General Calculus
Lia Vas

Linear Approximation, Implicit Differentiation. Related Rates

Linear Approximation.

Although different in general, the change in y-values $\Delta y = f(x+h) - f(x)$ and the differential $dy = f'(x)dx$, obtained from the formula $\frac{dy}{dx} = f'(x)$, are similar in size when dx is small. If we denote $h+x$ by x and x by a, in which case $dx = x-a$, and consider Δy as $f(x) - f(a)$, then we have that

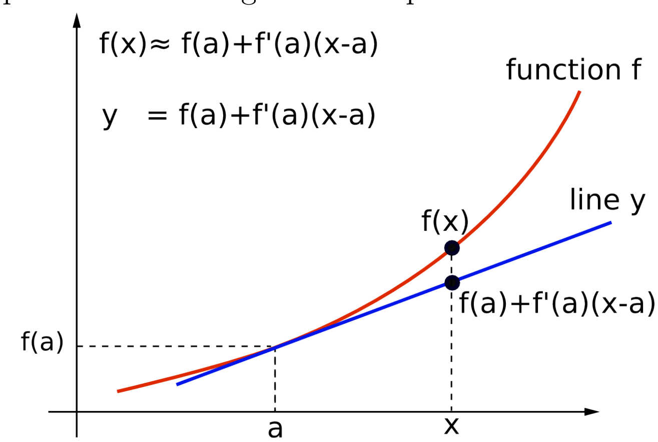
$$\Delta y \approx dy \Rightarrow f(x) - f(a) \approx f'(a)(x-a) \Rightarrow f(x) \approx f(a) + f'(a)(x-a).$$

The last formula on the right represents the point-slope equation of the tangent line at point $x = a$.

The expression $f(a)+f'(a)(x-a)$ is called the **linear approximation** of $f(x)$ at $x = a$. Thus, approximating Δy by the differential dy amounts to approximating $f(x)$ by its linear approximation $f(a) + f'(a)(x-a)$.

$$\Delta y \approx dy \Leftrightarrow f(x) - f(a) \approx f'(a)(x-a) \Leftrightarrow$$

$$\boxed{f(x) \approx f(a) + f'(a)(x-a)}$$



In applications, you can think of the value $f(a)$ as of the **present value**, the value $f(x)$ then represents the **future value**, $(x-a)$ the **time lapsed** and $f'(a)$ the **change rate**. Thus

$f(x)$	$\approx$	$f(a)$	$+$	$f'(a)$	$(x-a)$
future value	$\approx$	present value	$+$	change rate	time elapsed

The linear approximation is particularly useful when certain phenomena is modeled by a function which is either too complex to be manipulated or evaluated exactly or such that its exact formulas is not known but its value and the value of its derivative are known at a point. We illustrate both scenarios in the next two examples.

Example 1. Find the linear approximation of e^x at 0 and use it to approximate $e^{0.01}$ by a rational number.

Solution. Let $f(x) = e^x$ so that $f'(x) = e^x$. Thus $f(0) = e^0 = 1$ and $f'(0) = e^0 = 1$. The linear approximation $f(x) \approx f(a) + f'(a)(x-a)$ for $f(x) = e^x$ and $a = 0$ gives you that

$$e^x \approx 1 + 1(x-0) = 1 + x.$$

Note that this answer also indicates that the line $1+x$ is tangent to the graph of e^x at $x=0$.

When $x = 0.01$ this approximation gives you $e^{0.01} \approx 1 + 0.01 = 1.01 = \frac{101}{100}$. Comparing with the calculator value $e^{0.01} \approx 1.01005\ldots$ we can see that our approximation is accurate to first four decimal places.

Example 2. Approximate $f(3.06)$ and $f(2.9)$ given that $f(3) = 1$ and $f'(3) = 0.5$.

Solution. The linear approximation formula for $f(x)$ at 3 gives us that $f(x) \approx f(3)+f'(3)(x-3)$. Thus

$f(3.06) \approx f(3) + f'(3)(3.01 - 3) = 1 + 0.5(0.06) = 1.03$ and
$f(2.9) \approx f(3) + f'(3)(2.9 - 3) = 1 + 0.5(-0.1) = 0.95$.

Let us also consider a more applied example.

Example 3. The profit P of a company increases at a rate of 30,000 dollars per year. If the company is presently making a profit of \$800,000, approximate the profit in four year time.

Solution. If x denotes the time in years and $x = 0$ denotes the present time, we are given that $P(0) = 800,000$ and $P'(0) = 30,000$ and the problem asks for an estimate for $P(4)$. Using the linear approximation, $P(4) \approx P(0) + P'(0)(4 - 0) = 800,000 + 30,000(4) = 920,000$ dollars.

Practice problems.

1. Find the linear approximation of $f(x) = \sin x$ at $x = 0$ and use it to approximate $\sin 0.1$.

2. If $f(2) = 5$ and $f'(2) = 3$, approximate $f(2.1)$ and $f(1.9)$.

3. If $f(1) = 1$ and $f'(1) = -2$, approximate $f(1.01)$.

4. Use the linear approximation to estimate $\sqrt[3]{26}$ with a rational number. Compare to the calculator value of $\sqrt[3]{26}$.

5. The cost of producing 10 items is \$200 and the cost of each new item produced is \$15. Approximate the cost of producing 12 items.

6. The number of bacteria five hours after the start of experiment is 2000 and the number is increasing by 100 bacteria per hour. Approximate the number of bacteria five and a half hours after the start of experiment.

Solutions.

1. If $f(x) = \sin x$, then $f'(x) = \cos x$. Thus $f(0) = 0$ and $f'(0) = \cos 0 = 1$. The linear approximation $f(x) \approx f(a) + f'(a)(x - a)$ for $a = 0$ gives you that

   $$\sin x \approx 0 + 1(x - 0) = x.$$

   Note that this answer also indicates that the line x is tangent to the graph of $\sin x$ at $x = 0$.

   When $x = 0.1$ this approximation gives you $\sin 0.1 \approx 0.1 = \frac{1}{10}$. To check out the accuracy, you can compare with the calculator value $\sin 0.1 \approx 0.998\ldots$

2. $f(2) = 5$ and $f'(2) = 3 \Rightarrow f(x) \approx f(2) + f'(2)(x - 2) = 5 + 3(x - 2)$ or $3x - 1$. Thus $f(2.1) \approx 5 + 3(2.1 - 2) = 5 + 0.3 = 5.3$ and $f(1.9) \approx 5 + 3(1.9 - 2) = 5 - 0.3 = 4.7$.

3. $f(1) = 1$ and $f'(1) = -2 \Rightarrow f(x) \approx f(1) + f'(1)(x-1) = 1 - 2(x-1)$ or $2x+3$. Thus $f(1.01) \approx 1 - 2(1.01-1) = 1 - 0.02 = 0.98$.

4. To approximate $\sqrt[3]{26}$, consider the function $f(x) = \sqrt[3]{x} = x^{1/3}$ and its linear approximation at $a = 27$ (since $\sqrt[3]{27} = 3$ is easy to determine and 26 is relatively close to 27. In this case, $f(x) = x^{1/3} \Rightarrow f'(x) = \frac{1}{3}x^{-2/3}$. Thus $f(27) = 3$ and $f'(27) = \frac{1}{3}\frac{1}{3^2} = \frac{1}{27}$ so that the linear approximation is $x^{1/3} \approx 3 + \frac{1}{27}(x-27)$. When $x = 26$, the linear approximation is $26^{1/3} \approx 3 - \frac{1}{27} = \frac{80}{27} \approx 2.962963$. The calculator value is $2.9624960\ldots$

5. If x denotes the number of items produced and $f(x)$ denotes the cost of producing x items, then $f(10) = 200$ and $f'(10) = 15$. Thus, the linear approximation is $f(x) \approx f(10)+f'(10)(x-10) = 200+15(x-10)$. When $x = 12$ this approximation gives you $f(12) \approx 200+15(2) = 230$ dollars.

6. If x denotes the number of hours after the start of experiment and $f(x)$ denotes the number of bacteria at that time, then $f(5) = 2000$ and $f'(5) = 100$. Thus, the linear approximation is $f(x) \approx f(5) + f'(5)(x-5) = 2000 + 100(x-5)$. When $x = 5.5$ this approximation gives you $f(5.5) \approx 2000 + 100(0.5) = 2050$ bacteria.

Implicit Differentiation

Implicit functions. If a function can be expressed as $y = f(x)$ it is said to be in the **explicit** form. However, in some cases, the variables x and y can be related with an equation $F(x, y) = 0$ which cannot be solved for y. The relation

$$F(x, y) = 0$$

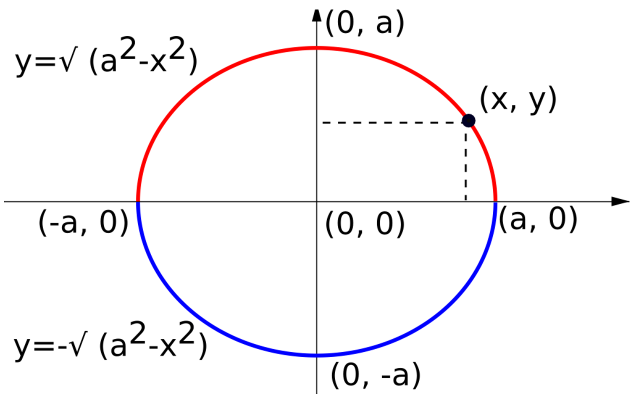
is said to define an **implicit function**. If a function is given in the form $f(x, y) = g(x, y)$ one can always consider it in the form $F(x, y) = 0$ when subtracting the right hand side from the whole equation.

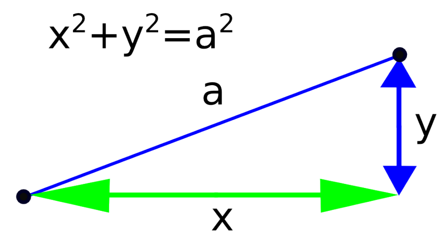
The graph of the relation $F(x, y) = 0$ may fail the vertical line and so it may not be a function. However, a collection of several pieces each of which passes the vertical line test and, for itself constitutes a function. Despite of this misnomer, the curve defined by $F(x, y) = 0$ is still refer to as an implicit *function*.

The most famous example. Probably the most widely encountered example of an implicit function is a circle. Recall that the equation of a circle of radius a centered at the origin is

$$x^2 + y^2 = a^2.$$

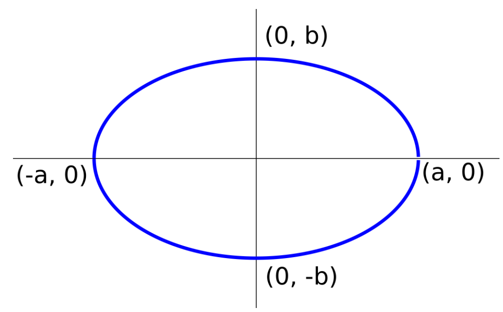
The circle fails the vertical line test so it is not a function. When solving for you, you have that $y^2 = a^2 - x^2$ and so $y = \pm\sqrt{a^2 - x^2}$. This also indicates that the circle cannot be described with a single explicit function. The upper half is given by $y = \sqrt{a^2 - x^2}$ and the lower half by $y = -\sqrt{a^2 - x^2}$. The fact that the circle cannot be described by a single function greatly contributes to introduction of parametric and polar coordinates which you will cover in Calculus 2.





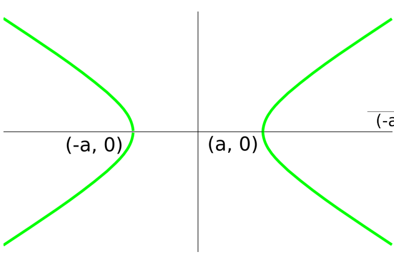
Other examples of implicit curves include:

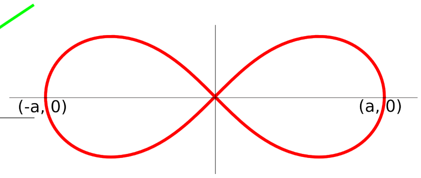
- ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,
- hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,
- lemniscate $(x^2 + y^2)^2 = a^2(x^2 - y^2)$,
- astroid $x^{2/3} + y^{2/3} = a^{2/3}$, and
- the folium of Descartes $x^3 + y^3 = 3axy$,

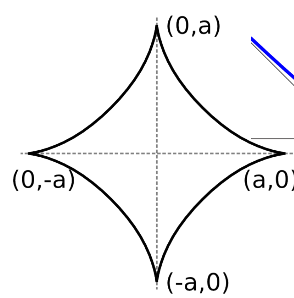


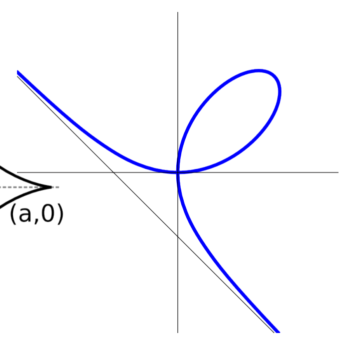
Ellipse

where a and b are positive constants.









Hyperbola, lemniscate, astroid, and the folium of Descartes

Derivative of an implicit function. It is possible to find the derivative of an implicit function $F(x, y) = 0$ even without solving the equation for y to do that, one can

1. Differentiate the whole equation $F(x, y) = 0$ using the chain rule for differentiation the terms containing y.
2. Solve for the derivative y'.

Let us illustrate this method on finding derivative of the circle centered at the origin.

Example 1. Find the derivative y' of the circle $x^2+y^2=5$ and use it to find an equation of the tangent line at the point $(2,1)$.

Solution. Start by differentiating the equation $x^2+y^2=5$.

$$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}5 \;\Rightarrow\; \frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)=0 \;\Rightarrow\; 2x+2y\cdot y'=0$$

Note that we used the chain rule when differentiating y^2. Treat the function y^2 as the composite of the outer y^2 and the inner y. The derivative of the outer is $2y$ and the derivative of the inner is y'. So, the derivative is $2y\cdot y'$.

Finally, solve the last equation for y' to get

$$2x+2yy'=0 \;\Rightarrow\; 2yy'=-2x \;\Rightarrow\; yy'=-x \;\Rightarrow\; y'=\frac{-x}{y}.$$

At the point $(2,1)$ the derivative has value $y'=\frac{-x}{y}=\frac{-2}{1}=-2$ giving you the slope of the tangent line. So, the tangent line is $y-1=-2(x-2)\Rightarrow y=-2x+5$.

Example 2. Find the derivative of the function $(3x+2y)^3=ye^{2x}+7$ and evaluate it at $(0,1)$.

Solution. Differentiate both side of the equation. Use the chain rule for the left side with inner function $3x+2y$. The derivative of this inner function is $3+2y'$. Use the product rule for the right hand side with $f(x)=y$ and $g(x)=e^{2x}$ so that $f'(x)=y'$ and $g'(x)=2e^{2x}$. Thus we have

$$\frac{d}{dx}(3x+2y)^3=\frac{d}{dx}(ye^{2x})+\frac{d}{dx}(7) \;\Rightarrow\; 3(3x+2y)^2(3+2y')=y'e^{2x}+2ye^{2x}$$

In order to solve for y', you need to keep all the terms with y' on the same side. Distribute the terms on the left side so that it is clear that it consists of two terms, one with and one without y'

$$9(3x+2y)^2+6y'(3x+2y)^2=y'e^{2x}+2ye^{2x}$$

Group the two terms with y' to the same side and the remaining two terms on the other side.

$$6y'(3x+2y)^2-y'e^{2x}=2ye^{2x}-9(3x+2y)^2$$

Finally, factor y' on the left side and solve for it.

Note that the left side consists of two terms, the first

$$y'\left(6(3x+2y)^2-e^{2x}\right)=2ye^{2x}-9(3x+2y)^2 \;\Rightarrow\; y'=\frac{2ye^{2x}-9(3x+2y)^2}{6(3x+2y)^2-e^{2x}}$$

When $x=0$ and $y=1$, $y'=\frac{2(1)e^0-9(0+2)^2}{6(0+2)^2-1}=\frac{2-36}{24-1}=\frac{-34}{23}\approx 1.48$.

Practice problems.

1. Find the derivative $\frac{dy}{dx}$ of the following implicit functions.

   (a) $x^2+xy^4=6$ (b) $x^3+12xy=y^3$ (c) $xe^y+x^2=y^2$

2. Find an equation of the line tangent to the graph of the given curves at the indicated point.

(a) $x^2+y^2=13;\ \ (3,2)$ (b) $x\ln y = 2x^3-2y;\ \ (1,1)$ (c) $x^2+y^2=e^y;\ \ (1,0)$

Solutions.

1. (a) $\frac{d}{dx}(x^2+xy^4) = \frac{d}{dx}(6) \Rightarrow 2x+1\cdot y^4+4y^3\cdot y'\cdot x = 0 \Rightarrow 4y^3xy' = -2x-y^4 \Rightarrow y' = \frac{-2x-y^4}{4xy^3} = -\frac{2x+y^4}{4xy^3}$.

   (b) $\frac{d}{dx}(x^3+12xy) = \frac{d}{dx}(y^3) \Rightarrow 3x^2+12\cdot y + y'\cdot 12x = 3y^2\cdot y' \Rightarrow 3x^2+12y = 3y^2y'-12xy' \Rightarrow 3x^2+12y = (3y^2-12x)y'\ y' = \frac{3x^2+12y}{3y^2-12x} = \frac{x^2+4y}{y^2-4x}$.

   (c) $\frac{d}{dx}(xe^y+x^2) = \frac{d}{dx}(y^2) \Rightarrow 1\cdot e^y + e^y y'\cdot x + 2x = 2y\cdot y' \Rightarrow e^y+2x = 2yy'-xy'e^y \Rightarrow e^y+2x = (2y-xe^y)y'\ y' = \frac{e^y+2x}{2y-xe^y}$.

2. (a) Find the derivative first using implicit differentiation. $\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(13) \Rightarrow 2x+2y\cdot y' = 0 \Rightarrow 2yy' = -2x \Rightarrow y' = \frac{-x}{y}$. At point $(3,2)$, the derivative is $y' = \frac{-3}{2}$. So, the tangent line is $y-2 = \frac{-3}{2}(x-3) \Rightarrow y = \frac{-3}{2}x+\frac{13}{2}$.

   (b) $\frac{d}{dx}(x\ln y) = \frac{d}{dx}(2x^3-2y) \Rightarrow \ln y + \frac{1}{y}\cdot y'\cdot x = 6x^2-2y' \Rightarrow 2y'+\frac{x}{y}y' = 6x^2-\ln y \Rightarrow (2+\frac{x}{y})y' = 6x^2-\ln y \Rightarrow y' = \frac{6x^2-\ln y}{2+\frac{x}{y}}$. At point $(1,1)$, the derivative is $y' = \frac{6-0}{2+1} = 2$. The tangent line is $y-1 = 2(x-1) \Rightarrow y = 2x-1$.

   (c) $\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(e^y) \Rightarrow 2x+2y\cdot y' = e^y y' \Rightarrow 2x = e^y y' - 2yy' \Rightarrow 2x = (e^y-2y)y' \Rightarrow y' = \frac{2x}{e^y-2y}$. At point $(1,0)$, the derivative is $y' = \frac{2}{1-0} = 2$. The tangent line is $y-0 = 2(x-1) \Rightarrow y = 2x-2$.

Related Rates

The most important reason for a non-mathematics major to learn mathematics is to be able to apply it to problems from other disciplines or real life. In this section, we focus on modeling real life scenarios using functions and then determining the rate of one quantity in terms of the rate of the other. Since the implicit differentiation formula relates the two rates, they are often referred to as the **related rates**.

In modeling a problem and then solving it, following the steps below is often useful.

1. Read the problem carefully. Sketch a diagram if possible in order to visualize the relevant information.

2. List the relevant quantities in the problem and assign them appropriate variables. Then write down all the information given.

3. Write down the **equation that relates all the variables**.

4. Differentiate the equation using the implicit differentiation if necessary.

5. Solve for the unknown rate. Substitute the given information into the relation and determine the unknown rate.

We illustrate this method with examples below.

Example 1. Suppose a spherical balloon is inflated by 10 cubic centimeters per minute. Determine how fast the radius of the balloon increases at the time when the radius is 5 cm.

Solution. *Determine the relevant quantities.* From the first sentence of the problem, we can conclude that the *volume* is increasing as *time* passes by. So, the volume and the time are two relevant quantities. In the next sentence, we are also given the size of the radius at a certain time, so the third relevant quantity is the *radius*. Let us use V, r and t for the volume, radius and time respectively. Note that the volume and the radius depend on time, so they are functions of time.

Using this notation, note that the *given information* is that $\frac{dV}{dt} = 10$ cm^3/min and $r = 5$ cm. The problem is *asking* you to calculate $\frac{dr}{dt}$ at the time when $r = 5$ cm.

Relating the variables. Recall that the formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$. Thus, this equation relates the quantities.

Differentiate the equation. Keep in mind that you are differentiating *with respect to time.*

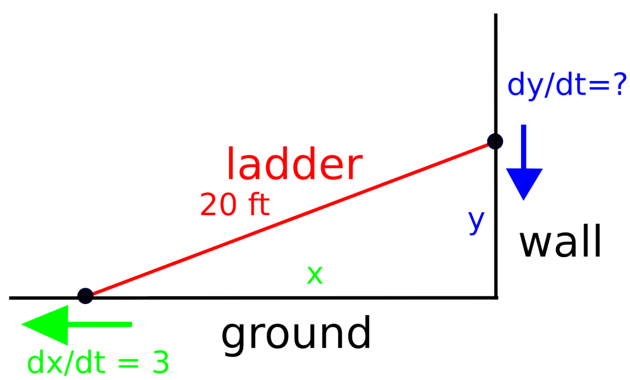
$$\frac{d}{dt}(V) = \frac{d}{dt}(\frac{4}{3}\pi r^3) \quad\Rightarrow\quad \frac{dV}{dt} = \frac{4\pi}{3}3r^2\frac{dr}{dt} \quad\Rightarrow\quad \frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

This last equation relates the rates of V and r.

Solve for the unknown rate. Substitute the given information $\frac{dV}{dt} = 10$ cm^3/min and $r = 5$ cm and then solve for the unknown $\frac{dr}{dt}$.

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} \quad\Rightarrow\quad 10 = 4\pi(5)^2\frac{dr}{dt} \quad\Rightarrow\quad \frac{dr}{dt} = \frac{10}{4\pi(25)} = \frac{10}{100\pi} = \frac{1}{10\pi} \approx 0.32 \text{ cm per min.}$$

Example 2. A 20-foot ladder is leaning against the wall. If the base of the ladder is sliding away from the wall at the rate of 3 feet per second, find the rate at which the top of the ladder is sliding down when the top of the ladder is 8 feet from the ground.



Solution. *Determine the relevant quantities.* The distance from the base of the ladder to the wall and the distance from the top of the ladder to the ground are the two relevant quantities.

Let x denotes the first distance and y the second one. Note that as the ladder is sliding down x is increasing so that $\frac{dx}{dt}$ is positive and y is decreasing so that $\frac{dy}{dt}$ is negative. Using this notation, note that the *given information* is that $\frac{dx}{dt} = 3$ ft/sec and $y = 8$ ft. The problem is *asking* you to calculate $\frac{dy}{dt}$.

Relating the variables. Since x and y are two sides of the right triangle with the 20-foot ladder as the hypotenuse, we have that

$$x^2 + y^2 = 20^2.$$

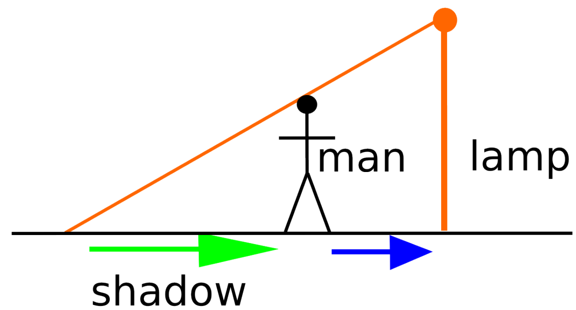
Differentiate the equation with respect to time and solve for the unknown rate.

$$\frac{d}{dt}(x^2+y^2) = \frac{d}{dt}(400) \quad\Rightarrow\quad 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \quad\Rightarrow\quad \frac{dy}{dt} = \frac{-x\frac{dx}{dt}}{y}$$

Find the missing value. Note that we have the values for $\frac{dx}{dt}$ and $y = 8$ but we are still missing the value of x. The x-value can be determined from the equation $x^2 + y^2 = 400$ using that $y = 8$. Thus $x^2 + 64 = 400 \Rightarrow x^2 = 336 \Rightarrow x = \pm\sqrt{336}$. Since we are looking for positive value, $x \approx 18.33$.

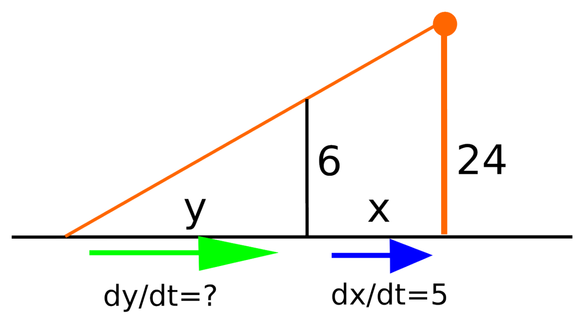
Find the unknown rate. Substitute $\frac{dx}{dt} = 3$ ft/sec, $y = 8$ ft and $x = 18.33$ into the rate equation to get $\frac{dy}{dt} = \frac{-x\frac{dx}{dt}}{y} \approx \frac{-18.33(3)}{8} = 6.87$ ft/sec. Thus, the distance from the top of the ladder to the ground is decreasing by 6.87 feet per second.

Example 3. A 6-feet-tall man walks at the rate of 5 feet per second towards a 24-feet-tall street lamp. Determine how fast is the tip of man's shadow moving along the ground.



Solution. *Determine the relevant quantities.* Sketch a diagram of this scenario as on figure on the right.

Note that the rate of 5 feet per second given in the problem is referring to the rate of change of the distance of the person from the lamp. So, let us denote this by x. The problem is asking for the rate of change of the length of the shadow, so let us denote this length by y.

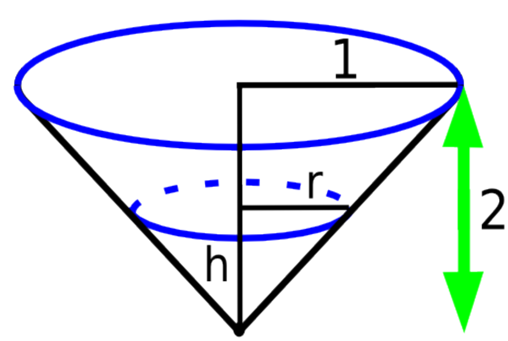


Relate the variables. The diagram consists of two similar triangles, one with sides y and 6 and the other with sides $x + y$ and 24. Thus

$$\frac{y}{6} = \frac{x+y}{24} \quad\Rightarrow\quad 4y = x + y \quad\Rightarrow\quad x = 3y \text{ or } y = \frac{1}{3}x.$$

Differentiate the equation and solve for the unknown rate. $y = \frac{x}{3} \Rightarrow \frac{dy}{dt} = \frac{1}{3}\frac{dx}{dt}$. Note that both x and y are decreasing as time passes by so both rates are negative. Thus $\frac{dx}{dt} = -5$ ft/sec and so $\frac{dy}{dt} = -\frac{5}{3} \approx -1.67$ ft/sec. Thus, the shadow is decreasing in length by 1.67 feet per second.

Example 4. A conical tank of height 2 meters is full of water. The radius of the surface is 1 meter. If the water evaporates at the rate of 30 centimeters cubic per day, determine the rate at which the water level decreases when the water is 0.5 meters deep. Discuss if this rate is increasing or decreasing as the depth of the water becomes smaller.



Solution. *Determine and relate the relevant quantities.* Sketch a diagram of this scenario as on the figure above.

The rate of 30 cubic centimeters per day refers to the rate of change of the *volume* of water. Let us denote the volume by V. The formula for the volume of the cone of height h with the radius of the base r is given by

$$V = \frac{1}{3} r^2 h \pi.$$

Since all three variables, V, r and h, are changing in time, we need to reduce the number of variables from three to two. So, we need to relate r and h.

Relate the variables. Consider the two similar triangles on the diagram. Since r and h are the sides of one and 1 and 2 are sides of the other, we have that

$$\frac{r}{h} = \frac{1}{2} \quad \Rightarrow \quad r = \frac{1}{2} h.$$

Thus, we obtain the formula for the volume in terms of the height of the water alone.

$$V = \frac{1}{3} \left(\frac{h}{2} \right)^2 h\pi = \frac{1}{12} h^3 \pi.$$

Differentiate the equation and solve for the unknown rate.

$$V = \frac{1}{12} h^3 \pi \Rightarrow \frac{dV}{dt} = \frac{1}{12} 3h^2 \frac{dh}{dt} \pi = \frac{\pi}{4} h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{4\frac{dV}{dt}}{h^2 \pi}.$$

We are given that $\frac{dV}{dt} = -30$ cm^3/day (negative sign since the volume is decreasing) and $h = 0.5$ m. To have uniform units, convert either h to centimeters or $\frac{dV}{dt}$ to cubic meters. With h as 50 cm, we have that $\frac{dh}{dt} = \frac{4(-30)}{50^2 \pi} \approx -0.015$ cm per day. So, the radius is decreasing by 0.015 cm per day.

The formula $\frac{dh}{dt} = \frac{4\frac{dV}{dt}}{h^2 \pi}$ also indicates that the rate increases when height is getting smaller. Looking at the rate at which the sand is pouring down in an hourglass just before the sand completely poured out supports this conclusion.

Practice problems.

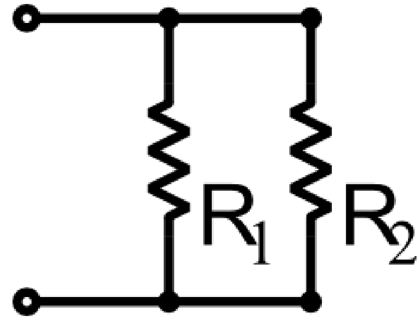
1. Water leaking onto a floor creates a circular puddle with an area that increases at the rate of 3 square centimeters per minute. Determine how fast the radius of the puddle increases when the radius is 10 cm.

2. Assume that the number of bass in the pond is related to the level of polychlorinated biphenyls (PCBs, a group of industrial chemicals used in plasticizers, fire retardants and other materials) in the pond. The bass population is modeled by

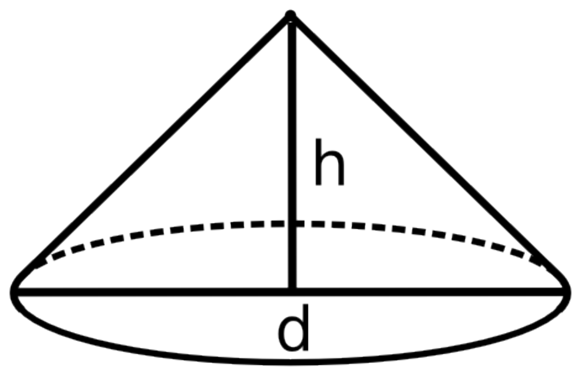
$$y = \frac{2500}{1 + x}$$

where x represents the PCB level in parts per million (ppm) and y represents the number of bass in the pond. If the level of PCBs is increasing at the rate of 40 ppm per year, find the rate at which is the number of bass changing when there are 100 bass in the pond.

3. Two resistors with resistances R_1 and R_2 are connected in parallel into an electrical circuit. The total resistance R in ohms is computed by the formula
$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$
If R_1 and R_2 are increasing by 0.25 ohms per second, determine how fast is R changing when $R_1 = 75$ and $R_2 = 100$ ohms.

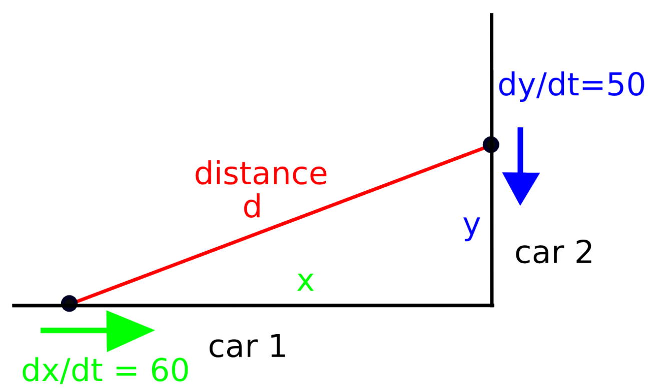


4. A conveyor belt is pouring sand down creating a conical pile whose diameter is twice as long as its height. If the belt is dumping sand at a rate of 40 cubic feet per minute, determine the rate at which the height is increasing when the pile is 7 feet high.



5. Water is leaking from a hole on the ceiling next to the wall. Sliding down the wall, the water is creating a semi-circular puddle on the floor next to the wall. The puddle is growing in surface area at a rate of 10 square inches per minute. Determine how rapidly the radius of the puddle is growing at the moment when the area of the puddle is 100 square inches.

6. One car is traveling east with speed of 60 miles per hour and the other car is traveling south with speed of 50 miles per hour. Both cars are headed towards the same spot. Determine the rate at which cars are approaching each other when the first car is 4 miles and the second car is 3 miles from the meeting spot.



7. A snowball is melting at the rate of 100 cubic centimeters per hour. Determine the rate at which the radius is decreasing at the time when radius is 5 centimeters.

Solutions.

1. *Determine the relevant quantities.* From the first sentence of the problem, conclude that the area is increasing as time passes by so that the volume and time are two relevant quantities. We are also given the size of the radius at certain time so the third relevant quantity is the radius. Let us use A, r and t for the area, radius and time respectively. Note that the area and

the radius depend on time. Using this notation, note that the *given information* is that $\frac{dA}{dt} = 3$ cm^2/min and $r = 10$ cm. The problem is *asking* you to calculate $\frac{dr}{dt}$.

Relating the variables. Recall that the formula for the area of the circle with radius r is $A = r^2\pi$. Thus, this equation relates the variables.

Differentiate the equation and solve for the unknown rate. Keep in mind that you are differentiation *with respect to time.*

$$\frac{d}{dt}(A) = \frac{d}{dt}(r^2\pi) \quad \Rightarrow \quad \frac{dA}{dt} = 2\pi r\frac{dr}{dt} \quad \Rightarrow \quad \frac{dr}{dr} = \frac{\frac{dA}{dt}}{2\pi r}.$$

Plug the given information in the equation. Substitute the given information $\frac{dA}{dt} = 3$ cm^2/min and $r = 10$ cm and solve for the unknown $\frac{dr}{dt}$. Get $\frac{dr}{dt} = \frac{\frac{dA}{dt}}{2\pi r} = \frac{3}{2\pi(10)} \approx 0.048$ cm per minute.

2. Since the variables are already named, you can start by denoting the given information as $\frac{dx}{dt} = 40$ ppm/ year, $y = 100$ bass and noting that the problem is asking for the unknown rate $\frac{dy}{dt}$. The variables are also already related so you can differentiate the given equation and solve for the unknown rate.

$$y = \frac{2500}{1+x} = 2500(1+x)^{-1} \Rightarrow \frac{dy}{dt} = -2500(1+x)^{-2}\frac{dx}{dt} = \frac{-2500\frac{dx}{dt}}{(1+x)^2}.$$

Since this last formula involves x, you need to determine x given that $y = 100$. Use the equation relating x and y again. $y = \frac{2500}{1+x} \Rightarrow 100 = \frac{2500}{1+x} \Rightarrow 1 + x = \frac{2500}{100} = 25 \Rightarrow x = 24$. Compute the unknown rate given this information as

$$\frac{dy}{dt} = frac{-}2500\frac{dx}{dt}(1+x)^2 = \frac{-2500(40)}{(1+24)^2} = \frac{-4000}{25} = -160 \text{ bass per year}$$

Thus the number of bass is decreasing by 160 each year.

3. Since the variables are already named, you can start by denoting the given information as $\frac{dR_1}{dt} = \frac{dR_2}{dt} = 0.25$ ohms/sec, $R_1 =$ and $R_2 =$ ohms and noting that the problem is asking for the unknown rate $\frac{dR}{dt}$. The variables are also already related so you can differentiate the given equation and solve for the unknown rate.

$$R^{-1} = R_1^{-1} + R_2^{-1} \Rightarrow -R^{-2}\frac{dR}{dt} = -R_1^{-2}\frac{dR_1}{dt} - R_2\frac{dR_2}{dt} \Rightarrow \frac{dR}{dt} = R^2\left(R_1^{-2}\frac{dR_1}{dt} + R_2^{-2}\frac{dR_2}{dt}\right).$$

Since this equation involves R, you need to determine R given that $R_1 = 75$ and $R_2 = 100$.

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \Rightarrow \frac{1}{R} = \frac{1}{75} + \frac{1}{100} \Rightarrow \frac{1}{R} = \frac{7}{300} \Rightarrow R = \frac{300}{7} \approx 42.86.$$

Thus the unknown rate is $\frac{dR}{dt} = R^2\left(R_1^{-2}\frac{dR_1}{dt} + R_2^{-2}\frac{dR_2}{dt}\right) = \frac{dR}{dt} = \frac{90000}{49}\left(\frac{0.25}{75^2} + \frac{0.25}{100^2}\right) \approx 1.128$ ohms per second.

4. If we denote the volume of the pile by V, then $\frac{dV}{dt} = 40$ ft^3/min. The formula $V = \frac{1}{3}r^2h\pi$ computes the volume of the cone of height h with the radius of the base r. Since the diameter $2r$ is twice as long as its height h, we have that $2r = 2h \Rightarrow r = h$. Thus, $V = \frac{1}{3}(h)^2h\pi = \frac{1}{3}h^3\pi$. Differentiate the equation and obtain $V = \frac{1}{3}3h^2\frac{dh}{dt}\pi = h^2\frac{dh}{dt}\pi \Rightarrow \frac{dh}{dt} = \frac{\frac{dV}{dt}}{h^2\pi}$. With $\frac{dV}{dt} = 40$ ft^3/min and $h = 7$ ft, we have that $\frac{dh}{dt} = \frac{40}{7^2\pi} \approx 0.26$ ft per min.

5. The formula for the area of a semi-circle of radius r is $A = \frac{1}{2}r^2\pi$. We are given that $\frac{dA}{dt} = 10$ square inches per minute and $A = 100$ square inches and the problem is asking for $\frac{dr}{dt}$.

   Differentiate the formula and solve for $\frac{dr}{dt}$. $A = \frac{1}{2}r^2\pi \Rightarrow \frac{dA}{dt} = \frac{1}{2}2r\pi\frac{dr}{dt} = r\pi\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{\frac{dA}{dt}}{r\pi}$.

   Since this last formula involves r, you need to determine r given that $A = 100$. $A = \frac{1}{2}r^2\pi \Rightarrow 100 = \frac{1}{2}r^2\pi \Rightarrow r^2 = \frac{200}{\pi} \Rightarrow r \approx \pm 7.98$. We need the positive solution so $r \approx 7.98$. Thus $\frac{dr}{dt} \approx \frac{10}{7.98\pi} \approx 0.40$ inches per minute.

6. Try to work out the problem on your own. You should get the final answer of $\frac{1}{5}(4(-60) + 3(-50)) = -78$ miles/hour. Thus, the cars are approaching each other at the rate of 78 mph.

7. Try to work out the problem on your own. You should get the final answer of $\frac{dr}{dt} = \frac{-100}{4\pi(5)^2} \approx -0.32$ cm per hour. Thus, the radius of the snowball is decreasing by 0.32 cm per hour.