Fundamentals of Calculus
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The Limit

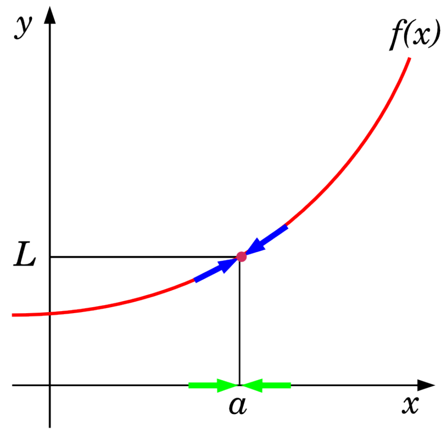
The **limit of a function** $f(x)$ **at a point** $x = a$ describes the **behavior of the function** $f(x)$ **near the point** $x = a$.

If the values of $f(x)$ accumulate near point $y = L$ when values of x approach a both from the left and the right side, we denote this fact by writing

$$\boxed{\lim_{x\to a} f(x) = L}$$

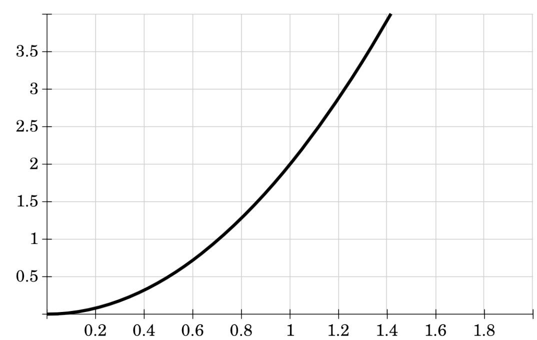
Sometimes we also denote this fact by writing

$$\boxed{f(x) \to L \text{ when } x \to a}$$



When x is near a, $f(x)$ is near L.

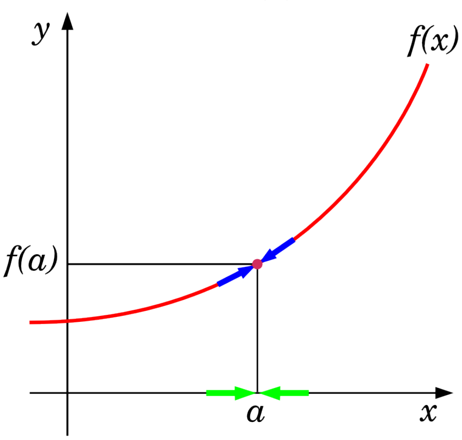
Example 1. If $f(x)$ has the following graph we can see that the y-values accumulate around 2 when the x values accumulate around 1. Thus we have that $\lim_{x\to a} f(x) = 2$.



When x is near 1, $f(x)$ is near 2.

The best case scenario: continuous functions. A function $f(x)$ is continuous at $x = a$ if its graph has no holes, jumps or breaks at and around $x = a$. For such functions, the limit at $x = a$ is equal to the value $f(a)$. Thus, to find the limit of a continuous function at a point, all you have to do is "plug and chug".

$$\boxed{\text{For } f \text{ continuous, } \lim_{x\to a} f(x) = f(a)}$$

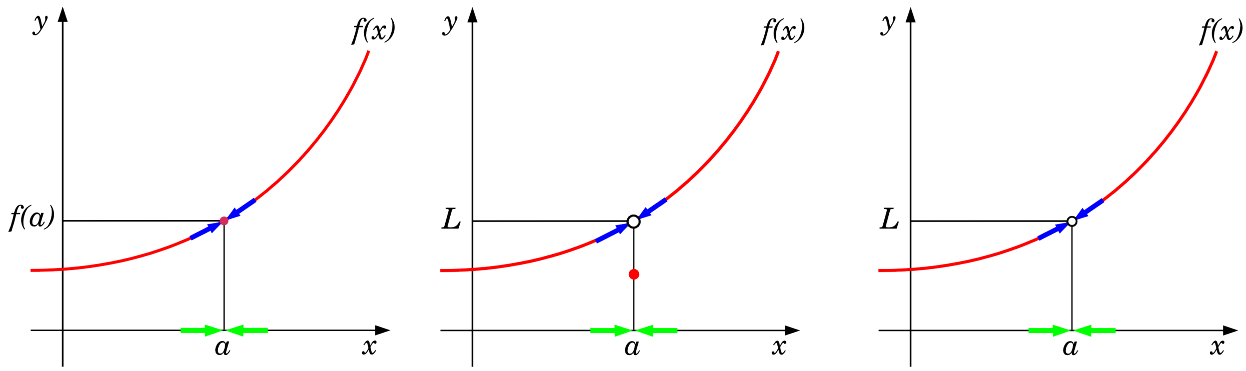


When f is continuous and x is near a, $f(x)$ is near $f(a)$.

Example 2. To find the limit $\lim_{x\to 1} 2x + 5$ of $f(x) = 2x + 5$ at $x = 1$, compute the value $f(1)$ to be $2(1)+5 = 7$. Thus $\lim_{x\to 1} 2x+5 = 7$. Graph the function to note that the y-values are close to 7 when x-values are close to 1.

The fact that $\lim_{x\to a} f(x) = L$ reflects that when x is near a $f(x)$ is near L *regardless what happens when x is exactly equal to a*. Thus the limit of all three functions below is L when x approaches a.

In both the second and the third case, when x-values are near a, y-values are near to L. Thus $\lim_{x\to a} f(x) = L$ regardless of the fact that $f(a) \neq L$.

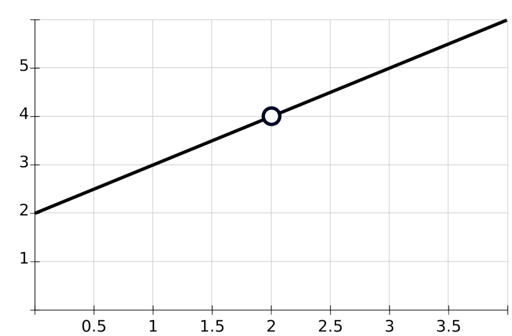


$\lim_{x\to a} f(x) = L$ in all three cases

Example 3. Let us consider the limit of $f(x) = \frac{x^2-4}{x-2}$ when $x \to 2$. Note that plugging 2 for x produces the indeterminate expression $\frac{0}{0}$ so that alone does not determine the limit. Note also that the numerator can be written as $x^2 - 4 = (x-2)(x+2)$ and so $f(x) = \frac{(x-2)(x+2)}{x-2}$ and the expression $x - 2$ can be canceled resulting just in $x + 2$ for all values of x for which the function is defined. And you can simply plug $x = 2$ into $x + 2$ to find the limit. Thus,

$$\lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2} x + 2 = 2 + 2 = 4.$$

Let us also consider the graph of $f(x)$. Plotting $f(x) = \frac{x^2-4}{x-2}$ produces a graph that on your calculator appears to be a line. And our earlier analysis indicates indicates that the equation of this line is $y = x + 2$. However, $f(x)$ is not defined at $x = 2$. So the graph of $f(x)$ is a line $x+2$ that has a hole at $x = 2$. Note that the graph also indicates that the limit of $f(x)$ at $x = 2$ is 4 since the y-values accumulate near 4 when x values accumulate about 2.



Graph of $f(x)$ is a line with a hole

Note also that $f(x)$ can be represented as follows $f(x) = \begin{cases} x+2 & x \neq 2 \\ \text{not defined} & x = 2 \end{cases}$.

The left and the right limits. In order for the limit of $f(x)$ at $x = a$ to exist, the y-values must approach *the same value* when x-values approach a both from the left and from the right. If that is not the case, the limit does not exist.

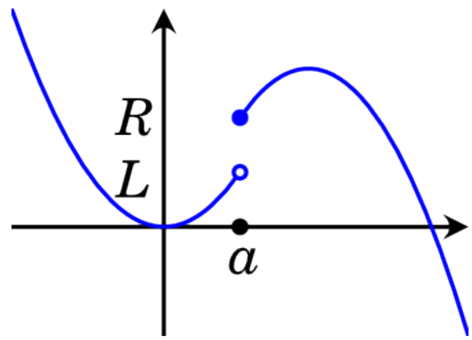
For example, for the function on the following graph, when x-values are approaching a from the left (written as $x \to a^-$), the y-values are approaching L. When x-values are approaching a from the right (written as $x \to a^+$), the y-values are approaching R. So we have that

$$\boxed{\lim_{x\to a^-} f(x) = L \text{ and } \lim_{x\to a^+} f(x) = R}$$

Since $L \neq R$ we have that

$$\boxed{\lim_{x\to a} f(x) \text{ does not exist.}}$$

From this graph we can also conclude that the value of $f(x)$ at $x = a$ is equal to R.



The left limit is L and the right limit is R

The notation $x \to a^-$ indicate that left from a, the numbers are smaller than a and the notation $x \to a^+$ indicate that right from a, the numbers are larger than a.

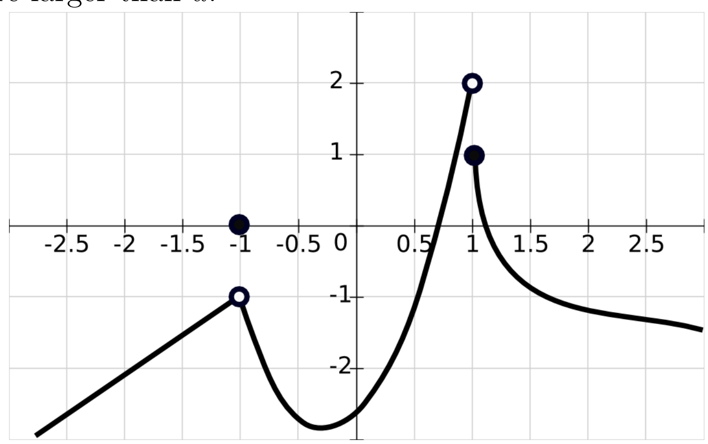
Example 4. Consider the function $f(x)$ given by the following graph. Determine the following.

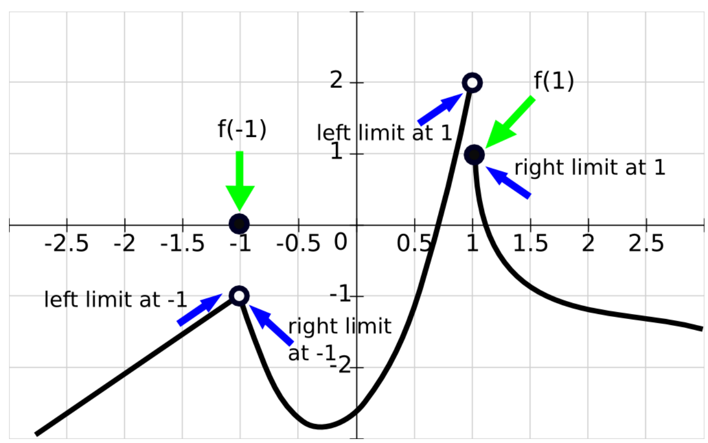
$$\lim_{x\to -1^-} f(x) \qquad \lim_{x\to -1^+} f(x)$$

$$\lim_{x\to -1} f(x) \qquad f(-1)$$

$$\lim_{x\to 1^-} f(x) \qquad \lim_{x\to 1^+} f(x)$$

$$\lim_{x\to 1} f(x) \qquad f(1)$$





Solution. From the graph on the right we conclude that $\lim_{x\to -1^-} f(x) = -1$ and $\lim_{x\to -1^+} f(x) = -1$. Since the left and the right limits are equal $\lim_{x\to -1} f(x)$ exists and it is equal to -1. From the graph we also see that $f(-1) = 0$. Similarly, $\lim_{x\to 1^-} f(x) = 2$ and $\lim_{x\to 1^+} f(x) = 1$ from the graph. Since the left and the right limits are different $\lim_{x\to 1} f(x)$ does not exist. The value $f(1)$ is equal to 1.

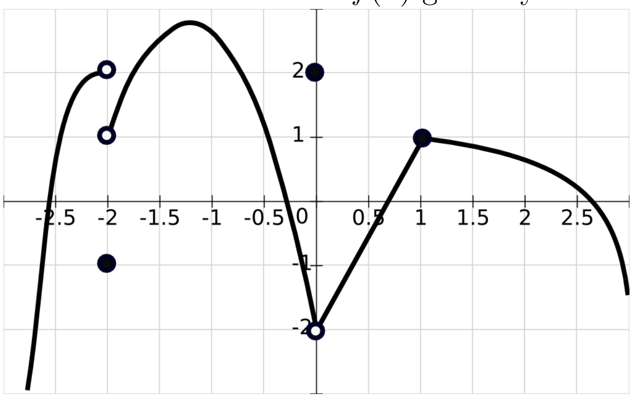
Practice problems. Evaluate the following limits.

1. (a) $\lim_{x\to 2} 3x^2 - 5x + 2$ (b) $\lim_{x\to 0} \frac{x-1}{x^2-3x+2}$

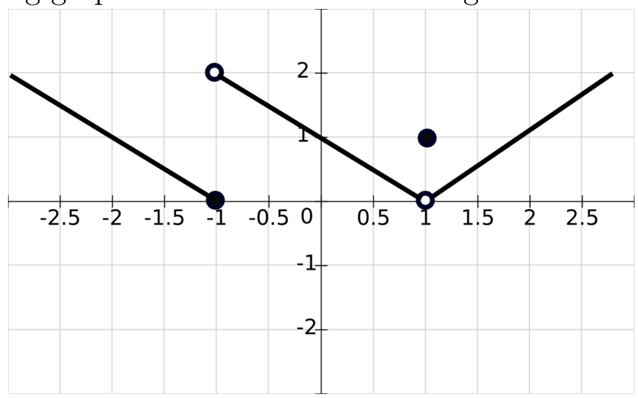
   (c) $\lim_{x\to 1} \frac{x-1}{x^2-3x+2}$ (d) $\lim_{x\to 3} \frac{x^2-x-6}{x^2-2x-3}$

(e) $\lim_{h\to 0} \frac{(h+2)^2-4}{h}$

2. Consider the function $f(x)$ given by the following graph. Determine the following limits.



part (a)



part (b)

(a)

$$\lim_{x\to -2^-} f(x) \qquad \lim_{x\to -2^+} f(x) \qquad \lim_{x\to -2} f(x) \qquad f(-2)$$
$$\lim_{x\to 0} f(x) \qquad f(0) \qquad \lim_{x\to 1^-} f(x) \qquad \lim_{x\to 1^+} f(x)$$

(b)

$$\lim_{x\to -1^-} f(x) \qquad \lim_{x\to -1^+} f(x) \qquad \lim_{x\to -1} f(x) \qquad f(-1)$$
$$\lim_{x\to 1^-} f(x) \qquad \lim_{x\to 1^+} f(x) \qquad \lim_{x\to 1} f(x) \qquad f(1)$$

Solutions.

1. (a) Plug 2 for x to obtain $\lim_{x\to 2} 3x^2 - 5x + 2 = 3(2)^2 - 5(2) + 2 = 4$.

   (b) Plug 0 for x to obtain $\lim_{x\to 0} \frac{x-1}{x^2-3x+2} = \frac{0-1}{0+2} = \frac{-1}{2}$.

   (c) Plugging 1 for x produces indeterminate expression $\frac{0}{0}$. Simplify the function $\frac{x-1}{x^2-3x+2}$ as $\frac{x-1}{(x-1)(x-2)}$. For $x \neq 1$ this function is equal to $\frac{1}{x-2}$. Plug 1 for x to obtain that $\lim_{x\to 1} \frac{1}{x-2} = \frac{1}{1-2} = -1$.

   (d) Plugging 3 for x produces indeterminate expression $\frac{0}{0}$. Simplify the function as $\frac{(x-3)(x+2)}{(x-3)(x+1)}$. For $x \neq 3$ this function is equal to $\frac{x+2}{x+1}$. Plug 3 for x to obtain that $\lim_{x\to 3} \frac{x+2}{x+1} = \frac{3+2}{3+1} = \frac{5}{4}$.

   (e) Plugging 0 for h produces indeterminate expression $\frac{0}{0}$. Simplify the function $\frac{(h+2)^2-4}{h}$ as $\frac{h^2+4h+4-4}{h} = \frac{h^2+4h}{h} = h + 4$. Plug 0 for h to obtain that $\lim_{h\to 0} h + 4 = 4$.

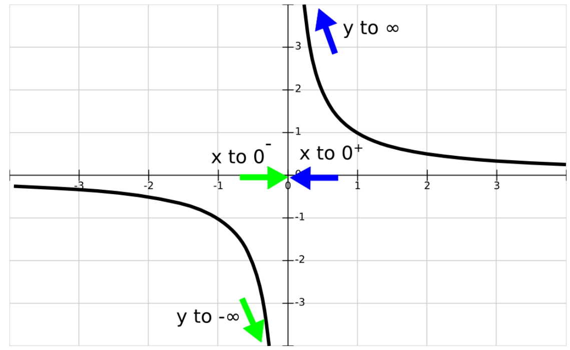
2. (a) From the graph, $\lim_{x\to -2^-} f(x) = 2, \lim_{x\to -2^+} f(x) = 1$ and $f(-2) = -1$ Since the left and the right limit are 2 and 1 respectively and thus are not equal, $\lim_{x\to -2} f(x)$ does not exist. When $x \to 0$ from either side, y-values are near -2 thus $\lim_{x\to 0} f(x) = -2$ $f(0) = 2$. $\lim_{x\to 1^-} f(x) = 1$ and $\lim_{x\to 1^+} f(x) = 1$.

   (b) From the graph, $\lim_{x\to -1^-} f(x) = 0, \lim_{x\to -1^+} f(x) = 2$ and $f(-1) = 0$. Since the left and the right limits are not equal $\lim_{x\to -1} f(x)$ does not exist. $\lim_{x\to 1^-} f(x) = 0, \lim_{x\to 1^+} f(x) = 0$ and so $\lim_{x\to 1} f(x) = 0$ while $f(1) = 1$.

Infinite Limits and Limits at Infinity. Horizontal and vertical asymptotes

Example 1. Determine the limit when $x \to 0$ of the function $f(x) = \frac{1}{x}$,

$$\lim_{x\to 0} f(x).$$



Solution. Graph the function to have a better sense of its *behavior near* $x = 0$. Using the graph, we can conclude that

$$\text{when } x \to 0^+, \quad y \to \infty \text{ and}$$
$$\text{when } x \to 0^-, \quad y \to -\infty.$$

Since the left limit is $-\infty$ and the right limit is $+\infty$, the limit $\lim_{x\to 0} \frac{1}{x}$ does not exist.

We can reach the same conclusion by considering the y-values when x-values become closer and closer to zero.

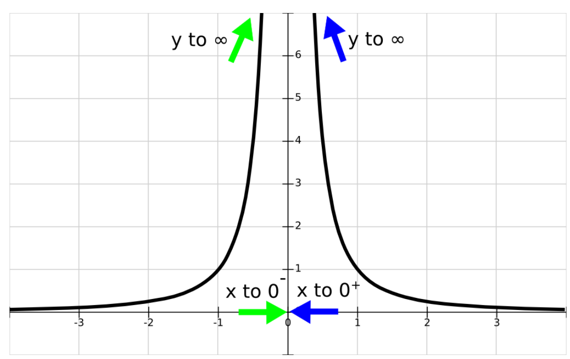
x	$y = \frac{1}{x}$
0.1	10
0.01	100
0.001	1000
0.0001	10000

x	$y = \frac{1}{x}$
-0.1	-10
-0.01	-100
-0.001	-1000
-0.0001	-10000

The first table indicates that when x is taking smaller and smaller positive values, the y-values increase without bound. So, $\lim_{x\to 0^+} \frac{1}{x} = \infty$. The second table indicates that when x is taking negative values closer and closer to 0, the y-values become larger and larger negative values. So, $\lim_{x\to 0^-} \frac{1}{x} = -\infty$.

Example 2. Determine the limit

$$\lim_{x\to 0} \frac{1}{x^2}.$$



Solution. You can use the graph of $y = \frac{1}{x^2}$ or the table. By considering the graph, note that when x is taking smaller and smaller values, both positive and negative, the y-values become larger and larger positive values. Thus, $y \to \infty$ both when $x \to 0^+$ and when $x \to 0^-$. Since the left and the right limits are equal, the limit $\lim_{x\to 0} \frac{1}{x^2}$ exists (bot not as a finite number) and it is equal to ∞. We can reach the same conclusion by considering the y-values when x-values become closer and closer to zero.

x	$y = \frac{1}{x^2}$
0.1	100
0.01	10000
0.001	1000000
0.0001	100000000

x	$y = \frac{1}{x^2}$
-0.1	100
-0.01	10000
-0.001	1000000
-0.0001	100000000

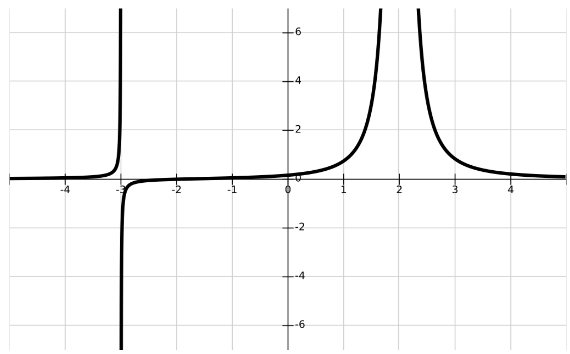
Infinite limits. Consider the first example again. When $x \to 0^+$, the function $\frac{1}{x}$ takes large positive values so the limit is ∞. We can write down this conclusion as $\frac{1}{0^+} = \infty$. Similarly, when $x \to 0^-$, $\frac{1}{x} \to -\infty$ which we can write as $\frac{1}{0^-} = -\infty$. From these observations, we can conclude the following.

> If the numerator is nonzero, the expressions 0^- or 0^+ in denominator cause a limit not to have a finite value: it is either ∞, $-\infty$, or it does not exist.

We illustrate the above observations by the following example.

Example 3. Determine $\lim_{x\to -3} f(x)$ and $\lim_{x\to 2} f(x)$ for $f(x) = \frac{x+2}{(x+3)(x-2)^2}$.

Solution. Let us find the left and right limits at -3 first. When $x \to -3^-$, $f(x) \to \frac{-3+2}{(-3^-+3)(-3-2)^2} = \frac{-1}{(0^-)(25)}$. The sign of the answer is positive since we are dividing a negative with a negative number. Thus, the answer is ∞.



When $x \to -3^+$, $f(x) \to \frac{-3+2}{(-3^++3)(-3-2)^2} = \frac{-1}{(0^+)(25)}$. The sign of the answer is negative since we are dividing a negative with a positive number. Thus, the answer is $-\infty$.

We conclude that $\lim_{x\to -3} f(x)$ does not exist because the left and the right limit are not equal to each other.

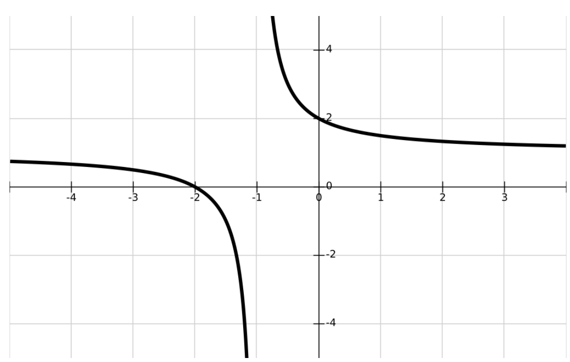
Let us determine $\lim_{x\to 2} f(x)$ now. When $x \to 2^-$, $f(x) \to \frac{2+2}{(2+3)(2^- -2)^2} = \frac{4}{(5)(0^-)^2} = \frac{4}{(5)(0^+)} = \infty$. When $x \to 2^+$, $f(x) \to \frac{2+2}{(2+3)(2^+ -2)^2} = \frac{4}{(5)(0^+)^2} = \frac{4}{(5)(0^+)} = \infty$. Thus, we conclude that $\lim_{x\to 2} f(x) = \infty$.

Considering the graph of $f(x)$ supports these conclusions.

Example 4. Find the limits of $f(x) = \frac{x^2-x-6}{x^2-2x-3}$ at points at which $f(x)$ is not defined.

Solution. To determine the points at which $f(x)$ is not defined, factor the numerator and denominator and obtain $f(x) = \frac{(x-3)(x+2)}{(x-3)(x+1)}$. This tells you that there are two potential vertical asymptotes $x = 3$ and $x = -1$. Determine limits at both values.

Note that when $x \neq 3$, the terms $x - 3$ cancel and $f(x)$ is equal to $\frac{x+2}{x+1}$. So the limit $\lim_{x\to 3}$ is $\frac{3+2}{3+1} = \frac{5}{4}$.

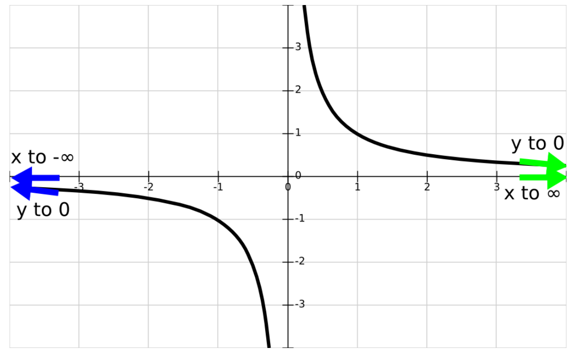


Using the graph, you can see that when $x \to -1^+$, $f(x) \to \frac{-1+2}{-1^++1} = \frac{1}{0^+} = \infty$. The graph also indicates that $\lim_{x\to -1^-} f(x) = \frac{1}{0^-} = -\infty$. Thus, $\lim_{x\to -1} f(x)$ does not exist.

Limits at infinity. When x-values become arbitrarily large, we say that x approaches infinity and write $x \to \infty$. The limits $\lim_{x\to\infty} f(x)$ are relevant for determining the **long term behavior** of $f(x)$.

When x-values become arbitrarily large negative value, we say that x approaches negative infinity and write $x \to -\infty$.

Example 1 revisited. Find limits $\lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to-\infty} \frac{1}{x}$.



Solution. Graph the function $\frac{1}{x}$. The graph indicates that when x is a large positive number, $\frac{1}{x}$ is a small (positive) number. Thus, $\lim_{x\to\infty} \frac{1}{x} = \frac{1}{\infty} = 0$. The graph also indicates that when x is a large negative number, $\frac{1}{x}$ is a small (negative) number. Thus $\lim_{x\to-\infty} \frac{1}{x} = \frac{1}{-\infty} = 0$ as well.

Example 3 revisited. Find limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for $f(x)$ from example 3.

Solution. Considering the graph of function from example 3, you can conclude that $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$.

Limits at infinity of polynomials and rational functions. While it is easy to use the graph to determine the limit at infinity of the function from example 3, it may note be as straightforward to do the same for the function from example 4. From the graph alone (see the graph of function in example 4), we can see that the limit of $f(x)$ is a finite number, but it is not as obvious what this number is as it was in example 3. For cases like this, the following analysis may be useful.

When x is a large number, the value of a polynomial function is impacted the most by the term with the highest power of x, known also as the **leading term**. For example, consider the values of each term of the polynomial $x^2 - 2x - 3$ for $x = 1000$. The value of x^2 is 1000000, the value of $-2x$ is -2000 while the last term -3 has the same value for any x. We can see that the value of the leading term x^2 impacts the total value of the polynomial the most.

This observation can help us simplify determination of limits at infinity of rational functions a great deal. Recall that a rational function is a quotient of two polynomial functions. For example, all functions from examples 1 to 4 are rational functions.

> If $p(x)$ and $q(x)$ are polynomials with leading terms $a_n x^n$ and $b_m x^m$, then $\lim_{x\to\infty} \frac{p(x)}{q(x)}$ is equal to $\lim_{x\to\infty} \frac{a_n x^n}{b_m x^m}$.

The same reasoning can be used for limits when $x \to -\infty$.

Example 4 revisited. Find the limit of $f(x) = \frac{x^2-x-6}{x^2-2x-3}$ when $x \to \pm\infty$.

Solution. Using the reasoning above $\lim_{x\to\pm\infty} \frac{x^2-x-6}{x^2-2x-3} = \lim_{x\to\pm\infty} \frac{x^2}{x^2} = 1$.

Example 5. Find the limit of the following functions when $x \to \infty$.

(a) $f(x) = \dfrac{57x^3 - 4x^2 + 9}{35x - 114x^3}$ (b) $f(x) = \dfrac{57x^5 - 4x^2 + 9}{35x - 114x^3}$ (c) $f(x) = \dfrac{57x^3 - 4x^2 + 9}{35x - 114x^4}$

Solution. (a) $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{57x^3}{-114x^3} = \frac{57}{-114} = \frac{-1}{2}$.
(b) $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{57x^5}{-114x^3} = \lim_{x\to\pm\infty} \frac{57x^2}{-114} = -\infty$.
(c) $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{57x^3}{-114x^4} = \lim_{x\to\pm\infty} \frac{57}{-114x} = 0$.

Practice problems. Evaluate the following limits.

(1) $\lim_{x\to 3^-} \frac{2}{x-3}$ (2) $\lim_{x\to 3} \frac{2}{x-3}$ (3) $\lim_{x\to\infty} \frac{2}{x-3}$
(4) $\lim_{x\to\infty} \frac{2x}{x-3}$ (5) $\lim_{x\to\infty} \frac{x-1}{x^2-3x+2}$ (6) $\lim_{x\to\infty} \frac{x^2-x-6}{x^2-2x-3}$
(7) $\lim_{x\to 0^+} \frac{x+2}{x(x-2)}$ (8) $\lim_{x\to 2} \frac{x+2}{x(x-2)}$ (9) $\lim_{x\to\infty} \frac{x+2}{x(x-2)}$

Solutions. For all the problems, you can use the **graphs** or **tables** instead or in addition to the following reasoning.

1. $\lim_{x\to 3^-} \frac{2}{x-3} = \frac{2}{3^- -3} = \frac{2}{0^-} = -\infty$
2. By part (1), the left limit is $-\infty$. The right limit is $\lim_{x\to 3^+} \frac{2}{x-3} = \frac{2}{3^+-3} = \frac{2}{0^+} = \infty$. Since the left and the right limits are different, the limit when $x \to 3$ does not exist.
3. $\lim_{x\to\infty} \frac{2}{x-3} = \frac{2}{\infty} = 0$.
4. $\lim_{x\to\infty} \frac{2x}{x-3} = \lim_{x\to\infty} \frac{2x}{x} = 2$.
5. $\lim_{x\to\infty} \frac{x-1}{x^2-3x+2} = \lim_{x\to\infty} \frac{x}{x^2} = \lim_{x\to\infty} \frac{1}{x} = \frac{1}{\infty} = 0$.
6. $\lim_{x\to\infty} \frac{x^2-x-6}{x^2-2x-3} = \lim_{x\to\infty} \frac{x^2}{x^2} = 1$.
7. $\lim_{x\to 0^+} \frac{x+2}{x(x-2)} = \frac{0+2}{0^+(0-2)} = \frac{2}{0^+(-2)} = \frac{1}{-0^+} = -\infty$.
8. Compare the left and the right limits $\lim_{x\to 2^+} \frac{x+2}{x(x-2)} = \frac{4}{2(0^+)} = \infty$ and $\lim_{x\to 2^-} \frac{x+2}{x(x-2)} = \frac{4}{2(0^-)} = -\infty$. Thus $\lim_{x\to 2} \frac{x+2}{x(x-2)}$ does not exist.
9. $\lim_{x\to\infty} \frac{x+2}{x(x-2)} = \lim_{x\to\infty} \frac{x}{x^2} = \lim_{x\to\infty} \frac{1}{x} = 0$.

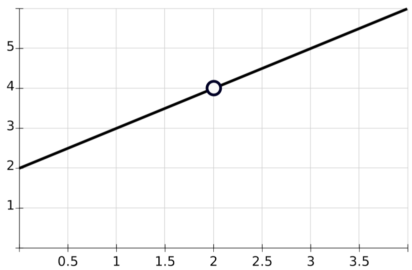
Continuous functions. Limits of non-rational functions. Applications of limits

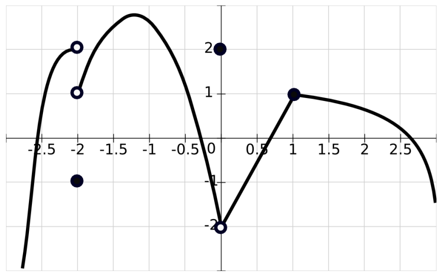
Continuous Functions. Recall that we say that a function $f(x)$ is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$. This means that the following three conditions hold.

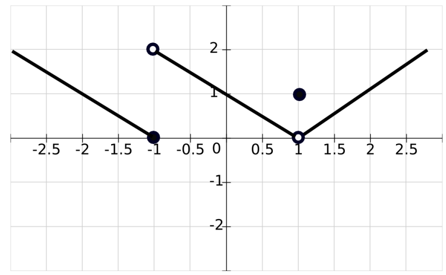
A function $f(x)$ is **continuous** at $x = a$ if
(1) $f(a)$ exists,
(2) $\lim_{x\to a} f(x)$ exists and
(3) $\lim_{x\to a} f(x) = f(a)$

These conditions hold when the graph of $f(x)$ has no holes, jumps or breaks at $x = a$.

For example, consider the following three functions.







The first function is not continuous at $x = 2$ since $f(2)$ does not exist. It is continuous at all other points displayed on the graph.

The second function is not continuous at -2 since $\lim_{x\to -2} f(x)$ does not exist. It is not continuous at 0 also since $\lim_{x\to 0} f(x) = -2 \neq 2 = f(0)$. It is continuous at 1 since $\lim_{x\to 1} f(x) = 1 = f(1)$. It is continuous at all other points displayed on the graph.

The third function is not continuous at -1 since $\lim_{x\to -1} f(x)$ does not exist. It is not continuous at 1 since $\lim_{x\to 1} f(x) = 0 \neq 1 = f(1)$. It is continuous at all other points displayed on the graph.

The property $\lim_{x\to a} f(x) = f(a)$ of a continuous function $f(x)$ can also be written as

$$\lim_{x\to a} f(x) = f(\lim_{x\to a} x)$$

Without explicitly stating it, we have used this property when evaluating limits in Example 2 of the previous section:

$$\lim_{x\to 1} 2x + 5 = 2(\lim_{x\to 1} x) + 5 = 2(1) + 5 = 7.$$

All the *elementary functions* (rational functions, power and exponential functions and their inverses, trigonometric functions) *are continuous at every value of their domains.* This fact together with the above property greatly simplifies determination of various limits as the next example illustrates.

Example 1. Find the following limits.

$$\text{(a)} \quad \lim_{x\to 2} \sqrt{3x^2 - 5x + 2} \qquad\qquad \text{(b)} \quad \lim_{x\to\infty} \ln\left(1 + \frac{1}{x}\right)$$

Solution. (a) Plug 2 for x to obtain $\lim_{x\to 2} \sqrt{3x^2 - 5x + 2} = \sqrt{3(2)^2 - 5(2) + 2} = \sqrt{4} = 2$.

(b) $\lim_{x\to\infty} \ln\left(1 + \frac{1}{x}\right) = \ln(1 + \frac{1}{\infty}) = \ln(1 + 0) = \ln 1 = 0$. Here we are using the above property when considering the limit just of the terms inside of the logarithm. This argument is valid since the logarithmic function is continuous.

Example 2. Find the limits for $x \to \infty$ and $x \to 0$ of the following functions.

$$\text{(a)} \quad f(x) = \sin x, \qquad\qquad \text{(b)} \quad f(x) = \sin\frac{1}{x}$$

Solution. (a) $\lim_{x\to 0} \sin x = \sin 0 = 0$. The graph of $\sin x$ indicates that there is no specific and unique value which $\sin x$ approaches when $x \to \infty$. Thus, the limit $\lim_{x\to\infty} \sin x$ does not exist.

(b) $\lim_{x\to 0} \sin\frac{1}{x} = \sin \lim_{x\to 0}\frac{1}{x} = \sin\infty$. By part (a), this limit does not exist. It is also interesting to consider the graph of $\sin\frac{1}{x}$. For small values of x, this function oscillates about x-axis faster and faster but does not approach any specific value illustrating also that the limit does not exist.

$\lim_{x\to\infty} \sin\frac{1}{x} = \sin\lim_{x\to\infty}\frac{1}{x} = \sin 0 = 0$.

Example 3. Find values of the constant a for which the functions below are continuous for every value of x.

(a) $f(x) = \begin{cases} x^2+1 & x<0 \\ 3x+a & x\geq 0 \end{cases}$ (b) $f(x) = \begin{cases} e^{5x}+1 & x<0 \\ x+2 & 0\leq x<2 \\ ax^2-8 & x\geq 2 \end{cases}$

Solutions. (a) Since x^2+1 is continuous for $x<0$ and $ax+3$ is continuous for $x>0$, the only x-value needing additional consideration is $x=0$. The limit of $f(x)$ when $x\to 0^-$ is the value of x^2+1 at 0 which is $0^2+1=1$. For the function to be continuous at zero, this limit needs to be equal to the limit $x\to 0^+$ as well as to the value of $f(x)$ at zero. This limit and this value are equal to the value of $3x+a$ at 0 which is $3(0)+a=a$. Thus, a should be equal to 1 if the function is to be continuous at zero.

(b) Since $e^{5x}+1$ is continuous for $x<0$ and $x+2$ is continuous for $0<x<2$, and ax^2-8 is continuous for $x>2$, the only x-values needing additional consideration are $x=0$ and $x=2$. The limit of $f(x)$ when $x\to 0^-$ is the value of $e^{5x}+1$ at 0 which is $e^{5(0)}+1=1+1=2$. For the function to be continuous at zero, this limit needs to be equal to the limit $x\to 0^+$ as well as to the value of $f(x)$ at zero. This limit and this value are equal to the value of $x+2$ at 0 which is $0+2=2$. So, the function is continuous at zero.

The limit of $f(x)$ when $x\to 2^-$ is the value of $x+2$ at 2 which is $2+2=4$. For the function to be continuous at 2, this limit needs to be equal to the limit $x\to 2^+$ as well as to the value of $f(x)$ at 2. This limit and this value are equal to the value of ax^2-8 at 2 which is $4a-8$. So, for the function to be continuous at $x=2$, $4a-8$ should be equal to 4. Hence, $4a-8=4\Rightarrow 4a=12\Rightarrow a=3$.

Calculator issues.

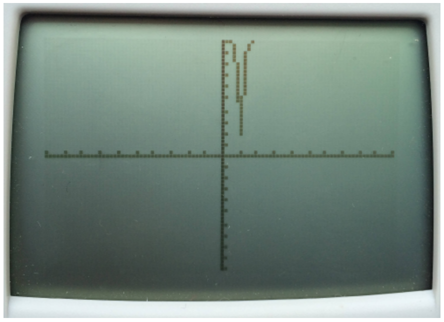
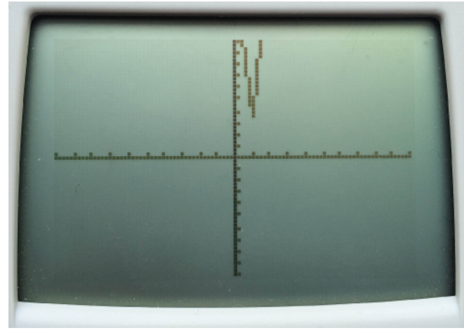
Using the calculator can greatly facilitate determination of limits. Still there are some issues one should keep in mind. We illustrate these issues on the following three examples.

Example 4. To find the limit of the function $f(x)=\frac{x+10}{10x}$ when $x\to\infty$, you may want to consider its graph first. The graph may appear to indicate that $y\to 0$ when $x\to\pm\infty$ so you may falsely conclude that $y=0$ is the horizontal asymptote.

However, a closer analysis reveals that the horizontal asymptote is in fact $y=\frac{1}{10}$ since

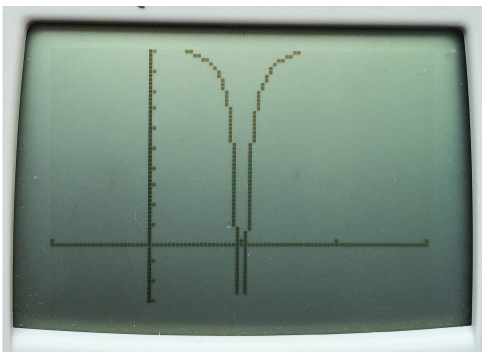
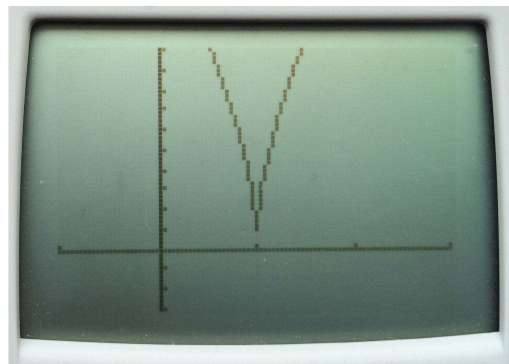
$$\lim_{x\to\pm\infty}\frac{x+10}{10x} = \lim_{x\to\pm\infty}\frac{x}{10x}+\frac{10}{10x} = \lim_{x\to\pm\infty}\frac{1}{10}+\frac{1}{x} = \frac{1}{10}+0=\frac{1}{10}.$$

Example 5. Consider the graphs of the following functions $f(x)=11-\sqrt{\frac{1}{3(x-1)^2}}$ and $g(x)=1+15(x-1)^{2/3}$. Graphed on the standard screen, the graphs look almost the same: both functions seem to have a downwards directed spike at $x=1$. So one may assume that their behavior for $x\to 1$ is the same and that either both have a finite value at 1 or that both have a vertical asymptote at 1.

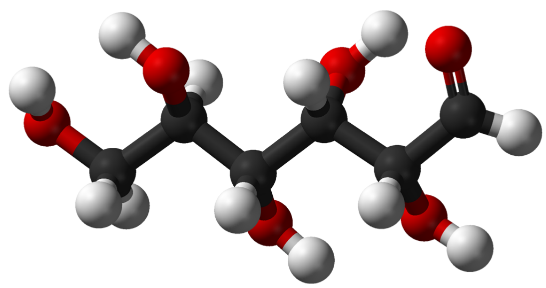
However, a closer analysis of the two functions (or simply examining them at different windows) reveals that the first one has a vertical asymptote at 1 while the second one does not.

$$\lim_{x\to\pm1} f(x) = 11-\sqrt{\frac{1}{3(0^{\pm})^2}} = 11-\sqrt{\frac{1}{0^+}} = 11-\infty = -\infty \text{ and } \lim_{x\to\pm1} g(x) = 1+15(0)^{2/3} = 1+15(0) = 1.$$

Applications of limits. Limit of the function when $x \to \infty$ indicates the long term behavior as the next example illustrates.

Example 6. A glucose solution is administered intravenously into the bloodstream at a constant rate of 4 mg/cm^3 per hour. As the glucose is added, it is converted into other substances and removed from the bloodstream at a rate proportional to the concentration at that time with proportionality constant 2.



In this case, the formula

$$C(t) = 2 - ce^{-2t}$$

describes the concentration of the glucose in mg/cm^3 as a function of time in hours. The constant c can be determined based on the initial concentration of glucose. Determine the limiting glucose concentration (the concentration of the glucose after a long period of time).

Solution. Note that the expression $2 - ce^{-2t}$ can also be written as $2 - \frac{c}{e^{2t}}$. Thus, when $t \to \infty$ the denominator e^{2t} increases to ∞ so the part $\frac{c}{e^{2t}}$ converges to 0 regardless of the value of c. The same conclusion can be reached by considering graph of $2 - ce^{-2t}$ for several different values of c. Thus, the limiting concentration is 2 mg/cm^3.

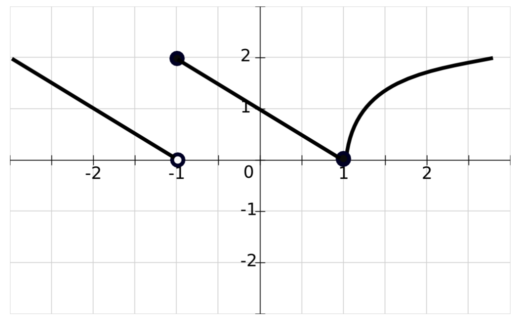
Practice problems.

1. Evaluate the following limits.

(a) $\lim_{x\to 0} 5^x + 3$	(b) $\lim_{x\to\infty} 5^x + 3$	(c) $\lim_{x\to-\infty} 5^x + 3$
(d) $\lim_{x\to-\infty} 3^{\frac{4}{x-2}} - 5$	(e) $\lim_{x\to 2^-} 3^{\frac{4}{x-2}} - 5$	(f) $\lim_{x\to 2^+} 3^{\frac{4}{x-2}} - 5$
(g) $\lim_{x\to 0} \ln(x+1) + 3$	(h) $\lim_{x\to -1^+} \ln(x+1) + 3$	(i) $\lim_{x\to\infty} \ln(x+1) + 3$
(j) $\lim_{x\to 1^-} \ln(1-x)$	(k) $\lim_{x\to\infty} \sin\frac{x^2-x}{3+2x^2}$	(l) $\lim_{x\to\infty} \cos\frac{x-1}{x^2}$

2. Determine if the following functions are continuous at the given points.

(a) $f(x) = \begin{cases} x+2 & x < -1 \\ x+1 & -1 \le x < 1 \\ 3-x & x \ge 1 \end{cases}$,
$x = -1$ and $x = 1$.



(b) $f(x) = \begin{cases} x^2 & x < 0 \\ x & 0 < x < 2 \\ 1 & x = 2 \\ 4-x & x > 2 \end{cases}$,
$x = 0$ and $x = 2$.

(c) Function given by the above graph, $x = -1$ and $x = 1$.

3. The function

$$B(t) = \frac{2 \cdot 10^7}{1 + 7e^{-3t/10}}$$

models the biomass (total mass of the members of the population) in kilograms of a Pacific halibut fishery after t years. Determine the biomass in the long run.

4. Brine that contains the solution of water and salt is pumped into a water tank. The concentration of salt is increasing according to the formula

$$C(t) = \frac{5t}{100+t}$$

grams per liter. Determine the concentration of salt after a substantial amount of time.

Solutions.

1. For all the problems, you can use the **graphs** or **tables** instead or in addition to the following reasoning.

   (a) $\lim_{x\to 0} 5^x + 3 = 5^0 + 3 = 4$. For part (b) and (c) you can use the graph of the function to note that (b) $\lim_{x\to\infty} 5^x + 3 = \infty$ and (c) $\lim_{x\to-\infty} 5^x + 3 = \frac{1}{5^\infty} + 3 = \frac{1}{\infty} + 3 = 0 + 3 = 3$.

   (d) $\lim_{x\to-\infty} 3^{\frac{4}{x-2}} - 5 = 3^0 - 5 = -4$. For parts (e) and (f) you can also use the graph in addition to the following reasoning. (e) $\lim_{x\to 2^-} 3^{\frac{4}{x-2}} - 5 = 3^{\frac{4}{0^-}} - 5 = 3^{-\infty} - 5 = \frac{1}{3^\infty} = \frac{1}{\infty} - 5 = 0 - 5 = -5$ and (f) $\lim_{x\to 2^+} 3^{\frac{4}{x-2}} - 5 = 3^{\frac{4}{0^+}} - 5 = 3^\infty + 5 = \infty$.

   (g) $\lim_{x\to 0} \ln(x+1) + 3 = \ln(0+1) + 3 = \ln(1) + 3 = 0 + 3 = 3$. (h) Use the graph or the following reasoning $\lim_{x\to-1^+} \ln(x+1) + 3 = \ln(0^+) + 3 = -\infty$. (i) Use the graph or the following reasoning $\lim_{x\to\infty} \ln(x+1) + 3 = \ln(\infty + 1) + 3 = \ln(\infty) + 3 = \infty + 3 = \infty$.

(j) Use the graph or the following reasoning. $\lim_{x\to 1^-} \ln(1-x) = \lim_{x\to 1^-} \ln(1-1^-) = \ln 0^+ = -\infty$.

(k) Consider the leading terms of the inner rational function. $\lim_{x\to\infty} \sin \frac{x^2-x}{3+2x^2} = \lim_{x\to\infty} \sin \frac{x^2}{2x^2} = \sin \frac{1}{2} = .479$.

(l) Consider the leading terms of the inner rational function. $\lim_{x\to\infty} \cos \frac{x-1}{x^2} = \lim_{x\to\infty} \cos \frac{x}{x^2} = \cos 0 = 1$.

2. (a) Continuity at $x = -1$. The left limit is $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} x + 2 = -1 + 2 = 1$ and the right limit is $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} x + 1 = -1 + 1 = 0$. So, $\lim_{x\to -1} f(x)$ doesn't exist and so the function is not continuous at -1.

   Continuity at $x = 1$. The left limit is $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x + 1 = 1 + 1 = 2$ and the right limit is $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 3 - x = 3 - 1 = 2$. So, $\lim_{x\to 1} f(x) = 2$. The value of $f(x)$ at $x = 1$ is computed by the third branch and it is 2 as well. Thus, all three conditions from the definition of continuous function hold and $f(x)$ is continuous at 1.

   (b) Continuity at 0. Note that the function is not defined for $x = 0$. Thus, although the limit $\lim_{x\to 0} f(x)$ exist (both the left and the right limits at 0 are 0 so $\lim_{x\to 0} f(x) = 0$), the function is not continuous at 0.

   Continuity at 2. The left limit is $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x = 2$ and the right limit is $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 4 - x = 4 - 2 = 2$. So, $\lim_{x\to 2} f(x) = 2$. However, $f(2) = 1 \neq 2 = \lim_{x\to 2} f(x)$ so the function is not continuous at 2.

   (c) The function is not continuous at -1 since the left and the right limits are different. The function is continuous at 1 since the left limit, the right limit and the value of function at 1 are all equal to 0.

3. The problem is asking you to find $\lim_{t\to\infty} \frac{2\cdot 10^7}{1+7e^{-3t/10}}$. The expression $e^{-3t/10}$ is equal to $\frac{1}{e^{3t/10}}$ so, when $t \to \infty$, $\frac{1}{e^{3t/10}} \to \frac{1}{\infty} = 0$. Thus, $\frac{2\cdot 10^7}{1+7e^{-3t/10}} \to \frac{2\cdot 10^7}{1+7(0)} = 2 \cdot 10^7$. So, the biomass eventually becomes $2 \cdot 10^7$ kilograms.

4. The problem is asking you to find $\lim_{t\to\infty} \frac{5t}{100+t}$. Considering the leading terms, this limit is equal to $\lim_{t\to\infty} \frac{5t}{t} = 5$. Hence, the concentration eventually becomes 5 grams per liter.