General Calculus
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The Indefinite Integral. Substitution

The idea of employing an inverse function is central when solving any equation:

- $x+2=4$ requires application of the function $x-2$, the inverse of $x+2$, to the right hand side. Thus $x=4-2=2$.
- $x^2=4$ requires application of the function $\sqrt{x}$ (and $-\sqrt{x}$ for the left branch), the inverse of x^2, to the right hand side. Thus $x=\pm\sqrt{4}=\pm 2$.
- $2^x=4$ requires application of the function $\log_2 x$, the inverse of 2^x, to the right hand side. Thus $x=\log_2 4=2$.

The same argument can be employed when looking for an unknown *function* such that its derivative is a given function.

Given a function $f(x)$, find function y such that $y'=f(x)$.

Finding a solution of this general problem requires the use of the *inverse of derivation*, that is the *process that produces a function whose derivative is known.* This process is known as the **antidifferentiation** and the outcome of this process is known as an **antiderivative**. Thus,

A function $F(x)$ is an **antiderivative** of a function $f(x)$ if $F'(x)=f(x)$.

The antidifferentiation is relevant when the rate of change of a quantity can be measured and the quantity size itself needs to be determined from the rate. For example, if velocity is known and we need to determine the function computing the position, the antiderivation is needed.

In some cases, an antiderivative can be easily guessed.

Example 1. Find a function with derivative 4.

Solution. To find the solution, one needs to note that any line with slope 4 has derivative 4. Thus, any line of the form $y=4x+c$ is the solution of this problem.

The previous example indicates that if $F(x)$ is an antiderivative of $f(x)$ then so is $F(x)+c$. Indeed if $\frac{d}{dx}F(x)=f(x)$ then $\frac{d}{dx}(F(x)+c)=f(x)$ as well since the derivative of a constant is zero.

Conversely, we claim that any antiderivative of $f(x)$ has to have the form $F(x)+c$ for some constant c. Indeed, if $G(x)$ is is another antiderivative of $f(x)$, then $\frac{d}{dx}(G(x)-F(x))=f(x-f(x))=0$. Since only a constant function has slope of the tangent equal to 0 at every point, we have that $G(x)-F(x)=c$ for some constant c and so $G(x)=F(x)+c$.

The collection of *all* antiderivatives of a function $f(x)$ is called the **indefinite integral** and it is denoted by

$$\int f(x)\,dx$$

Thus, we have the following.

> If $F(x)' = f(x)$ then $\int f(x)dx = F(x) + c$.

The process of producing the indefinite integral for a given function $f(x)$ is known as **integration** and the function $f(x)$ is known as the **integrand.**

The definition above implies that to integrate a function, one needs to find any antiderivative of the function and adds a constant to it. The definition above also gives you that you can **cancel** the derivative and integral in the following way.

$$\int \left(\frac{d}{dx}F(x)\right)\,dx = F(x) + c \quad \text{and} \quad \frac{d}{dx}\left(\int f(x)\,dx\right) = f(x).$$

While guessing an antiderivative of a function may work for some simple functions, this method is not always effective. Because of that we develop several rules which are used to simplify the process of finding antiderivative.

The first rule we present follows the idea of Example 2. To find an antiderivative of x^n, we divide the function x^{n+1} with $n+1$. Then indeed derivative of $\frac{1}{n+1}x^{n+1}$ is $\frac{1}{n+1}(n+1)x^n = x^n$ as needed. Thus $\frac{1}{n+1}x^{n+1} + c$ is an antiderivative of x^n for any constant c.

> **The Power Rule.** $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$

This rule can be applied to any real value of n except $n = -1$. We shall return to the case $n = -1$ later.

The second two rules follow from the sum and constant multiple rules of differentiation. If $F(x)$ is an antiderivative of $f(x)$ and $G(x)$ an antiderivative of $g(x)$, then $(F(x)+G(x))' = F'(x)+G'(x) = f(x) + g(x)$ so that $F(x) + G(x)$ is an antiderivative of $f(x) + g(x)$. This gives us the following.

> **The Sum Rule.** $\int(f(x) + g(x))dx = \int f(x)\,dx + \int g(x)dx$

Using similar arguments, one shows that if k is a constant and $F(x)$ is an antiderivative of $f(x)$, then $kF(x)$ is an antiderivative of $kf(x)$ so that the following holds.

> **The Constant Multiple Rule.** $\int kf(x)dx = k\int f(x)\,dx$

Note that the sum and the constant multiple rule also imply that $\int(f(x) - g(x))dx = \int f(x)\,dx + \int -g(x)dx = \int f(x)\,dx + (-1)\int g(x)dx = \int f(x)\,dx - \int g(x)dx$.

These two rules enables us to find antiderivative of any polynomial function by **integrating term by term**. It is important to keep the following in mind.

1. Keep carrying *both* the integral sign $\int$ and the symbol dx until you use an integration rule (just the power rule for the time being) which eliminates $\int$ and dx *simultaneously.*

2. When $\int$ and dx are eliminated, *do not forget to add a constant* to your answer.

The next example illustrates integration of polynomials.

Example 2. Find the integral $\int(6x^2 - 4x + 5)dx$.

Solution. Using the sum and constant multiple rule, we have that

$$\int(6x^2 - 4x + 5)dx = \int 6x^2dx - \int 4xdx + \int 5dx = 6\int x^2dx - 4\int xdx + 5\int dx$$

The first two integrals can be evaluated using the power rule. The last integral can be evaluated either using the power rule and representing the integrand 1 as x^0 and using the power rule (producing $\frac{1}{0+1}x^{0+1} = 1x^1 = x$) or simply noting that x is an antiderivative of 1 since derivative of 1 is x. In either case, we obtain

$$6\int x^2dx - 4\int xdx + 5\int dx = 6\frac{1}{2+1}x^{2+1} - 4\frac{1}{1+1}x^{1+1} + 5x + c = 2x^3 - 2x^2 + 5x + c.$$

Keep in mind the following algebra rules can be handy when integrating non-polynomial power functions.

$\frac{1}{x^n} = x^{-n}$	$\sqrt[n]{x} = x^{1/n}$

Example 3. Find the integral $\int \left(\frac{\sqrt{x^3}}{3} + \frac{3}{\sqrt[3]{x}}\right) dx$.

Solution. Write the function $\frac{\sqrt{x^3}}{3} + \frac{3}{\sqrt[3]{x}}$ as $\frac{1}{3}x^{3/2} + 3x^{-1/3}$ so that the power rule applies to each term. Integrate each term and obtain

$$\frac{1}{3}\int x^{3/2}dx + 3\int x^{-1/3}dx = \frac{1}{3}\frac{1}{\frac{3}{2}+1}x^{3/2+1} + 3\frac{1}{\frac{-1}{3}+1}x^{-1/3+1} + c = \frac{2}{15}x^{5/2} - \frac{9}{2}x^{2/3} + c.$$

You can also express your answer as $\frac{2}{15}\sqrt{x^5} - \frac{9}{2}\sqrt[3]{x^2} + c$.

The next example illustrates that in some cases you need to simplify the function before you can apply the above rules.

Example 4. Find the integral $\int \frac{3x^4+7}{x^2}dx$.

Solution. Write the function as $\frac{3x^4}{x^2} + \frac{7}{x^2} = 3x^2 + 7x^{-2}$ so that the power rule applies to each term. Integrate each term and obtain

$$3\int x^2dx + 7\int x^{-2}dx = 3\frac{1}{2+1}x^{2+1} + 7\frac{1}{-2+1}x^{-2+1} + c = x^3 - 7x^{-1} + c = x^3 - \frac{7}{x}.$$

In some cases, we need to find the particular antiderivative satisfying particular condition instead of all the antiderivatives. The next example illustrates this.

Example 5. Find the function $f(x)$ with derivative $x^2 - \sqrt{x}$ such that $f(1) = 3$.

Solution. We can find $f(x)$ as the antiderivative of $x^2 - \sqrt{x}$ which passes (1,3). Thus, we can

1. Find $f(x)$ as the indefinite integral of the derivative $x^2 - \sqrt{x}$;

2. Then find the value of c such that $f(1) = 3$.

Step 1. gives us that $f(x) = \int (x^2 - \sqrt{x})dx = \frac{1}{2+1}x^{2+1} - \frac{1}{\frac{1}{2}+1}x^{1/2+1} + c = \frac{1}{3}x^3 - \frac{2}{3}x^{3/2} + c$. To do step 2, note that since $f(1) = 3$, $3 = \frac{1}{3}1^3 - \frac{2}{3}1^{3/2} + c = \frac{1}{3} - \frac{2}{3} + c \Rightarrow 3 = \frac{-1}{3} + c \Rightarrow 10 = 3c \Rightarrow c = \frac{10}{3}$. Thus

$$f(x) = \frac{1}{3}x^3 - \frac{2}{3}x^{3/2} + \frac{10}{3}.$$

Applications.

Finding an antiderivative applies to every situation when the rate is given and a function with the given rate is needed. For example, recall that the velocity is the derivative of the position function thus, the *position is an antiderivative of the velocity.* Similarly, since the acceleration is the derivative of velocity, the *velocity is an antiderivative of the acceleration.*

Example 6. An object is dropped from a cliff 300 meters high and it is falling with the velocity of $v(t) = -9.8t$ (the distance is measured from the ground so it is *decreasing* as time passes by which explains the negative sign of the velocity). Determine the time the object hits the ground and find the speed at the time of the impact.

Solution. If we consider $s(t)$ to be the height of the object, then $v(t) = s'(t)$. Hence,

$$v(t) = s'(t) = -9.8t \quad \Rightarrow \quad s(t) = \int -9.8tdt = -9.8\,\frac{1}{2}t^2 + c = -4.9t^2 + c$$

Since $s(0) = 300$, we have that $300 = -4.9(0) + c \Rightarrow c = 300$. Thus

$$s(t) = -4.9t^2 + 300.$$

The object hits the ground when $s(t) = 0 \Rightarrow 300 = 4.9t^2 \Rightarrow t^2 = \frac{300}{4.9} \Rightarrow t \approx 7.82$ seconds. The velocity at the time of the impact is $v(7.82) = -76.68$ m/s so the speed is 76.68 meters per second.

Practice Problems.

1. Evaluate the following integrals.

   (a) $\int (x^5 + 2)\, dx$ (b) $\int (4x^3 + 6x^2 - 4x + 3)\, dx$ (c) $\int (\sqrt{x} - \frac{4}{x^2})\, dx$

   (d) $\int \frac{1+x^3}{x^2}\, dx$ (e) $\int \frac{8\sqrt[3]{x}+x^2}{4}\, dx$ (f) $\int \frac{x+3}{\sqrt{x}}\, dx$

2. Find the function $f(x)$ which has the given derivative and satisfies the given condition.

   (a) $f'(x) = \frac{1}{\sqrt[3]{x}}$ and $f(8) = 9$ (b) $f'(x) = 5\sqrt{x^3} + 3$ and $f(1) = 4$

3. Suppose that the velocity of an object is given by the function $v(t) = \frac{t}{2}$ where t is the time in seconds and v is the velocity in feet per second. Knowing that when $t = 2$ seconds, the position function $s(t) = 5$ feet, determine the position function $s(t)$.

Solutions.

1. (a) $\int(x^5+2)\,dx = \int x^5 dx + \int 2dx = \frac{1}{6}x^6 + 2x + c.$

   (b) $\int(4x^3+6x^2-4x+3)\,dx = 4\frac{1}{4}x^4 + 6\frac{1}{3}x^3 - 4\frac{1}{2}x^2 + 3x + c = x^4 + 2x^3 - 2x^2 + 3x + c.$

   (c) $\int(x^{1/2} - 4x^{-2})\,dx = \frac{2}{3}x^{3/2} + \frac{4}{x} + c$

   (d) $\int \frac{1+x^3}{x^2}\,dx = \int(\frac{1}{x^2} + \frac{x^3}{x^2})\,dx = \int(x^{-2}+x)\,dx = -\frac{1}{x} + \frac{x^2}{2} + c$

   (e) $\int \frac{8\sqrt[3]{x}+x^2}{4}\,dx = \int(2x^{1/3} + \frac{1}{4}x^2)dx = 2\,\frac{3}{4}x^{4/3} + \frac{1}{4}\cdot\frac{1}{3}x^3 + c = \frac{3}{2}\sqrt[3]{x^4} + \frac{1}{12}x^3 + c$

   (f) $\int \frac{x+3}{\sqrt{x}}\,dx = \int(\frac{x}{\sqrt{x}} + \frac{3}{\sqrt{x}})dx = \int x^{1/2}\,dx + \int 3x^{-1/2}\,dx = \frac{1}{3/2}x^{3/2} + 3\frac{1}{1/2}x^{1/2} = \frac{2}{3}\sqrt{x^3} + 6\sqrt{x} + c$

2. (a) $f(x) = \int f'(x)\,dx = \int x^{-1/3}dx = \frac{3}{2}\sqrt[3]{x^2} + c$. Using $f(8) = 9$ to solve for c, you have that $9 = \frac{3}{2}\sqrt[3]{8^2} + c = \frac{3}{2}\cdot 4 + c = 6 + c \Rightarrow 9 = 6 + c \Rightarrow c = 3$. Thus, $f(x) = \frac{3}{2}\sqrt[3]{x^2} + 3$.

   (b) $f(x) = \int f'(x)\,dx = \int(5x^{3/2}+3)dx = 5\,\frac{2}{5}x^{5/2} + 3x + c = 2\sqrt{x^5} + 3x + c$. Using $f(1) = 4$ to solve for c, you have that $4 = 2\sqrt{1^5} + 3(1) + c = 2 + 3 + c = 5 + c \Rightarrow 4 = 5 + c \Rightarrow c = -1$. Thus, $f(x) = 2\sqrt{x^5} + 3x - 1$.

3. Recall that $s(t) = \int v(t)dt$. Thus $s(t) = \frac{1}{2}\,\frac{t^2}{2} + c = \frac{t^2}{4} + c$. Using that $s(2) = 5$, we have that $5 = \frac{2^2}{4} + c = 1 + c \Rightarrow c = 4$. Thus $s(t) = \frac{t^2}{4} + 4$.

The Substitution

Many integrals cannot be evaluated using the rules we covered so far (and many others cannot be evaluated even by methods covered in higher calculus courses). Still there is a class of integrals which can be evaluated using a method known as the **substitution**. Namely, consider the case when the integrand is a constant multiple of a function of the form

$$f(g(x))\;g'(x)$$

and an antiderivative of $f(x)$ can be found to be $F(x)$. In this case $F'(x) = f(x)$ and the Chain Rule applied to the function $F(g(x))$ produces the function $F'(g(x))g'(x) = f(g(x))g'(x)$. This implies that $F(g(x)))$ is an antiderivative of $f(g(x))g'(x)$.

$$\frac{d}{dx}(F(g(x))) = f(g(x))\;g'(x) \quad \Rightarrow \quad \int f\,(g(x))\;\;g'(x)\;dx = F(g(x)) + c.$$

The last formula is known as the Substitution Rule. You can think of the function $g(x)$ as the inner function and consider the Substitution Rule to be applicable if the integrand is of the form

constant multiple of	$f(g(x))$	$\cdot$	$g'(x)$
	the composite	$\cdot$	the derivative of the inner

To check if the integrand is of this form and perform substitution, use the following steps.

1. Start by analyzing the integrand, determining whether you can evaluate the integral directly, without the substitution, or, if the substitution is appropriate, follow the steps below.

2. **Identify the inner function** $u = g(x)$ (usually the term under a radical, term in parenthesis, denominator, exponent...).

3. **Find the differential** $du = g'(x)dx$. and solve for $dx = \frac{du}{g'(x)}$.

4. **Substitute** the inner function $g(x)$ with u and dx with dx from previous step.

5. The substitution is successful if there are **<u>no terms with x left</u>** in the integrand. You may need to simplify the integrand and use the relation $u = g(x)$ again.

6. If you obtain a simpler integrand than the initial one and can evaluate it using the rules of integration, the substitution method worked.

7. Then **integrate** the integrand.

8. Finally, **substitute back** using that $u = g(x)$ so that your final answer is in terms of x again.

We illustrate this method in the next examples.

Example 1. Use substitution to evaluate the integral

$$\int \frac{3}{(4x+1)^2}\, dx$$

Solution. Follow the steps of the substitution method.

1. The power rule cannot be applied directly since the integrand $3(4x+1)^{-2}$ is a composite of a power function $3u^{-2}$ and a linear function $4x+1$. Try the substitution.

2. Identify the inner function $u = 4x + 1$.

3. Find the differential $du = 4dx$ and solve for $dx = \frac{du}{4}$.

4. Substitute $4x + 1$ with u and dx with $\frac{du}{4}$. Obtain the following

$$\int 3(4x+1)^{-2}\, dx = \int 3(u)^{-2}\, \frac{du}{4}$$

5. The substitution is successful since there are no terms with x left.

6. Factor the constants out to simplify the integration. The integrand u^{-2} is now simple and you can integrate it using the Power Rule.

7. Then integrate the integrand.

$$\int 3u^{-2}\, \frac{du}{4} = \frac{3}{4}\int u^{-2}\, du = \frac{3}{4}\, \frac{1}{-2+1}\, u^{-2+1} + c = \frac{-3}{4}\, u^{-1} + c = \frac{-3}{4u} + c.$$

8. Finally, substitute back using that $u = 4x + 1$ so that your final answer is in terms of x again.

$$\frac{-3}{4u} + c = \frac{-3}{4(4x+1)} + c.$$

Example 2. Evaluate the following integral.

$$\int \frac{x}{\sqrt[3]{x^2+3}}\, dx$$

Solution. Note that the integral cannot be evaluated directly since the integrand is complex function.

1. Try the substitution.
2. Identify the inner function $u = x^2 + 3$.
3. Find the differential $du = 2xdx$ and solve for $dx = \frac{du}{2x}$.
4. Substitute $x^3 + 3$ with u and dx with $\frac{du}{2x}$. Obtain the following

$$\int x(x^2+3)^{-1/3}\, dx = \int x(u)^{-1/3}\, \frac{du}{2x}$$

5. The substitution is successful since the x terms cancel and we are left with the x-free integrand.

$$\int (u)^{-1/3}\, \frac{du}{2}$$

6. Factor the constant $\frac{1}{2}$ out to simplify the integration. The integrand $u^{-1/3}$ is now simple and you can integrate it using the Power Rule.
7. Then integrate.

$$\int u^{-1/3}\, \frac{du}{2} = \frac{1}{2}\int u^{-1/3}\, du = \frac{1}{2}\, \frac{1}{-\frac{1}{3}+1}\, u^{-1/3+1} + c = \frac{3}{4}\, u^{2/3} + c = \frac{3}{4}\sqrt[3]{u^2} + c.$$

8. Finally, substitute back using that $u = x^2 + 3$ so that your final answer is in terms of x again.

$$\frac{3}{4}\sqrt[3]{u^2} + c = \frac{3}{4}\sqrt[3]{(x^2+3)^2} + c.$$

Practice Problems.

1. Determine if the substitution method is appropriate for evaluating the following integrals and, if it is, evaluate them using substitution.

   (a) $\int (3x+5)^6\ dx$ (b) $\int (2x+1)^3\, dx$ (c) $\int \frac{x}{(x^2+3)^2}\, dx$

   (d) $\int \frac{(x^2+1)^2}{x^2}\, dx$ (e) $\int \frac{x^2}{\sqrt{x^3-5}}\, dx$ (f) $\int \frac{6}{\sqrt[3]{3x+5}}\, dx$

2. Find the function $f(x)$ which has the given derivative and satisfies the given condition.

(a) $f'(x) = \sqrt{2x+9}$ and $f(0) = 5$ (b) $f'(x) = \frac{10}{\sqrt{4x+1}}$ and $f(0) = 3$

3. Suppose that the velocity of an object is given by the function

$$v(t) = \frac{t}{\sqrt{t^2+9}}$$

where t is the time in seconds and v is the velocity in feet per second. Knowing that when $t = 4$ seconds, the position function $s(t) = 8$ feet, determine the position function $s(t)$.

Solutions.

1. (a) Use substitution $u = 3x+5$. Then $du = 3dx$ so $dx = \frac{du}{3}$. $\int (3x+5)^6\, dx = \int u^6 \frac{du}{3} = \frac{1}{3}\int u^6 du = \frac{1}{3}\frac{u^7}{7} + c = \frac{u^7}{21} + c = \frac{(3x+5)^7}{21} + c$.
(b) Use substitution $u = 2x+1$. Then $du = 2dx$ so $dx = \frac{du}{2}$. $\int (2x+1)^3\, dx = \int u^3 \frac{du}{2} = \frac{1}{2}\int u^3 du = \frac{1}{2}\frac{u^4}{4} + c = \frac{u^4}{8} + c = \frac{(2x+1)^4}{8} + c$.
(c) Use substitution $u = x^2+3$. Then $du = 2xdx$ so $dx = \frac{du}{2x}$. $\int \frac{x}{(x^2+3)^2}\, dx = \int \frac{x}{u^2}\frac{du}{2x} = \frac{1}{2}\int u^{-2} du = \frac{1}{2}\frac{1}{-1}u^{-1} + c = \frac{-1}{2u} + c = \frac{-1}{2(x^2+3)} + c$.
(d) Note that substitution $u = x^2+1$ does not work since there is a remaining term x in denominator after substituting. The integral can be evaluated without substitution since the integrand simplifies as $\frac{(x^2+1)^2}{x^2} = \frac{x^4+2x^2+1}{x^2} = x^2+2+x^{-2}$ so that the integral becomes $\int (x^2 + 2 + x^{-2})dx = \frac{1}{3}x^3 + 2x + \frac{1}{-1}x^{-1} + c = \frac{x^3}{3} + 2x - \frac{1}{x} + c$.
(e) Use substitution $u = x^3-5$. Then $du = 3x^2dx$ so $dx = \frac{du}{3x^2}$. $\int \frac{x^2}{\sqrt{x^3-5}}\, dx = \int \frac{x^2}{\sqrt{u}}\frac{du}{3x^2} = \frac{1}{3}\int u^{-1/2} du = \frac{1}{3}\frac{u^{1/2}}{1/2} + c = \frac{2\sqrt{u}}{3} + c = \frac{2\sqrt{x^3-5}}{3} + c$ or $\frac{2}{3}\sqrt{x^3-5} + c$.
(f) Use substitution $u = 3x+5$. Then $du = 3dx$ so $dx = \frac{du}{3}$. $\int \frac{6}{\sqrt[3]{3x+5}}\, dx = 6\int \frac{1}{u^{1/3}}\frac{du}{3} = \frac{6}{3}\int u^{-1/3} du = 2\frac{u^{2/3}}{2/3} + c = 3u^{2/3} + c = 3(3x+5)^{2/3} + c$ or $3\sqrt[3]{(3x+5)^2} + c$.

2. (a) $f(x) = \int f'(x)\, dx = \int \sqrt{2x+9}\, dx$. Use substitution $u = 2x+9$. Then $du = 2dx$ so $dx = \frac{du}{2}$. $\int \sqrt{2x+9}\, dx = \int u^{1/2}\frac{du}{2} = \frac{1}{2}\frac{u^{3/2}}{3/2} + c = \frac{1}{3}u^{3/2} + c = \frac{1}{3}(2x+9)^{3/2} + c$. Using $f(0) = 5$ to solve for c, you have that $5 = \frac{1}{3}\sqrt{9^3} + c = \frac{27}{3} + c \Rightarrow 5 = 9 + c \Rightarrow c = -4$. Thus, $f(x) = \frac{1}{3}(2x+9)^{3/2} - 4$.
(b) $f(x) = \int f'(x)\, dx = \int \frac{10}{\sqrt{4x+1}}\, dx$. Use substitution $u = 4x+1$. Then $du = 4dx$ so $dx = \frac{du}{4}$. $\int \frac{10}{\sqrt{4x+1}}\, dx = 10\int u^{-1/2}\frac{du}{4} = \frac{10}{4}\frac{u^{1/2}}{1/2} + c = 5u^{1/2} + c = 5\sqrt{4x+1} + c$. Using $f(0) = 3$ to solve for c, you have that $3 = 5\sqrt{0+1} + c = 5 + c \Rightarrow 3 = 5 + c \Rightarrow c = -2$. Thus, $f(x) = 5\sqrt{4x+1} - 2$.

3. $s(t) = \int v(t)\, dt = \int \frac{t}{\sqrt{t^2+9}}\, dt$. Use substitution $u = t^2+9$. Then $du = 2tdt$ so $dt = \frac{du}{2t}$. $\int \frac{t}{\sqrt{t^2+9}}\, dt = \int \frac{t}{u^{1/2}}\frac{du}{2t} = \frac{1}{2}\int u^{-1/2} du = \frac{1}{2}\frac{u^{1/2}}{1/2} + c = u^{1/2} + c = \sqrt{t^2+9} + c$. Using $s(4) = 8$ to solve for c, you have that $8 = \sqrt{4+9} + c = 5 + c \Rightarrow 8 = 5 + c \Rightarrow c = 3$. Thus, $s(t) = \sqrt{t^2+9} + 3$.

Integrals of Exponential and Trigonometric Functions. Integrals Producing Logarithmic Functions.

Integrals of exponential functions. Since the derivative of e^x is e^x, e^x is an antiderivative of e^x. Thus

$$\int e^x\,dx = e^x + c$$

Recall that the exponential function with base a^x can be represented with the base e as $e^{\ln a^x} = e^{x\ln a}$. With substitution $u = x\ln a$ and using the above formula for the integral of e^x, we have that

$$\int a^x dx = \int e^{x\ln a} dx = \int e^u \frac{du}{\ln a} = \frac{1}{\ln a}\int e^u du = \frac{1}{\ln a}e^u + c = \frac{1}{\ln a}e^{x\ln a} + c = \frac{1}{\ln a}a^x + c.$$

Integrals producing logarithmic functions. Recall that the Power Rule formula for integral of x^n is valid just for $n \neq 1$ because of zero in denominator of $\frac{1}{n+1}x^{n+1}$ when $n = -1$. Thus, this rule does not apply to the integral $\int \frac{1}{x}dx$. However, this integral can be evaluated using the fact that derivative of $\ln x$ is $\frac{1}{x}$. Since $\ln x$ is defined just for $x > 0$, we have that $\ln x$ is an antiderivative of $\frac{1}{x}$ for $x > 0$.

If x is negative, the derivative of $\ln(-x)$ is $\frac{1}{-x}(-1) = \frac{1}{x}$ so that we can conclude that $\ln|x|$ is an antiderivation of $\frac{1}{x}$ both for $x > 0$ and $x < 0$. Thus,

$$\int \frac{1}{x}\,dx = \ln|x| + c.$$

Be careful about the following.

1. The formula $\int \frac{1}{x}\,dx = \ln|x| + c$ does not imply that $\int \frac{1}{x^2}\,dx = \ln|x^2| + c$. Use the power rule for $\int x^{-2}dx$ to get the answer $\frac{-1}{x} + c$.

2. The fact that $\int \frac{1}{x^2}\,dx = \frac{1}{-2+1}x^{-2+1} + c$ does not imply that $\int \frac{1}{x}\,dx = \frac{1}{-1+1}x^{-1+1} + c$. Use the formula $\int \frac{1}{x}\,dx = \ln|x| + c$ for the integrand $\frac{1}{x}$.

Integrals producing trigonometric functions.

Since the derivative of $\sin x$ is $\cos x$, $\sin x$ is an antiderivative of $\cos x$. Also, since the derivative of $\cos x$ is $-\sin x$, $\cos x$ is an antiderivative of $-\sin x$ so that $-\cos x$ is an antiderivative of $\sin x$.

$$\int \sin x\,dx = -\cos x + c \quad \text{and} \quad \int \cos x\,dx = \sin x + c$$

To integrate other trigonometric functions, you can convert them to sine and cosine functions and use the formulas above.

We summarize the formulas for integration of functions in the table below and illustrate their use in examples below.

y	x^n	e^x	a^x	$\frac{1}{x}$	$\sin x$	$\cos x$
$\int y\,dx$	$\frac{1}{n+1}x^{n+1}$	e^x	$\frac{1}{\ln a}\,a^x$	$\ln|x|$	$-\cos x$	$\sin x$

Example 1. Find the integral $\int xe^{x^2+1}\,dx$.

Solution. Identify the inner function $u = x^2 + 1$. Find the differential $du = 2xdx$ and solve for $dx = \frac{du}{2x}$.

Substitute x^2+1 with u and dx with $\frac{du}{2x}$. Obtain the following $\int xe^u \, \frac{du}{2x}$. Cancel x and factor $\frac{1}{2}$ out of the integral. The integrand e^u is now simple and you can integrate it using the formula for integral of e^x. Obtain

$$\frac{1}{2}\int e^u \, du = \frac{1}{2}\, e^u + c = \frac{1}{2}\, e^{x^2+1} + c.$$

Example 2. Find the integral $\int 2^{3x+1}\, dx$.

Solution. Use the substitution $u = 3x+1 \Rightarrow du = 3dx \Rightarrow \frac{du}{3} = dx$. The integral becomes $\int 2^u \frac{du}{3} = \frac{1}{3}\int 2^u \, du$. The integrand 2^u is now simple and you can integrate it using the formula for integral of a^x with $a=2$. Obtain

$$\frac{1}{3}\int 2^u \, du = \frac{1}{3}\,\frac{1}{\ln 2} 2^u + c = \frac{1}{3\ln 2} 2^{3x+1} + c.$$

Example 3. Find the integral $\int \frac{x^2+4}{x} dx$.

Solution. Simplify the integral as $\int \frac{x^2+4}{x} dx = \int \frac{x^2}{x} + \frac{4}{x} dx = \int (x + \frac{4}{x}) dx$. You can integrate term by term and factor 4 in front of the second integral. Evaluate the second integral using the formula that produces $\ln|x|$.

$$\int (x+\frac{4}{x})dx = \int xdx + 4\int \frac{1}{x}dx = \frac{x^2}{2} + 4\ln|x| + c.$$

Example 4. Find the integral $\int (9 + 2\sin\frac{\pi t}{5})\, dt$.

Solution. Use the substitution $u = \frac{\pi t}{5} \Rightarrow du = \frac{\pi}{5} dt \Rightarrow \frac{5du}{\pi} = dt$. The integral becomes

$$\int (9+2\sin u)\frac{5du}{\pi} = \frac{5}{\pi}\int (9+2\sin u) du = \frac{5}{\pi}(9u - 2\cos u) + c = \frac{5}{\pi}(9\frac{\pi t}{5} - 2\cos\frac{\pi t}{5}) + c = 9t - \frac{10}{\pi}\cos\frac{\pi t}{5} + c.$$

Alternatively, separate the integral into a sum of two as $\int 9dt + 2\int \sin\frac{\pi t}{5})\, dt$ and use the substitution $u = \frac{\pi t}{5}$ just for the second part. Obtain the same answer as above.

Example 5. Find the integral $\int \tan x dx$.

Solution. Recall that $\tan x = \frac{\sin x}{\cos x}$. The denominator $\cos x$ has derivative $-\sin x$ which is (up to a constant multiple) in the numerator. This points out to using the substitution $u = \cos x$. Then $du = -\sin x dx \Rightarrow dx = \frac{du}{-\sin x}$ and the integral reduces to

$$\int \frac{\sin x}{\cos x} dx = \int \frac{\sin x}{u}\frac{du}{-\sin x} = -\int \frac{1}{u}\, du = -\ln|u| + c = \ln|\cos x| + c.$$

Practice Problems.

1. Evaluate the following integrals. In problems (d) and (k) a and b are arbitrary constants.

   (a) $\int e^{2x}\, dx$ (b) $\int 5^{4x+7}\, dx$ (c) $\int x\, 3^{2x^2+1} dx$

(d) $\int bx\, e^{ax^2+1}\, dx$ (e) $\int (e^{2x}+e^{-2x})\, dx$ (f) $\int \frac{e^x+1}{e^x}\, dx$

(g) $\int \frac{e^x}{e^x+1}\, dx$ (h) $\int \frac{e^{2x}}{e^x+1}\, dx$ (i) $\int \frac{1}{3x+5}\, dx$

(j) $\int \frac{x-1}{x^2}\, dx$ (k) $\int \frac{ax^2}{bx^3+1}\, dx$ (l) $\int \cos(3x+1)\, dx$

(m) $\int x \sin x^2\, dx$ (n) $\int \sin^3 x \cos x\, dx$ (o) $\int \frac{\cos x}{\sin x+3} dx$

2. An oscillator's velocity depends on time t in seconds as $v(t) = \sin \omega t$ m/s where $\omega = 2$ (in 1/second). If the oscillator is at a distance of 1 m from the equilibrium position when $t = 0$, determine the position as a function of the time t.

Solutions.

1. (a) Use the substitution $u = 2x \Rightarrow du = 2dx \Rightarrow \frac{du}{2} = dx$. The integral becomes $\int e^u \frac{du}{2} = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + c = \frac{1}{2}e^{2x} + c$.

(b) Use the substitution $u = 4x+7 \Rightarrow du = 4dx \Rightarrow \frac{du}{4} = dx$. The integral becomes $\int 5^u \frac{du}{4} = \frac{1}{4}\int 5^u du = \frac{1}{4}\frac{1}{\ln 5}5^u + c = \frac{1}{4\ln 5}5^{4x+7} + c$.

(c) Use the substitution $u = 2x^2+1 \Rightarrow du = 4xdx \Rightarrow \frac{du}{4x} = dx$. The integral becomes $\int x3^u \frac{du}{4x} = \frac{1}{4}\int 3^u du = \frac{1}{4\ln 3}3^u + c = \frac{1}{4\ln 3}3^{2x^2+1} + c$.

(d) Use the substitution $u = ax^2+1 \Rightarrow du = 2axdx \Rightarrow \frac{du}{2ax} = dx$. The integral becomes $b\int xe^u \frac{du}{2ax} = \frac{b}{2a}\int e^u du = \frac{b}{2a}e^u + c = \frac{b}{2a}e^{ax^2+1} + c$.

(e) Separate into two integrals $\int e^{2x}dx + \int e^{-2x}dx$ and use the substitution $u = 2x$ for the first and the substitution $v = -2x$ for the second. Obtain $\int e^u \frac{du}{2} + \int e^v \frac{dv}{-2} = \frac{e^u}{2} + \frac{e^v}{-2} = \frac{1}{2}(e^{2x} - e^{-2x}) + c$.

(f) Simplify the function as $\int \left(\frac{e^x}{e^x} + \frac{1}{e^x}\right) dx = \int (1+e^{-x})dx$. You can use substitution $u = -x$ for the second term. Obtain $x - e^{-x} + c$ or $x - \frac{1}{e^x} + c$.

(g) Consider the denominator as the inner function and use the substitution $u = e^x+1 \Rightarrow du = e^x dx \Rightarrow dx = \frac{du}{e^x}$. The integral becomes $\int \frac{e^x}{e^x+1}\, dx = \int \frac{e^x}{u}\frac{du}{e^x} = \int \frac{1}{u}du = \ln|u| + c = \ln|e^x+1| + c$.

(h) Similarly as in the previous problem, consider the denominator as the inner function and use the substitution $u = e^x+1 \Rightarrow du = e^x dx \Rightarrow dx = \frac{du}{e^x}$. The integral becomes $\int \frac{e^{2x}}{e^x+1}\, dx = \int \frac{e^{2x}}{u}\frac{du}{e^x} = \int \frac{e^x}{u}du$. Use the substitution relation $u = e^x+1$ to solve for e^x and express it in terms of u as $e^x = u-1$. Thus the integral becomes $\int \frac{u-1}{u}du = \int (1-\frac{1}{u})du$. Integrate term by term to get $u - \ln|u| + c = e^x + 1 - \ln|e^x+1| + c$.

(i) Use the substitution $u = 3x+5 \Rightarrow du = 3dx \Rightarrow \frac{du}{3} = dx$. The integral becomes $\int \frac{1}{u}\frac{du}{3} = \frac{1}{3}\ln|u| + c = \frac{1}{3}\ln|3x+5| + c$.

(j) Simplify the integral as follows and integrate term by term. $\int \frac{x-1}{x^2}\, dx = \int (\frac{x}{x^2} - \frac{1}{x^2})dx = \int (\frac{1}{x} - x^{-2})dx = \ln|x| - \frac{1}{-1}x^{-1} + c = \ln|x| + \frac{1}{x} + c.$

(k) Use the substitution $u = bx^3 + 1 \Rightarrow du = 3bx^2 dx \Rightarrow \frac{du}{3bx^2} = dx$. The integral becomes $\int \frac{ax^2}{u} \frac{du}{3bx^2} = \frac{a}{3b} \int \frac{1}{u} du = \frac{a}{3b} \ln|u| + c = \frac{a}{3b} \ln|bx^3 + 1| + c.$

(l) Use the substitution $u = 3x + 1$. Obtain $\frac{1}{3}\sin(3x + 1) + c$.

(m) Use the substitution $u = x^2$. Obtain $\frac{-1}{2}\cos x^2 + c$.

(n) Note that the integrand can be written as $(\sin x)^3 \cos x$ and $\cos x$ is derivative of the inner $\sin x$. This points out to using the substitution $u = \sin x$. Then $du = \cos x dx \Rightarrow dx = \frac{du}{\cos x}$ and the integral reduces to $\int u^3 \cos x \frac{du}{\cos x} = \int u^3\, du = \frac{u^4}{4} + c = \frac{\sin^4 x}{4} + c.$

(o) The denominator $\sin x + 3$ has derivative $\cos x$ which is in the numerator so this points out to using the substitution $u = \sin x + 3$. Then $du = \cos x dx \Rightarrow dx = \frac{du}{\cos x}$ and the integral reduces to $\int \frac{\cos x}{u} \frac{du}{\cos x} = \int \frac{1}{u}\, du = \ln|u| + c = \ln|\sin x + 3| + c.$

2. $s(t) = \int v(t)dt = \int \sin 2x\, dx$. Using the substitution $u = 2x$ obtain $\frac{1}{2}\int \sin u du = \frac{-1}{2}\cos u + c = \frac{-1}{2}\cos(2x) + c$. Determine c using that $s(0) = 1 \Rightarrow 1 = \frac{-1}{2}\cos(0) + c \Rightarrow c = 1 + \frac{1}{2} = \frac{3}{2}$. Thus $s(t) = \frac{-1}{2}\cos(2x) + \frac{3}{2}$.