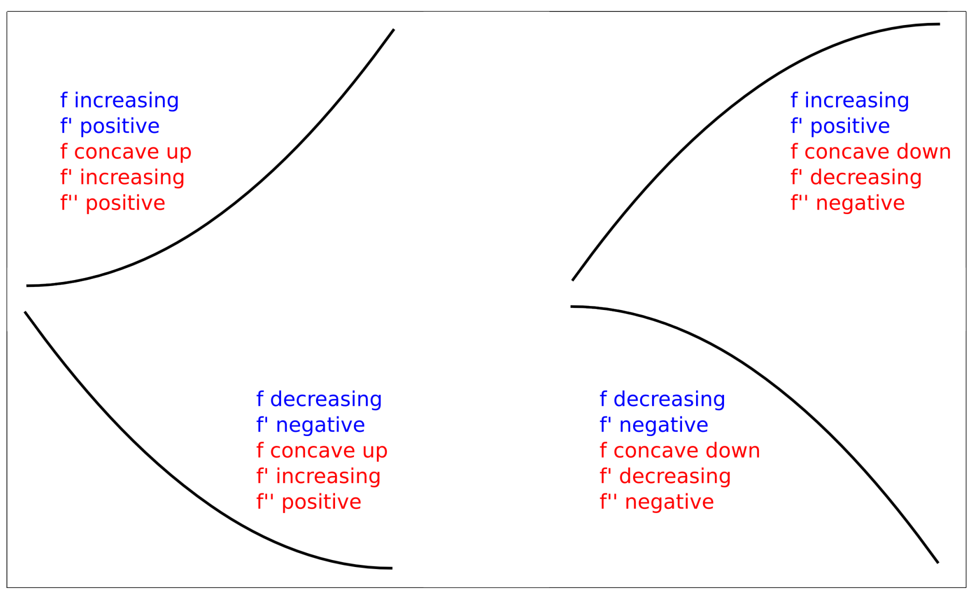




The first decreasing function *decreases at an increasing rate* (see how the slopes of tangent lines become less and less negative as x-values increase). So, the *first derivative of the first function is increasing*. Thus, the derivative of the first derivative, the **second derivative is positive**. Note also that this function is **concave up**.

The second function *decreases at an decreasing rate* (see how the slope of tangent lines become more and more negative as x-values increase). So, the *first derivative of the second function is decreasing*. Thus, the derivative of the first derivative, **second derivative is negative**. Note also that this function is **concave down**.

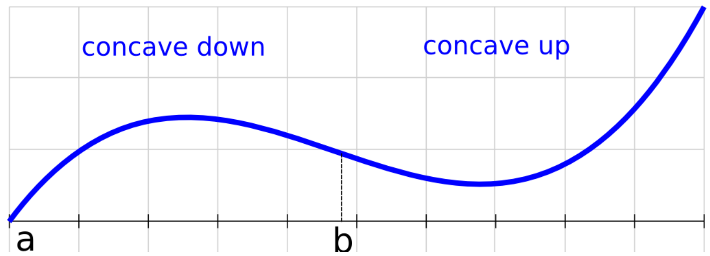
The chart below summarizes our conclusions regarding the four functions we considered so far.



Thus, we have seen that the concavity exactly corresponds to the rate of change of the first derivative which, in turn, exactly corresponds to the sign of the second derivative by the Increasing/Decreasing Test applied to the derivative. This correlation is referred to as the Concavity test.

The Concavity Test. For a function $f(x)$ with derivatives f' and f'' on an interval the following holds.
- f is concave up $\Leftrightarrow f'$ is increasing $\Leftrightarrow f''(x)$ is positive,
- f is concave down $\Leftrightarrow f'$ is decreasing $\Leftrightarrow f''(x)$ is negative.

Example 1. Determine the intervals on which the function with the graph on the right defined on interval (a, ∞) is concave up/down.



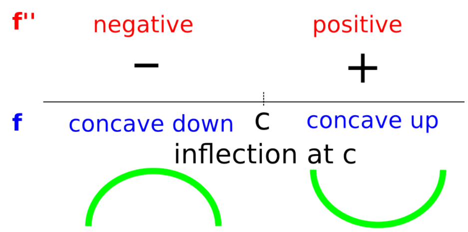
Solution. The function is concave up on the interval (a, b) and concave down on the interval (b, ∞).

The point at which a function is changing concavity is called the **inflection point**. In the example above, the point $(b, f(b))$ is an inflection point.

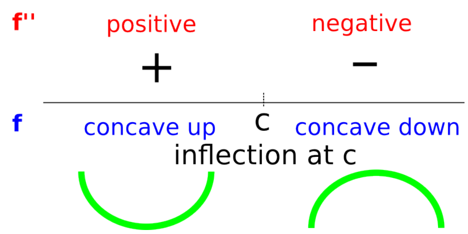
If $f(x)$ has an inflection point at $x = c$, then $\quad f''(c) = 0 \quad$ or $\quad f''(c)$ does not exist.

Note that if point c is such that $f''(c)$ is either zero or undefined, then c is the **critical point of** f'. Thus, the inflection points and the critical points of f' are in analogous relationship as the critical points of f and the extreme values. So, if f is such that f' and f'' exist, the following three scenarios are possible.

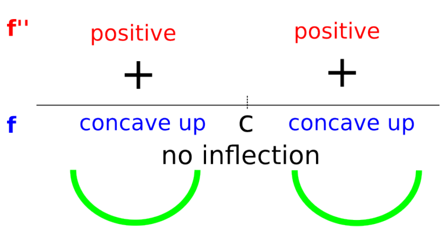
Case 1 The sign of f'' is changing from negative to positive. This means that the function is concave down before c and concave up after c. If $f(c)$ exists, then f has an inflection point at $x = c$.

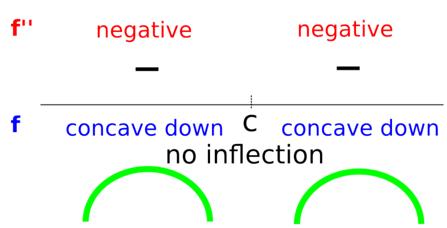


Case 2 The sign of f'' is changing from positive to negative. This means that the function is concave up before c and concave down after c. If $f(c)$ exists, then f has an inflection point at $x = c$.



Case 3 The sign of f'' is not changing at $x = c$ (it is either positive both before or after c or negative both before or after c). In this case, f does not have an inflection point at $x = c$.





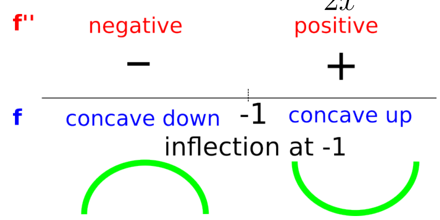
The existence of the third case demonstrates that *a function does not necessarily have an inflection point at a critical point of* f'. For example, the function x^4 is such that $f' = 4x^3$ and $f''(x) = 12x^2$. $f''(x) = 12x^2 = 0 \Rightarrow x = 0$ so 0 is the critical point of f'. However, $f''(x) \geq 0$ for all x so the sign of f'' does not change at 0. Hence, there is no inflection point at $x = 0$. Looking at the graph of x^4, you can also see that it is concave up on the entire domain.

Using the number line test for f'' one can both determine the intervals on which f is concave up/down as well as classify the critical point of f' into three categories matching the three cases above and determine the inflection points. Let us summarize.

> **The Inflection Points Test.** To determine the inflection points a differentiable function $f(x)$:
> 1. Find $f''(x)$.
> 2. Set it to zero and find all the critical points of $f'(x)$.
> 3. Use the number line to classify the critical points of f' into the three cases.
>   - if $f''(x)$ changes sign at c and $f(c)$ is defined, then f has an inflection point at c,
>   - if $f''(x)$ does not change the sign at c, f does not have an inflection point at c.

Example 2. Determine the concavity and the inflection points of the following functions.

(a) $f(x) = x^3 + 3x^2 - 9x - 8$ (b) $f(x) = \dfrac{x^2+4}{2x}$



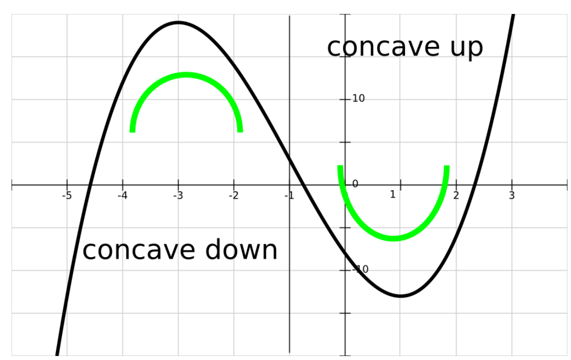
Solutions. (a) $f(x) = x^3 + 3x^2 - 9x - 8 \Rightarrow f'(x) = 3x^2 + 6x - 9 \Rightarrow f''(x) = 6x + 6$. Find the critical points of f', $f''(x) = 0 \Rightarrow 6x + 6 = 0 \Rightarrow 6x = -6 \Rightarrow x = -1$. Put -1 on the number line and test both intervals to which it divides the number line. For example, with -2 and 0 as the test points, you obtain that $f''(-2) = -6 < 0$, and $f''(0) = 6 > 0$.

Thus, the function is concave down before -1 and concave up after -1.

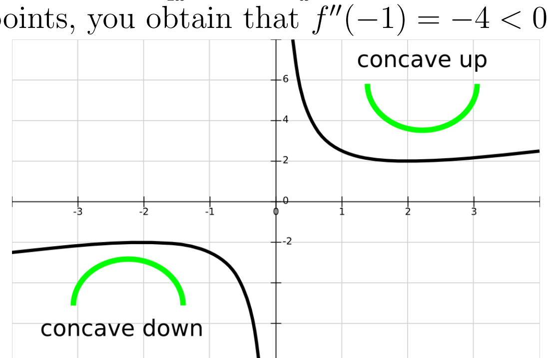
$f(x)$ is concave up on $(-1, \infty)$,

$f(x)$ is concave down on $(-\infty, -1)$.

Find that $f(-1) = 3$ and conclude that $(-1, 3)$ is an inflection point. You can graph the function to check that the graph agrees with your findings.

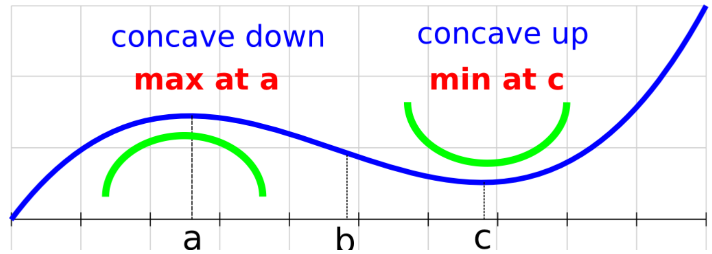


(b) $f(x) = \frac{x^2+4}{2x} \Rightarrow f'(x) = \frac{x^2-4}{2x^2} \Rightarrow f''(x) = \frac{2x(2x^2)-4x(x^2-4)}{4x^4} = \frac{4x(x^2-x^2+4)}{4x^4} = \frac{4}{x^3}$. Thus the critical point of f' is 0. For example, with -1 and 1 as the test points, you obtain that $f''(-1) = -4 < 0$, and $f''(1) = 4 > 0$. Hence, the number line test for f'' is $\begin{array}{c|c} f'' & - \qquad + \\ \hline f & \cap \quad 0 \quad \cup \end{array}$. Thus, f is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$. Since $f(x)$ is not defined at 0, there are no inflection points. Finally, graph the function and make sure that the graph agrees with your findings.



Concavity of a function can be used also to determine if there is a maximum or a minimum at a critical point of f. Note that a function with a relative minimum is concave up on an interval around it. Similarly, a function with a relative maximum is concave down on an interval around it.

Thus, if c is a critical point and the second derivative at c is positive, that means that the function is concave up around c. In this case, there is a relative minimum at c. Analogously, if c is a critical point and the second derivative at c is negative, that means that the function is concave down around c. Thus, there is a relative maximum at c.



This procedure of determining the extreme values is known as the **Second Derivative Test.**

The Second Derivative Test. To determine the extreme values of a function $f(x)$ with derivatives f' and f'':

1. Find $f'(x)$.
2. Set it to zero and find all the critical points.
3. Find $f''(x)$. Plug each critical point c in f''.
   - if $f''(c) > 0$, f has a minimum at c,
   - if $f''(c) < 0$, f has a maximum at c,
   - if $f''(c) = 0$ or $f''(c)$ is not defined, this test is inconclusive.

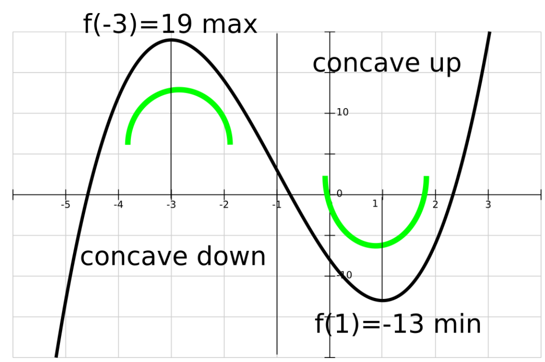
Example 3. Determine the extreme values of the following functions using the Second Derivative Test.

(a) $f(x) = x^3 + 3x^2 - 9x - 8$ (b) $f(x) = \dfrac{x^2+4}{2x}$

Solutions. (a) $f'(x) = 3x^2 + 6x - 9 = 3(x-1)(x+3)$ so that $x = 1$ and $x = -3$ are the critical points and $f''(x) = 6x + 6$.

$f''(1) = 12 > 0$ so there is a minimum at $x = 1$ by the Second Derivative Test. The minimal value is $f(1) = -13$.

$f''(-3) = -12 < 0$ so there is a maximum at $x = -3$ by the Second Derivative Test. The maximal value is $f(-3) = 19$.

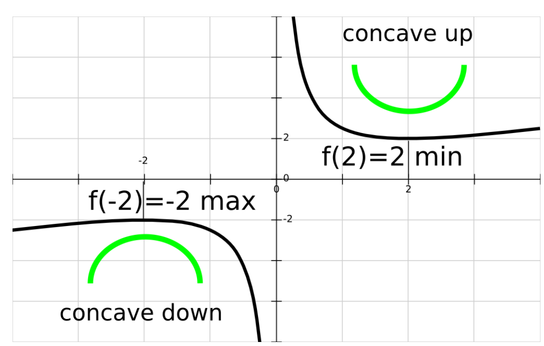


(b) $f'(x) = \frac{x^2-4}{2x^2} = \frac{(x-2)(x+2)}{2x^2}$ so that the critical points are $x = 2, x = 0$ and $x = -2$ and $f''(x) = \frac{4}{x^3}$.

$f''(2) = \frac{1}{2} > 0$ so there is a minimum at $x = 2$ by the Second Derivative Test. The minimal value is $f(2) = 2$.

$f''(2) = \frac{-1}{2} < 0$ so there is a maximum at $x = -2$ by the Second Derivative Test. The maximal value is $f(-2) = -2$.

$f(x)$ is not defined at 0 so there is no extreme



point at 0. Graph the function to check that it agrees with your findings.

The existence of both the First and the Second Derivative Tests gives you an option to choose which one you prefer. The following are some general pros and cons for both tests. In each problem, determine first your best course of action.

Use the First Derivative test in the following cases.

- The Second Derivative Test is inconclusive at a critical point.
- The second derivative is not easy to determine.
- Besides asking for the extreme values, the problem is asking for the increasing/decreasing intervals. In this case, you need to have the number line for the first derivative anyway.

Use the Second Derivative test in the following cases.

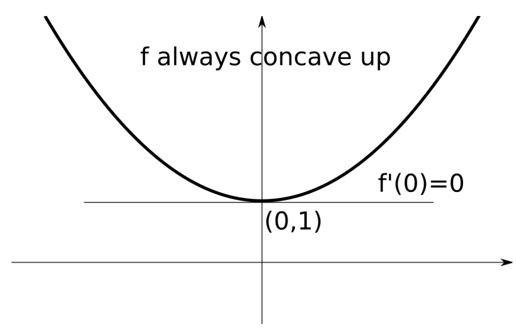
- The First Derivative Test cannot be performed (see the following example).
- Besides asking for the extreme values, the problem is asking for concavity and inflection points. In this case, you need to find the second derivative anyway.
- If both tests can be performed, plugging the critical points in f'' may be computationally easier than performing the line test for f'.

Example 4. Determine the extreme values of $f(x)$ given that

$$f(0) = 1; \quad f'(0) = 0; \quad f''(x) > 0, \text{ for all values of } x.$$

Sketch the graph of a function with these properties.

Solution. The first condition implies that the function is passing the point (0,1). The second condition asserts that 0 is a critical point and the tangent at (0,1) is horizontal. The last condition implies that f is always concave up. In particular, $f''(0) > 0$ so the Second Derivative Test implies that there is a minimum at 0. Hence $f(0) = 1$ is the minimum value.



Example 5. Given that

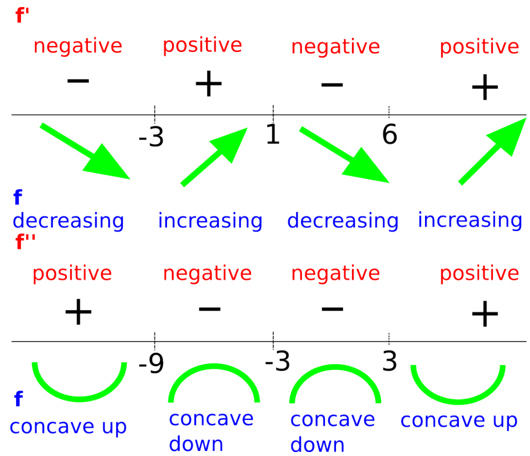
$$f'(x) = \frac{(x-6)(x-1)}{(x+3)}, \quad f''(x) = \frac{(x+9)(x-3)}{(x+3)^2}, \quad \text{and} \quad f(-3) \text{ not defined,}$$

for a function $f(x)$, determine the intervals on which f increases/decreases, the intervals on which f is concave up/down and the x-values at which the function has maximums, minimums, and inflections.

Solutions. The critical points are 1, 6 and -3. From the number line test below, f is increasing on $(6, \infty)$ and $(-3, 1)$ and decreasing on $(-\infty, -3)$ and $(1, 6)$.

At $x = 1$, there is a maximum and, at $x = 6$, there is a minimum. f is not defined at -3, so there is no extreme value at -3.

The critical points of f' are 3, -9 and -3. From the number line tests for f'', on the right, we conclude that f is concave up on $(3, \infty)$ and $(-\infty, -9)$ and concave down on $(-9, -3)$ and $(-3, 3)$. Thus, there are inflection points at $x = 3$ and $x = -9$ and there is no inflection point at $x = -3$.

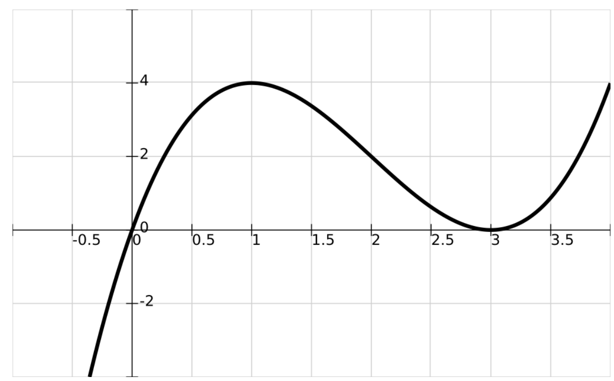


Example 6. Assume that the graph below is the graph of the *derivative* $f'(x)$ of a function $f(x)$.

(a) Determine the intervals on which f is increasing/decreasing and the intervals on which f is concave up/down.

(b) Determine the extreme values and the inflection points of f.

(c) Determine whether the function f is increasing or decreasing and whether it is concave up or down at $x = -0.2$



(d) Determine whether the function f is increasing or decreasing and whether it is concave up or down at $x = 2$.

Solutions. (a) From the graph, conclude that the critical points of f are 0 and 3. For increasing/decreasing intervals, look where f' is positive/negative. According to the graph, the number line test for f' is $\begin{array}{c|ccccc} f' & - & & + & & + \\ \hline f & \searrow & 0 & \nearrow & 3 & \nearrow \end{array}$. So, f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$.

For concavity, use the graph to conclude that the critical points of f' are 1 and 3. Recall that f is concave up exactly when $f'' > 0$ and that happens exactly when f' is increasing. Analogously, f is concave down exactly on the intervals on which f' is decreasing. So, use the graph to see where f' is increasing/decreasing. Conclude the following. $\begin{array}{c|ccccc} f' & \nearrow & & \searrow & & \nearrow \\ f'' & + & & - & & + \\ \hline f & \cup & 1 & \cap & 3 & \cup \end{array}$. Hence, f is concave up on $(-\infty, 1)$ and on $(3, \infty)$ and f is concave down on (1,3).

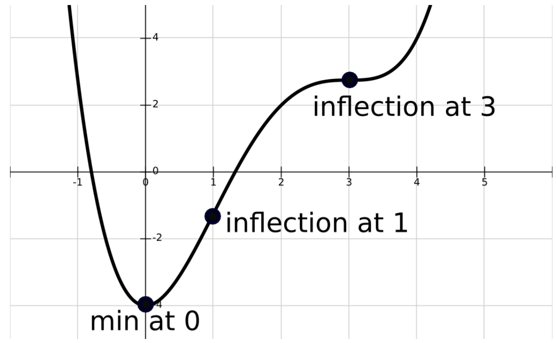
(b) Since f' changes sign just at 0, there is an extreme value just at $x = 0$. As f' changes from negative to positive at 0, so there is a minimum at 0.

There are two inflection points, at 1 and at 3.

(c) Using the first number line and the fact that -0.2 is smaller than 0, conclude that f' is negative at -0.2. So, f is decreasing at $x = -0.2$. Using the second number line and the fact that -0.2 is smaller than 1, conclude that f'' is positive at -0.2. So, f is concave up at $x = -0.2$.

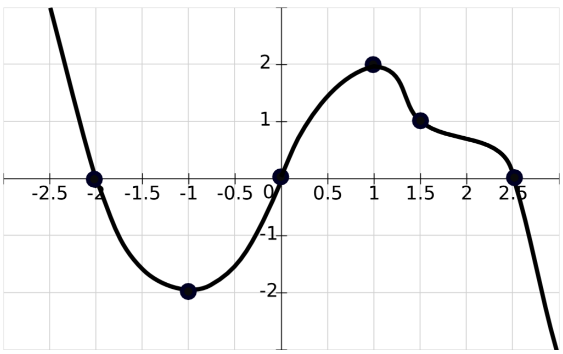
(d) Using the first number line and the fact that 2 is in the interval $(0, 3)$, conclude that f' is positive at $x = 2$. Thus, f is increasing at $x = 2$. Using the second number line and the fact that 2 is in the interval $(1, 3)$, conclude that f'' is negative at $x = 2$. So, f is concave down at $x = 2$.

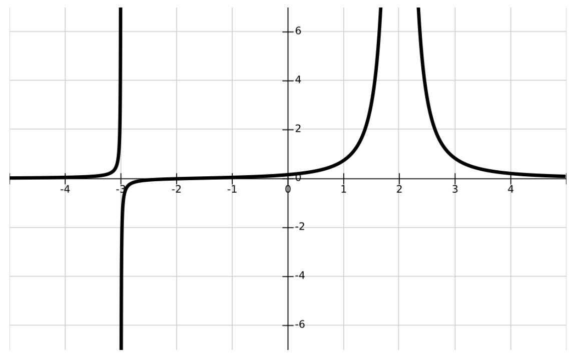
A graph of one possible function f with the given derivative f' is given on the right.

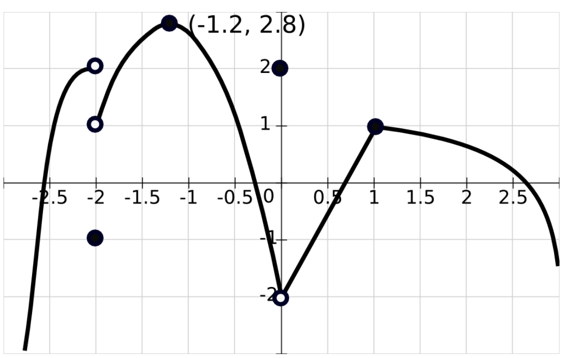


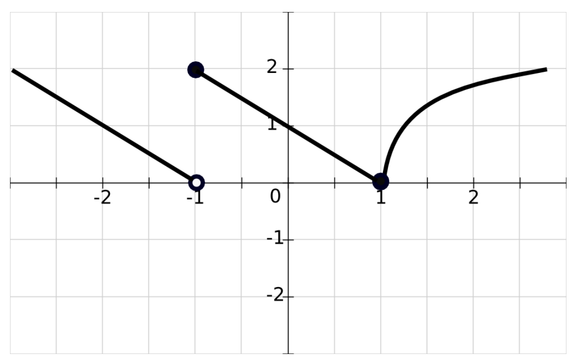
Practice Problems.

1. Find the intervals where the following functions given by their graphs are increasing/decreasing. Determine the critical points and the relative minimum and maximum values (if any).









2. Find the intervals where $f(x)$ is increasing/decreasing and the relative minimum and maximum values (if any).

   (a) $f(x) = \frac{1}{3}x^3 + x^2 - 15x + 3$
   (b) $f(x) = 6\sqrt[3]{(x-2)^2}$
   (c) $f(x) = \frac{2x}{x^2+4}$
   (d) $f(x) = e^x(x^2 - x - 5)$
   (e) $f(x) = 3\sqrt[3]{x^4} - 6\sqrt[3]{x}$
   (f) $f(x) = \ln(x + 2) - x$

3. A company determines that its revenue function is $R(x) = 15.22xe^{-.015x}$. Determine the production level which produces the maximal revenue and find that maximal revenue.

4. Find the intervals where $f(x)$ is increasing/decreasing and the intervals where $f(x)$ is concave up/down. Find the relative minimum and maximum values (if any) and the inflection points (if any).

(a) $f(x) = \frac{1}{3}x^3 + x^2 - 15x + 3$

(b) $f(x) = \frac{1}{x} + \frac{x}{16}$

(c) $f(x) = 3xe^{2x}$

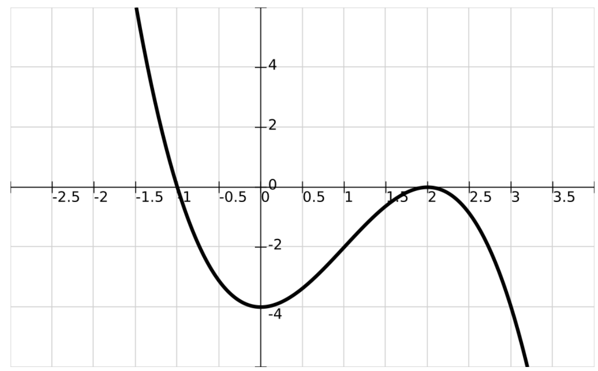
(d) $f(x) = \frac{\ln x}{x}$

5. Given that

$$f'(x) = \frac{(x-8)(x+1)}{(x+4)}, \quad f''(x) = \frac{(x+10)(x-2)}{(x+4)^2} \quad \text{and} \quad f(-4) \text{ not defined},$$

for a function $f(x)$, determine the intervals on which f increases/decreases, the intervals on which f is concave up/down and the x-values at which the function has a maximum, a minimum and an inflection.

6. Assume that the graph below is the graph of the *derivative* of a function $f(x)$.



(a) Determine the intervals on which f is increasing/decreasing and the intervals on which f is concave up/down.

(b) Determine the extreme values and the inflection points of f.

(c) Determine whether the function f is increasing or decreasing and whether it is concave up or down at $x = -1.5$

(d) Determine whether the function f is increasing or decreasing and whether it is concave up or down at $x = 1$.

Solutions.

1. (a) The function is increasing on $(-1, 1)$ and decreasing on $(-\infty, -1)$ and $(1, \infty)$. The critical points are $x = 1$ and $x = -1$ and the function has extreme values at ± 1. At 1, f has a maximum value $f(1) = 2$ and at -1, f has a minimum value $f(-1) = -2$.

(b) The function is increasing on $(-\infty, -3)$ and $(-3, 2)$ and decreasing on $(2, \infty)$. The critical points are $x = -3$ and $x = 2$ but since the function is not define at them, there are no extreme values.

(c) The function is increasing on $(-\infty, -2)$, $(-2, -1.2)$ and $(0, 1)$. The function is decreasing on $(-1.2, 0)$ and $(1, \infty)$. The critical points are $-2, -1.2, 0$ and 1. At -1.2 and 1, there are maximum values of $f(-1.2) = 2.8$ and $f(1) = 1$. At -2 and 0, there are no extreme values: at -2 the derivative does not change the sign and at 0 the function is not continuous because $\lim_{x\to 0} = -2$ and $f(0) = 2$. So the function never reaches the lowest value of -2 which it is approaching when $x \to 0$.

(d) The function is increasing on $(1, \infty)$ and decreasing on $(-\infty, -1)$ and $(-1, 1)$. The critical points are -1 and 1. At 1, there is a minimum values of $f(1) = 0$. At -1, there is no extreme value since the derivative does not change the sign.

2. (a) $f(x) = \frac{1}{3}x^3 + x^2 - 15x + 3 \Rightarrow f'(x) = x^2 + 2x - 15 = (x+5)(x-3)$. Setting the derivative to zero gives you the critical points $x = -5$ and $x = 3$. The number line test for f' produces $\begin{array}{c|ccccc} f' & + & & - & & + \\ \hline f & \nearrow & -5 & \searrow & 3 & \nearrow \end{array}$. Hence, f is increasing on $(-\infty, -5)$ and $(3, \infty)$, and decreasing on $(-5, 3)$. There is a relative minimum of -24 at $x = 3$ and a relative maximum of $\frac{184}{3} \approx 61.33$ at $x = -5$.

(b) $f(x) = 6\sqrt[3]{(x-2)^2} \Rightarrow f'(x) = 6\frac{2}{3}(x-2)^{-1/3} = \frac{4}{\sqrt[3]{x-2}}$. f' is not defined at 2 and it is never zero, so 2 is the only critical point. The number line test for f' produces $\begin{array}{c|ccc} f' & - & & + \\ \hline f & \searrow & 2 & \nearrow \end{array}$. Thus, f is increasing on $(2, \infty)$ and decreasing on $(-\infty, 2)$ and there is a minimum of $f(2) = 0$ at $x = 2$.

(c) $f(x) = \frac{2x}{x^2+4} \Rightarrow f'(x) = \frac{2(x^2+4)-2x(2x)}{(x^2+4)^2} = \frac{8-2x^2}{(x^2+4)^2} = \frac{2(2-x)(2+x)}{(x^2+4)^2}$. Since the denominator is never zero, the only critical points are 2 and -2. The number line test gives you $\begin{array}{c|ccccc} f' & - & & + & & - \\ \hline f & \searrow & -2 & \nearrow & 2 & \searrow \end{array}$ and so f is increasing on $(-2, 2)$ and decreasing on $(-\infty, 2)$ and $(2, \infty)$. There is a maximum of $\frac{1}{2}$ at $x = 2$ and a minimum of $\frac{-1}{2}$ at $x = -2$.

(d) $f(x) = e^x(x^2 - x - 5) \Rightarrow f'(x) = e^x(x^2 - x - 5) + e^x(2x - 1) = e^x\left(x^2 - x - 5 + 2x - 1\right) = e^x(x^2+x-6) = e^x(x+3)(x-2)$. Since e^x is never zero, 2 and -3 are the only critical points. From the number line test $\begin{array}{c|ccccc} f' & + & & - & & + \\ \hline f & \nearrow & -3 & \searrow & 2 & \nearrow \end{array}$, we have that f is increasing on $(-\infty, -3)$ and $(2, \infty)$ and decreasing on $(-3, 2)$. At $x = -3$ there is a maximum of $f(-3) = 7e^{-3} \approx .348$ and at $x = 2$ there is a minimum of $f(2) = -3e^2 \approx -22.17$.

(e) $f(x) = 3\sqrt[3]{x^4} - 6\sqrt[3]{x} = 3x^{4/3} - 6x^{1/3} \Rightarrow f'(x) = 3\cdot\frac{4}{3}x^{1/3} - 6\cdot\frac{1}{3}x^{-2/3} = 4x^{1/3} - 2x^{-2/3}$. Factor the term with the negative power and obtain $f'(x) = x^{-2/3}(4x-2) = \frac{4x-2}{x^{2/3}}$. As $4x - 2 = 0 \Rightarrow x = \frac{1}{2}$ and $x^{2/3} = 0 \Rightarrow x = 0$, there are two critical points, 0 and $\frac{1}{2}$. From the number line test $\begin{array}{c|ccccc} f' & - & & - & & + \\ \hline f & \searrow & 0 & \searrow & \frac{1}{2} & \nearrow \end{array}$, we have that f is increasing on $(\frac{1}{2}, \infty)$ and decreasing on $(-\infty, \frac{1}{2})$. At $x = \frac{1}{2}$, there is a minimum of $f(\frac{1}{2}) \approx -3.57$.

(f) $f(x) = \ln(x+2) - x \Rightarrow f'(x) = \frac{1}{x+2} - 1 = \frac{1-(x+2)}{x+2} = \frac{-x-1}{x+2}$. As $-x - 1 = 0 \Rightarrow x = -1$ and $x + 2 = 0 \Rightarrow x = -2$, there are two critical points, -1 and -2. Note that the function is not defined for $x < -2$ so the number line test on that interval is not needed. From the number line test $\begin{array}{c|ccccc} f' & \text{n/a} & & + & & - \\ \hline f & \text{n/a} & -2 & \nearrow & -1 & \searrow \end{array}$, we have that f is increasing on $(-2, -1)$ and decreasing on $(-1, \infty)$. At $x = -1$, there is a maximum of $f(-1) = \ln 1 - (-1) = 0 + 1 = 1$.

3. $R(x) = 15.22xe^{-.015x} \Rightarrow R'(x) = 15.22e^{-.015x} + 15.22xe^{-.015x}(-.015) = 15.22e^{-.015x}\left(1 - .015x\right)$. Since the term in front of parenthesis is never zero, the only critical point is $1 - .015x = 0 \Rightarrow x = \frac{200}{3} \approx 66.67$. Use either the First or the Second Derivative Test to show that there is a maximum at the critical point. For example, using the First Derivative Test, we have that $\begin{array}{c} f' \\ f \end{array}$

$\begin{array}{ccc} + & & - \\ \hline \nearrow & \frac{200}{3} & \searrow \end{array}$, hence there is a maximum at $\frac{200}{3}$. The maximum value is $R(66.67) \approx 373.27$. When interpreting this answer you can round the integer x-value assuming that the entire items are produced. So, the maximal revenue of \$373.27 is obtained when 67 items are sold.

4. (a) For increasing/decreasing intervals, see the solutions of problem 2(a). For concavity, find the second derivative. $f(x) = \frac{1}{3}x^3 + x^2 - 15x + 3 \Rightarrow f'(x) = x^2 + 2x - 15 \Rightarrow f''(x) = 2x + 2$. $f''(x) = 2x + 2 = 0 \Rightarrow 2x = -2 \Rightarrow x = -1$. The number line test for f'' produces $\begin{array}{c} f'' \\ f \end{array}$ $\begin{array}{ccc} - & & + \\ \hline \cap & -1 & \cup \end{array}$. Thus, f is concave up on $(-1, \infty)$ and f is concave down on $(-\infty, -1)$. At $x = -1$, f'' changes sign so $(-1, f(-1)) = (-1, \frac{56}{3})$ is an inflection point.

(b) $f(x) = \frac{1}{x} + \frac{x}{16} \Rightarrow f'(x) = -x^{-2} + \frac{1}{16} = \frac{-1}{x^2} + \frac{1}{16} = \frac{-16+x^2}{16x^2}$. The critical points are the solutions of $-16 + x^2 = 0 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$ and $16x^2 = 0 \Rightarrow x = 0$. The number line test gives you $\begin{array}{c} f' \\ f \end{array}$ $\begin{array}{ccccccc} - & & + & & + & & - \\ \hline \searrow & -4 & \nearrow & 0 & \nearrow & 4 & \searrow \end{array}$ and so f is increasing on $(-4, 4)$ and decreasing on $(-\infty, 4)$ and $(4, \infty)$. There is a maximum of $\frac{1}{2}$ at $x = 4$ and a minimum of $\frac{-1}{2}$ at $x = -4$.

For concavity, find f''. $f(x) = \frac{1}{x} + \frac{x}{16} \Rightarrow f'(x) = -x^{-2} + \frac{1}{16} \Rightarrow f''(x) = 2x^{-3} = \frac{2}{x^3}$. f'' changes the sign at $x = 0$. The number line test for f'' produces $\begin{array}{c} f'' \\ f \end{array}$ $\begin{array}{ccc} - & & + \\ \hline \cap & 0 & \cup \end{array}$. Thus, f is concave up on $(0, \infty)$ and f is concave down on $(-\infty, 0)$. At $x = 0$, f'' changes sign but f is not defined at $x = 0$ and so there is no inflection point at $x = 0$.

(c) $f(x) = 3xe^{2x} \Rightarrow f'(x) = 3e^{2x} + e^{2x}(2)(3x) = 3e^{2x}(1 + 2x)$. Since $3e^{2x} = 0 \Rightarrow e^{2x} = 0$ has no solutions as the exponential function is positive, the only critical point is when $1 + 2x = 0$ and so $x = \frac{-1}{2}$. The number line test for f' is $\begin{array}{c} f' \\ f \end{array}$ $\begin{array}{ccc} - & & + \\ \hline \searrow & \frac{-1}{2} & \nearrow \end{array}$ so f is increasing on $(\frac{-1}{2}, \infty)$ and decreasing on $(-\infty, \frac{-1}{2})$. There is a minimum at $\frac{-1}{2}$. $f(\frac{-1}{2}) = \frac{-3}{2e} \approx -0.552$ is the minimum value.

$f'(x) = 3e^{2x}(1+2x) \Rightarrow f''(x) = 3e^{2x}(2)(1+2x) + 2 \cdot 3e^{2x} = 6e^{2x}(1+2x+1) = 6e^{2x}(2x+2)$. Since $6e^{2x} = 0$ has no solutions, $f''(x) = 0$ when $2x + 2 = 0 \Rightarrow 2x = -2 \Rightarrow x = -1$. The number line test for f'' is $\begin{array}{c} f'' \\ f \end{array}$ $\begin{array}{ccc} - & & + \\ \hline \cup & -1 & \cap \end{array}$. Thus, f is concave up on $(-1, \infty)$ and f is concave down on $(-\infty, -1)$. At $x = -1$, f'' changes sign so $(-1, f(-1)) = (-1, \frac{-3}{e^2}) \approx (-1, -0.406)$ is the only inflection point.

(d) $f(x) = \frac{\ln x}{x} \Rightarrow f'(x) = \frac{\frac{1}{x}x - \ln x}{x^2} = \frac{1-\ln x}{x^2}$. Note that $\ln x$ is defined only for $x > 0$, so only positive values of x are relevant.

The critical points are solutions of $1 - \ln x = 0 \Rightarrow \ln x = 1 \Rightarrow x = e$ and of $x^2 = 0 \Rightarrow x = 0$. Since f is not defined at 0, only $x = e$ is relevant. The line test for f' is $\begin{array}{c} f' \\ f \end{array}$ $\begin{array}{ccc} + & & - \\ \hline \nearrow & e & \searrow \end{array}$ and so f is increasing on $(0, e)$, decreasing on (e, ∞), and it has a maximum of $f(e) = \frac{1}{e} \approx 0.37$ at $x = e$.

For concavity, find the second derivative. $f(x) = \frac{\ln x}{x} \Rightarrow f'(x) = \frac{\frac{1}{x}x - \ln x}{x^2} = \frac{1-\ln x}{x^2} \Rightarrow f''(x) =$

$\frac{-\frac{1}{x}x^2-2x(1-\ln x)}{x^4} = \frac{-x-2x+2x\ln x}{x^4} = \frac{x(-3+2\ln x)}{x^4} = \frac{-3+2\ln x}{x^3}$. The second derivative changes sign when $-3+2\ln x = 0$ and $x^3 = 0$. The first equation gives you $\ln x = \frac{3}{2} \Rightarrow x = e^{3/2}$ and the second gives you $x = 0$. As $\ln x$ is not defined at 0, only $e^{3/2}$ is relevant.

The number line test for f'' is $\begin{array}{c|ccc} f'' & - & & + \\ \hline f & \cap & e^{3/2} & \cup \end{array}$. Thus, f is concave up on $(e^{3/2}, \infty)$ and concave down for $(0, e^{3/2})$. Since f'' changes the sign at $e^{3/2}$, $(e^{3/2},\ f(e^{3/2})) \approx (4.48, 0.33)$ is an inflection point.

5. The critical points of f are $x = 8, x = -1$ and $x = -4$. Considering the number line for f', obtain $\begin{array}{c|ccccccc} f' & - & & + & & - & & + \\ \hline f & \searrow & -4 & \nearrow & -1 & \searrow & 8 & \nearrow \end{array}$. Conclude that f is increasing on $(8, \infty)$ and $(-4, -1)$, and that f is decreasing on $(-\infty, -4)$ and $(-1, 8)$. By the First Derivative Test, there is a maximum at $x = -1$ and a minimum at $x = 8$. The function is not defined at -4 so there is no extreme value at -4.

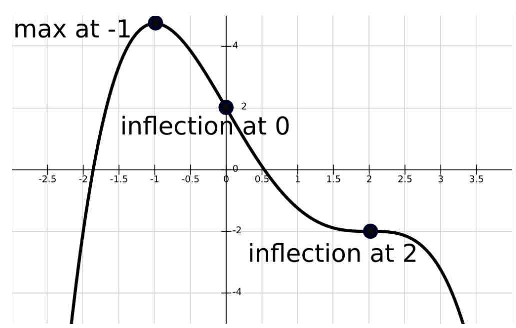
   The critical points of f' are $x = 2, x = -4$, and $x = -10$. Considering the number line for f'', obtain $\begin{array}{c|ccccccc} f'' & + & & - & & - & & + \\ \hline f & \cup & -10 & \cap & -4 & \cap & 2 & \cup \end{array}$ and conclude that f is concave up on $(2, \infty)$ and $(-\infty, -10)$ and that f is concave down on $(-10, -4)$ and $(-4, 2)$. Thus, there are inflection points at $x = 2$ and $x = -10$. As $f(-4)$ is not defined, there is no inflection point at $x = -4$.

6. (a) From the graph, deduce that the critical points are -1 and 2. For increasing/decreasing intervals, look where f' is positive/negative. The number line test for f' is $\begin{array}{c|ccccc} f' & + & & - & & - \\ \hline f & \nearrow & -1 & \searrow & 2 & \searrow \end{array}$. So, f is increasing on $(-\infty, -1)$ and decreasing on $(-1, 2)$ and $(2, \infty)$.

   For concavity, use the graph to conclude that the critical points of f' are 1 and 3. Recall that f is concave up exactly when $f'' > 0$ and that happens exactly when f' is increasing. Analogously, f is concave down exactly on the intervals on which f' is decreasing. So, use the graph to see where f' is increasing/decreasing. Conclude the following. $\begin{array}{c|ccccc} f' & \searrow & & \nearrow & & \searrow \\ f'' & - & & + & & - \\ \hline f & \cap & 0 & \cup & 2 & \cap \end{array}$.

   Hence, f is concave up on (0,2) and f is concave down on $(-\infty, 0)$ and on $(2, \infty)$.

   (b) Since f' changes sign just at -1, there is an extreme value just at -1. f' changes from positive to negative at -1, so there is a maximum at -1. There are two inflection points, at 0 and at 2. A graph of a possible function f with the given derivative f' is on the right.



   (c) Using the first number line and the fact that -1.5 is smaller than -1, deduce that f' is positive at -1.5. So, f is increasing at $x = -1.5$. Using the second number line and the fact that -1.5 is smaller than 0, deduce that f'' is negative at -1.5. So, f is concave down at -1.5.

   (d) Using the first number line and the fact that $x = 1$ is in the interval $(-1, 2)$, deduce that f' is negative at 1. So, f is decreasing at $x = 1$. Using the second number line and the fact that $x = 1$ is in the interval $(0, 2)$, deduce that f'' is positive at 1. So, f is concave up at $x = 1$.