Genral Calculus
Lia Vas

Finding Derivative

More Rules - Product, Quotient and Chain

The Product Rule. Given the Sum rule, one may expect that the derivative of a product is a product of the derivatives. However, this is not true.

$$\boxed{(fg)' \neq f'g'}$$

For example, if $f(x) = x^3$ and $g(x) = x^2$, the product fg is x^5 and its derivative is $(fg)'(x) = 5x^4$. On the other hand, $f'(x) = 3x^2$ and $g'(x) = 2x$ so that $f'(x)g'(x) = (3x^2)(2x) = 6x^3 \neq 5x^4$.

The correct formula for the derivative of the product is

$$\boxed{\text{The Product Rule} \quad (fg)' = f'g + g'f}$$

In $\frac{d}{dx}$ notation, this formula can be written as

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + \frac{dg}{dx}f$$

In our previous example with $f(x) = x^3$ and $g(x) = x^2$, the product rule gives the correct answer for $(fg)'(x) = 5x^4$ since $(f'g + g'f)(x) = (3x^2)(x^2) + (2x)(x^3) = 3x^4 + 2x^4 = 5x^4$.

The validity of the formula can be demonstrated using the definition of derivative of fg.

The Quotient Rule. Analogously to the product, the derivative of a quotient is *not* a quotient of the derivatives, thus

$$\boxed{\left(\frac{f}{g}\right)' \neq \frac{f'}{g'}}$$

For example, if $f(x) = x^3$ and $g(x) = x^2$, then $\frac{f}{g}(x) = \frac{x^3}{x^2} = x$ so that $(\frac{f}{g})'(x) = 1$ and $\frac{f'(x)}{g'(x)} = \frac{3x^2}{2x} = \frac{3}{2}x$.

The correct formula for the derivative of the quotient is

$$\boxed{\text{The Quotient Rule} \quad \left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}}$$

The validity of the quotient rule can also be demonstrated using the definition of the derivative.

Example 1. Find the derivative of $y = \frac{x^2+5}{x-x^3}$ and simplify your answer.

Solution. Let $f(x) = x^2 + 5$ and $g(x) = x - x^3$. Thus $f'(x) = 2x$ and $g'(x) = 1 - 3x^2$. Plugging this in the quotient rule formula we have that

$$y' = \left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2} = \frac{2x(x - x^3) - (1 - 3x^2)(x^2 + 5)}{(x - x^3)^2}$$

This answer simplifies as $\frac{2x^2-2x^4-(x^2+5-3x^4-15x^2)}{(x-x^3)^2} = \frac{2x^2-2x^4-x^2-5+3x^4+15x^2}{(x-x^3)^2} = \frac{x^4+16x^2-5}{(x-x^3)^2}$.

The Chain Rule.

Recall that the Leibniz notation $\frac{dy}{dx}$ for derivative of a function y. If y is a composite function $y = f(g(x))$, neither of the differentiation rules we covered so far can be used in order to find the derivative. If we represent the inner function by $u = g(x)$, the composite function y is $y = f(u)$. The rate of change of y with respect to x is impacted by two factors

- the rate of change $\frac{dy}{du}$ of y with respect to u and
- the rate of change $\frac{du}{dx}$ of u with respect to x

and the total rate of change $\frac{dy}{dx}$ is the **product** of these two rates. Thus, even the algebra of fractions applied to the derivatives in Leibniz form is valid.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

The formula above is known as the chain rule.

Since $u = g(x)$, the formula above can be written without the Leibniz notation as follows.

$$(f(g(x)))' = f'(g(x))\, g'(x)$$

To analyze the formula above, let us think of the function f as the **outer** and the function g as the **inner** function in the composite $y = f(g(x))$. The chain rule formula then can be stated as follows.

The Chain Rule for $y = f(g(x))$	$\Rightarrow$	$y' =$	$f'(g(x))$	$\cdot$	$g'(x)$
		derivative of the composite	derivative of the outer, keep the inner unchanged		derivative of the inner

Example 2. Find the derivative of the following functions

(a) $y = (x^2 + 6)^5$ (b) $y = \sqrt{x^3 + x}$

Solution. (a) This function can be considered as the composite of $f(u) = u^5$ and $g(x) = x^2 + 6$. Since $f'(u) = 5u^4$ and $g'(x) = 2x$, the derivative can be found as follows using the chain rule.

$y = (x^2+6)^5$	$\Rightarrow$	$y' =$	$5(x^2+6)^4$	$\cdot$	$2x$
		derivative of the composite	derivative of the outer, keep the inner unchanged		derivative of the inner

Thus the derivative is $y' = 10x(x^2+6)^4$.

(b) This function can be considered as the composite of $f(u) = u^{1/2}$ and $g(x) = x^3 + x$. Since $f'(u) = \frac{1}{2}u^{-1/2}$ and $g'(x) = 3x^2 + 1$, the derivative can be found as follows using the chain rule.

$y = (x^3+x)^{1/2}$	$\Rightarrow$	$y' =$	$\frac{1}{2}(x^3+x)^{-1/2}$	$\cdot$	$(3x^2+1)$
		derivative of the composite	derivative of the outer, keep the inner unchanged		derivative of the inner

Thus the derivative is $y' = \frac{1}{2}(x^3+x)^{-1/2}(3x^2+1) = \frac{3x^2+1}{2\sqrt{x^3+x}}$.

Some functions may require the use of more than one rule, for example, product and chain, quotient and chain or more than one chain rule. The next example illustrate all three cases.

Example 3. Find derivatives of the following functions.

(a) $y = 2x\sqrt{x^3+2}$ (b) $y = \dfrac{(x^2+3)^4}{(3x^2+1)^5}$ (c) $y = \sqrt{x + \sqrt{x^2+1}}$

Solution. (a) Note that the function is a product of $f(x) = 2x$ and $g(x) = \sqrt{x^3+2}$ so that we will need to employ the product rule. Also, note that the function $g(x)$ is the composite of the outer function $\sqrt{u} = u^{1/2}$ and the inner function $u = x^3+2$. Since the derivative of the outer is $\frac{1}{2}(x^3+2)^{-1/2}$ and the derivative of the inner is $3x^2$, the chain rule produces $g'(x) = \frac{1}{2}(x^3+2)^{-1/2}3x^2 = \frac{3x^2}{2\sqrt{x^3+2}}$. Since $f'(x) = 2$, the product rule gives us

$$y' = f'g + g'f = 2\sqrt{x^3+2} + \frac{3x^2}{2\sqrt{x^3+2}}2x = 2\sqrt{x^3+2} + \frac{3x^3}{\sqrt{x^3+2}}.$$

(b) Note that the function is a quotient of $f(x) = (x^2+3)^4$ and $g(x) = (3x^2+1)^5$ so that we will need to employ the quotient rule. Also, for both f and g we will need to use the chain rule. Using chain rule we find that

$$f'(x) = 4(x^2+3)^3(2x) = 8x(x^2+3)^3 \quad \text{and} \quad g'(x) = 5(3x^2+1)^4(6x) = 30x(3x^2+1)^4.$$

The quotient rule then gives us that

$$y' = \frac{f'g - g'f}{g^2} = \frac{8x(x^2+3)^3(3x^2+1)^5 - 30x(3x^2+1)^4(x^2+3)^4}{(3x^2+1)^{10}}.$$

(c) Note that the function is a composite of $f(u) = u^{1/2}$ and $g(x) = 1 + \sqrt{x^2+1}$. However, the function $g(x)$ is the sum of x and another composite function $\sqrt{x^2+1}$ which consists of the outer function $v^{1/2}$ and the inner function $1 + x^2$. Thus, the derivative of g is $1 + \frac{1}{2}(x^2+1)^{-1/2}(2x) = 1 + \frac{2x}{2\sqrt{x^2+1}} = 1 + \frac{x}{\sqrt{x^2+1}}$ and the derivative of y is

$$y' = f'(g(x))g'(x) = \frac{1}{2}\left(x + \sqrt{x^2+1}\right)^{-1/2}\left(1 + \frac{x}{\sqrt{x^2+1}}\right) = \frac{1}{2\sqrt{x+\sqrt{x^2+1}}}\left(1 + \frac{x}{\sqrt{x^2+1}}\right).$$

Practice problems.

1. Find the derivative of the given functions. You do not have to simplify the answers.

   (a) $y = (6x^2+5)^{10}$

   (b) $y = \frac{x^2+3x}{5x-2}$

   (c) $y = \frac{1}{\sqrt[3]{3x^3-7}}$

   (d) $y = \frac{(3x^2+1)(x+4)}{5-x^2}$

   (e) $y = (x^3 + (x^3+1)^5)^7$

   (f) $y = \frac{\sqrt{x}(2x^3-5)}{3x+2}$

2. Find an equation of the line tangent to the curve at the indicated point.

   (a) $f(x) = \frac{x^2+3x-5}{x}$ at $x = 1$.

   (b) $f(x) = \frac{3}{\sqrt{1+2x}}$ at $x = 4$.

3. The concentration of a certain medication in a patient's bloodstream (in mg per cm^3) is given by $C(t) = \frac{5t}{t^2+4}$, where t is the number of hours after the medication has been administered.

   (a) Determine the concentration 3 hours after the medication is administered.

   (b) Determine how fast is the concentration changing 3 hours after the medication is administered.

   (c) Determine fast is the concentration changing on average between 2nd and 4th hour.

Solutions.

1. (a) Chain rule with the outer u^{10} and the inner $6x^2+5$ gives you $y' = 10(6x^2+5)^9(12x) = 120x(6x^2+5)^9$.

   (b) Use the quotient rule with $f(xc) = x^2+3x$ and $g(x) = 5x-2$ to get $y' = \frac{(2x+3)(5x-2)-5(x^2+3x)}{(5x-2)^2}$.

   (c) Note that the function can be written as $y = (3x^3-7)^{-1/3}$ so you need to use the chain rule with the outer function $u^{-1/3}$ and the inner $u = 3x^3-7$. Obtain that $y' = \frac{-1}{3}(3x^3-7)^{-4/3}(9x^2) = \frac{-9x^2}{3\sqrt[3]{(3x^3-7)^4}} = \frac{-3x^2}{\sqrt[3]{(3x^3-7)^4}}$.

   (d) The function is a quotient of $f(x) = (3x^2+1)(x+4)$ and $g(x) = 5-x^2$. Using the product rule for derivative of f and the quotient rule for the derivative of the whole function, you obtain $y' = \frac{[6x(x+4)+(3x^2+1)](5-x^2)+2x(3x^2+1)(x+4)}{(5-x^2)^2}$.

   (e) Note that the function is a composite of $f(u) = u^7$ and $g(x) = x^3 + (x^3+1)^5$. However, the function $g(x)$ is the sum of x^3 and another composite function $(x^3+1)^5$ which consists of the outer function v^5 and the inner function x^3+1. Thus, the derivative of g is $3x^2+5(x^3+1)^4(3x^2) = 3x^2 + 15x^2(x^3+1)^4$ and the derivative of y is $y' = 7(x^3+(x^3+1)^5)^6(3x^2+15x^2(x^3+1)^4)$.

(f) The function is a quotient of $f(x) = \sqrt{x}(2x^3 - 5)$ and $g(x) = 3x + 2$. Using the product rule for derivative of f and the quotient rule for the derivative of the whole function, you obtain $y' = \frac{[\frac{1}{2}x^{-1/2}(2x^3-5)+\sqrt{x}6x^2](3x+2)-3\sqrt{x}(2x^3-5)}{(3x+2)^2}$.

2. (a) You can find the derivative of $f(x)$ using the quotient rule, or you can simplify $f(x)$ as follows $f(x) = x + 3 - 5x^{-1}$ and differentiate term by term. Obtain that $f'(x) = 1 + \frac{5}{x^2} = \frac{x^2+5}{x^2}$. Plugging $x = 1$ in the derivative, obtain the slope $m = 6$. Since $f(1) = -1$, find an equation of the line passing $(1, -1)$ with slope 6 to be $y = 6x - 7$.

   (b) Note that the function can be written as $f(x) = 3(1 + 2x)^{-1/2}$ so that the derivative can be found using the chain rule as $f'(x) = 3\,\frac{-1}{2}(1 + 2x)^{-3/2}(2) = \frac{-3}{(1+2x)^{3/2}}$. Thus the slope is $f'(4) = \frac{-3}{9^{3/2}} = \frac{-3}{27} = \frac{-1}{9}$. Since $f(4) = \frac{3}{3} = 1$, the tangent line is $y - 1 = \frac{-1}{9}(x - 4) \Rightarrow y = \frac{-1}{9}x + \frac{13}{9}$.

3. (a) $C(3) = \frac{5(3)}{3^2+4} = \frac{15}{13} \approx 1.15$ mg/cm^3

   (b) Find $C'(t)$ using the quotient rule to be $C'(t) = \frac{5(t^2+4)-2t(5t)}{(t^2+4)^2} = \frac{20-5t^2}{(t^2+4)^2}$. So that $C'(3) = \frac{-25}{169} \approx -.148$. Thus, the concentration is *decreasing* by .148 mg/cm^3 per hour.

   (c) $\frac{C(4)-C(2)}{4-2} = \frac{1-1.25}{2} = -.125$, thus the concentration is decreasing on average by .124 mg/cm^3 per hour between hour 2 and 4.

Derivatives of Exponential, Logarithmic and Trigonometric Functions

Derivative of the inverse function. If $f(x)$ is a one-to-one function (i.e. the graph of $f(x)$ passes the horizontal line test), then $f(x)$ has the inverse function $f^{-1}(x)$. Recall that f and f^{-1} are related by the following formulas

$$y = f^{-1}(x) \Longleftrightarrow x = f(y).$$

Also, recall that the graphs of $f^{-1}(x)$ and $f(x)$ are symmetrical with respect to line $y = x$.

Some pairs of inverse functions you encountered before are given in the table below where n is a positive integer and a is a positive real number.

f	x^2	x^n	e^x	a^x
$\mathbf{f^{-1}}$	$\sqrt{x}$	$\sqrt[n]{x}$	$\ln x$	$\log_a x$

With $y = f^{-1}(x)$, $\frac{dy}{dx}$ denotes the derivative of f^{-1} and since $x = f(y)$, $\frac{dx}{dy}$ denotes the derivative of f. Since the reciprocal of $\frac{dy}{dx}$ is $\frac{dx}{dy}$ we have that

$$(f^{-1})'(x) = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{f'(y)}.$$

Thus, the derivative of the inverse function of f is reciprocal of the derivative of f.

Another way to see this is to consider relation

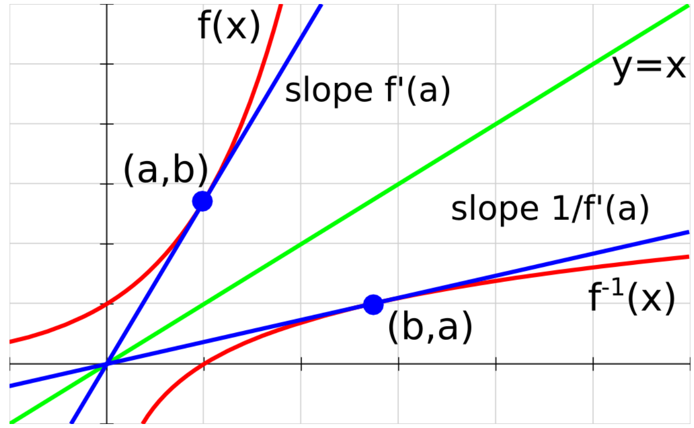
$$f(f^{-1}(x)) = x \text{ or } f^{-1}(f(x)) = x,$$

and to differentiate any of these identities. For example, differentiating $f^{-1}(f(x)) = x$ and using the chain rule for the left hand side produces

$$(f^{-1})'(f(x)) \cdot f'(x) = 1 \Longrightarrow (f^{-1})'(f(x)) = \frac{1}{f'(x)}.$$

Graphically, this rule means that

The slope of the tangent to $f^{-1}(x)$ at point (b, a) is reciprocal to the slope of the tangent to $f(x)$ at point (a, b).



Exponential Functions and their derivatives.

In a pre-calculus course you have encountered exponential function a^x of any base $a > 0$ and their inverse functions . All these functions can be considered to be a composite of e^u and $x \ln a$ since

$$a^x = e^{\ln a^x} = e^{x \ln a}$$

Thus, using the chain rule and formula for derivative of e^x, you can obtain the formula for derivative of any a^x.

The derivative of e^x can be computed by

$$\frac{de^x}{dx} = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x e^h - e^x}{h} = e^x \lim_{h\to 0} \frac{e^h - 1}{h}$$

Before you see a proof of $\lim_{h\to 0} \frac{e^h-1}{h} = 1$ in higher calculus courses, you can convince yourself that the limit $\lim_{h\to 0} \frac{e^h-1}{h}$ is 1 by evaluating the quotient $\frac{e^h-1}{h}$ at several values of h close to 0 as in the table below.

$\mathbf{h}$	0.1	0.01	0.001	0.0001
$\mathbf{\frac{e^h-1}{h}}$	1.0517	1.0050	1.0005	1.00005

This indicates that $\lim_{h\to 0} \frac{e^h-1}{h} = 1$ and so $\frac{de^x}{dx} = e^x \lim_{h\to 0} \frac{e^h-1}{h} = e^x$. Thus,

the derivative of e^x is e^x.

Example 1. Find the derivative of the following functions

(a) $y = e^{3x}$ (b) $y = x^2 e^{3x}$

Solution. (a) We can consider the function $y = e^{3x}$ as a composite of the outer function e^u and the inner function $3x$.

$y = e^{3x}$	$\Rightarrow$	$y' =$ derivative of the composite	e^{3x} $\cdot$ derivative of the outer, keep the inner unchanged	3 derivative of the inner

Thus the derivative is $y' = 3e^{3x}$.

(b) The function is a product of $f(x) = x^2$ and $g(x) = e^{3x}$. By part (a), $g'(x) = 3e^{3x}$. Since $f'(x) = 2x$, the product rule produces $y' = f'g + g'f = 2xe^{3x} + x^2(3)e^{3x} = (2x + 3x^2)e^{3x}$.

Using the formula $\frac{de^x}{dx} = e^x$ we can obtain the derivative of a^x. Recall that $y = a^x = e^{\ln a^x} = e^{x \ln a}$. In the last step we used the rule $\log_a(x^r) = r \log_a x$. Thus the function $y = a^x = e^{x \ln a}$ can be consider as a composite of e^u and $u = x \ln a$. Since $\ln a$ is a constant $u' = \ln a$ (just as in part (a) of the previous example we had $u = 3x \Rightarrow u' = 3$). Thus, the derivative of $y = a^x$ is $y' = e^{x \ln a} \ln a$. Note that $e^{x \ln a}$ is the original function $y = a^x$. Thus, $y' = a^x \ln a$ and we have that

the derivative of a^x is $a^x \ln a$.

Example 2. Find the derivative of the following functions

(a) $y = 2^{5x+7}$ (b) $y = \dfrac{3^x - 3^{-x}}{2}$

Solution. (a) Consider the function as a composite of 2^u and $u = 5x + 7$. Using the formula for a^x with $a = 2$ obtain $2^u \ln 2 = 2^{5x+7} \ln 2$ for the derivative of the outer. Since the derivative of the inner is 5, $y' = 5 \ln 2\, 2^{5x+7}$.

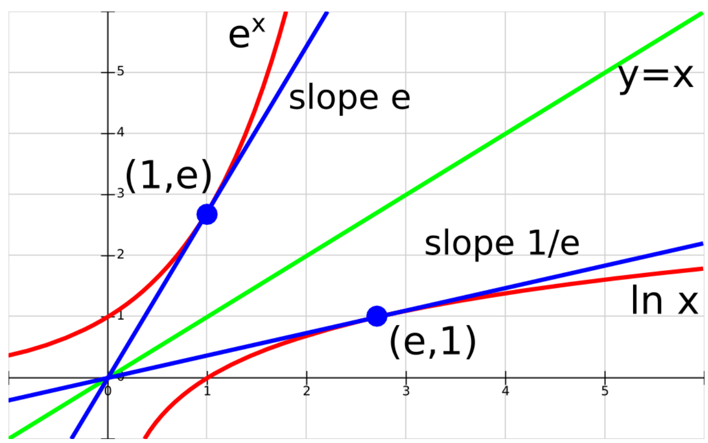
(b) Note that the function can be written as $y = \frac{1}{2}(3^x - 3^{-x})$. The derivative of the first term in the parenthesis is $3^x \ln 3$ by the formula for derivative of a^x with $a = 3$. Using the chain rule with inner function $-x$, the derivative of the second part 3^{-x} is $3^{-x} \ln 3(-1) = -3^{-x} \ln 3$. Thus $y' = \frac{1}{2}(3^x \ln 3 + 3^{-x} \ln 3) = \frac{\ln 3}{2}(3^x + 3^{-x})$.

Logarithmic function and their derivatives.

Recall that the function $\log_a x$ is the inverse function of a^x : thus $\log_a x = y \Leftrightarrow a^y = x$.

If $a = e$, the notation $\ln x$ is short for $\log_e x$ and the function $\ln x$ is called the **natural logarithm.**

The derivative of $y = \ln x$ can be obtained from derivative of the inverse function $x = e^y$. Note that the derivative x' of $x = e^y$ is $x' = e^y = x$ and consider the reciprocal:



$$y = \ln x \quad \Rightarrow \quad y' = \frac{1}{x'} = \frac{1}{e^y} = \frac{1}{x}.$$

The derivative of logarithmic function of any base can be obtained converting $\log_a$ to $\ln$ as $y = \log_a x = \frac{\ln x}{\ln a} = \ln x \frac{1}{\ln a}$ and using the formula for derivative of $\ln x$. So we have

$$\frac{d}{dx} \log_a x = \frac{1}{x} \frac{1}{\ln a} = \frac{1}{x \ln a}.$$

The derivative of $\ln x$ is $\frac{1}{x}$ and the derivative of $\log_a x$ is $\frac{1}{x \ln a}$.

To summarize,

y	e^x	a^x	$\ln x$	$\log_a x$
y′	e^x	$a^x \ln a$	$\frac{1}{x}$	$\frac{1}{x \ln a}$

Example 3. Find the derivative of the following functions

(a) $y = \ln(x^2 + 2x)$ (b) $y = \log_2(3x + 4)$

(c) $y = x \ln(x^2 + 1)$ (d) $y = \ln(x + 5e^{3x})$

Solution. (a) Using the chain rule with the outer $\ln u$ and the inner $x^2 + 2x$, you have $y' = \frac{1}{x^2+2x}(2x + 2) = \frac{2x+2}{x^2+2x} = \frac{2(x+1)}{x(x+2)}$.

(b) Using the chain rule with the outer $\log_2 u$ and the inner $3x + 4$, you have $y' = \frac{1}{\ln 2(3x+4)} 3 = \frac{3}{\ln 2(3x+4)}$.

(c) Use the product rule with $f(x) = x$ and $g(x) = \ln(x^2+1)$ and the chain rule with derivative of g with the outer $\ln u$ and the inner $x^2 + 1$. Obtain that $y' = f'g + g'f = 1 \ln(x^2 + 1) + \frac{1}{x^2+1}(2x)(x) = \ln(x^2 + 1) + \frac{2x^2}{x^2+1}$.

(d) Use the chain rule with the outer $\ln u$ and the inner $u = x + 5e^{3x}$. For derivative of the part e^{3x}, you will need to use the chain rule again to obtain $u' = 1 + 5e^{3x}(3) = 1 + 15e^{3x}$. Thus, $y' = \frac{1}{x+5e^{3x}} (1 + 15e^{3x}) = \frac{1+15e^{3x}}{x+5e^{3x}}$.

Trigonometric functions and their derivatives.

The derivative of $\sin x$ can be determined using the trigonometric identity $\sin(x+h) = \sin x \cos h + \cos x \sin h$ and calculating the limits $\lim_{h\to 0} \frac{\sin h}{h} = 1$ and $\lim_{h\to 0} \frac{\cos h-1}{h} = 0$ using tables. One obtains the formula

$$\frac{d \sin x}{dx} = \cos x.$$

Example 4. Find derivatives of the following functions.

(a) $y = \sin^3 x$ (b) $y = \sin x^3$ (c) $y = x^3 \sin x$

Solution. (a) In order to see better the inner and outer function in this composite, note that the function can be represented also as $y = (\sin x)^3$. In this representation it is more obvious that the outer function is u^3 and the inner is $\sin x$. Thus the chain rule produces $y' = 3(\sin x)^2 \cos x = 3 \sin^2 x \cos x$.

(b) This function is a composite of the outer $\sin u$ and the inner x^3. Thus the chain rule produces $y' = \cos x^3(3x^2) = 3x^2 \cos x^3$.

(c) Using the product rule with $f(x) = x^2$ and $g(x) = \sin x$ we obtain $y' = f'g + g'f = 3x^2 \sin x + \cos x(x^3) = 3x^2 \sin x + x^3 \cos x$.

The derivative of $\cos x$ can be found to be $-\sin x$. The derivatives of the remaining trigonometric functions can be obtained by expressing these functions in terms of sine or cosine. In general, you can always express a trigonometric function in terms of sine, cosine or both and then use just the following two formulas.

The derivative of $\sin x$ is $\cos x$ and the derivative of $\cos x$ is $-\sin x$.

Example 5. Find derivatives of $\tan x = \frac{\sin x}{\cos x}$ and $\sec x == \frac{1}{\cos x}$. Simplify your answers.

Solution. As $\tan x = \frac{\sin x}{\cos x}$, use the quotient rule with $f(x) = \sin x$ and $g(x) = \cos x$ to obtain $\frac{d\tan x}{dx} = \frac{\cos x \cos x - (-\sin x)\sin x}{\cos^2 x} = \frac{\cos^2 x + \sin^x x}{\cos^2 x} = \frac{1}{\cos^2 x}$ or $\sec^2 x$.

As $\sec x = (\cos x)^{-1}$, use the chain rule to obtain

$$\frac{d\sec x}{dx} = -1(\cos x)^{-2}(-\sin x) = (\cos x)^{-2}\sin x \text{ or } \sec^2 x \sin x.$$

Practice problems.

1. Find the derivative of the given functions.

(a) $y = e^{3x}(x^3 + 2x - 5)$
(b) $y = 3^{2x^2+5}$
(c) $y = x\ 5^{3x}$
(d) $y = (2x + e^{x^2})^4$
(e) $y = \frac{e^{2x}+e^{-2x}}{x^2}$
(f) $y = \ln(5x - e^{5x})$
(g) $y = \log_3(x^2 + 5)$
(h) $y = \log_2(x^2 + 7x)$
(i) $y = \sin(2x^2 + 4)$
(j) $y = x^2 \cos x^2$
(k) $y = \sin 3x \cos 5x$
(l) $y = \log_2 x + 3\sin x - xe^x$

2. Find an equation of the line tangent to the curve at the indicated point.

(a) $f(x) = \frac{e^{2x}-1}{e^{2x}+1}$ at $x = 0$.
(b) $f(x) = \ln\sqrt{2x-1}$ at $x = 1$.

3. The concentration of pollutants (in grams per liter) in a river is approximated by $C(x) = .04e^{-4x}$ where x is the number of miles downstream from a place where the measurements are taken.

(a) Determine the initial pollution and the pollution 2 miles downstream.
(b) Determine how much the concentration changed on average within the first two miles.
(c) Determine how fast the concentration changes 2 miles downstream.

Solutions.

1. (a) Use the product rule with $f(x) = e^{3x}$ and $g(x) = x^3+2x-5$ and the chain for $f'(x) = e^{3x}(3)$ so that $y' = 3e^{3x}(x^3 + 2x - 5) + (3x^2 + 2)e^{3x}$.

(b) Use the chain rule with inner $2x^2 + 5$ so that $y' = 3^{2x^2+5} \cdot \ln 3 \cdot 4x = 4x \ln 3\, 3^{2x^2+5}$.

(c) Use the product rule with $f(x) = x$ and $g(x) = 5^{3x}$ and the chain for $g'(x) = 5^{3x}\ln 3(3)$ so that $y' = 5^{3x} + 3x \ln 5\, 5^{3x}$.

(d) The chain rule with inner $2x + e^{x^2}$ and another chain rule for derivative of e^{x^2} produces $y' = 4(2x + e^{x^2})^3 \cdot (2 + e^{x^2}2x) = 8(1 + xe^{x^2})(2x + e^{x^2})^3$.

(e) The quotient rule with $f(x) = e^{2x}+e^{-2x}$ and $g(x) = x^2$ and the chain for $f'(x) = 2e^{2x}-2e^{-2x}$ produces $y' = \frac{(2e^{2x}-2e^{-2x})x^2-2x(e^{2x}+e^{-2x})}{x^4} = \frac{2x((x-1)e^{2x}-(x+1)e^{-2x})}{x^4} = \frac{2((x-1)e^{2x}-(x+1)e^{-2x})}{x^3}$.

(f) Use the chain rule with inner $5x - e^{5x}$ and another chain rule for derivative of e^{5x} so that $y' = \frac{1}{5x-e^{5x}}(5 - 5e^{5x}) = \frac{5-5e^{5x}}{5x-e^{5x}}$.

(g) The chain rule with the outer $\log_3 u$ and the inner x^2+5 produces $y' = \frac{1}{\ln 3(x^2+5)}2x = \frac{2x}{\ln 3(x^2+5)}$.

(h) The chain rule with the outer $\log_2 u$ and the inner x^2+7x produces $y' = \frac{1}{\ln 2(x^2+7x)}(2x+7) = \frac{2x+7}{\ln 2(x^2+7x)}$.

(i) The chain rule with the outer $\cos u$ and the inner $2x^2+4$ produces $y' = \cos(2x^2+4)\cdot(4x) = 4x\cos(2x^2+4)$.

(j) The product rule with $f(x) = x^2$ and $g(x) = \cos x^2$ and the chain for $g'(x) = -\sin x^2(2x)$ so that $y' = 2x\cos x^2 - \sin x^2(2x)(x^2) = 2x\cos x^2 - 2x^3\sin x^2$.

(k) The product rule with $f(x) = \sin 3x$ and $g(x) = \cos 5x$ and the chain for $f'(x) = \cos 3x(3)$ and $g'(x) = -\sin 5x(5)$ so that $y' = 3\cos 3x\cos 5x - 5\sin 5x\sin 3x$.

(l) Since the function is a sum of three terms, you can differentiate term by term. Use the product rule for the last term. Obtain $y' = \frac{1}{x\ln 2} + 3\cos x - e^x - xe^x$.

2. (a) Use the quotient rule to find $f'(x) = \frac{2e^{2x}(e^{2x}+1)-2e^{2x}(e^{2x}-1)}{(e^{2x}+1)^2}$ and evaluate it at $x=0$ to get the slope $f'(0) = \frac{2(1+1)-2(1-1)}{(1+1)^2} = \frac{4-0}{4} = 1$ so the slope is 1. Since $f(0) = \frac{1-1}{1+1} = 0$, the tangent is $y-0 = 1(x-0) \Rightarrow y = x$.

   (b) Either represent the function as $f(x) = \ln(2x-1)^{1/2} = \frac{1}{2}\ln(2x-1)$ and use a single chain rule to get $f'(x) = \frac{1}{2}\frac{1}{2x-1}(2) = \frac{1}{2x-1}$ or keep it as $f(x) = \ln(2x-1)^{1/2}$ and use two chain rules to obtain $f'(x) = \frac{1}{(2x-1)^{1/2}}\frac{1}{2}(2x-1)^{-1/2}(2) = \frac{1}{(2x-1)^{1/2}}\frac{1}{(2x-1)^{1/2}} = \frac{1}{2x-1}$. In either case $f'(1) = \frac{1}{2-1} = 1$ so the slope is 1. Since $f(1) = \ln\sqrt{2-1} = \ln 1 = 0$, the tangent line is $y-0 = 1(x-1) \Rightarrow y = x-1$.

3. (a) The initial pollution is $C(0) = 0.04$ and the pollution 2 miles downstream is $C(2) \approx 1.3\cdot 10^{-5}$ grams per liter.

   (b) The average rate of change within first two miles is $\frac{C(2)-C(0)}{2-0} \approx -.01999 \approx -.02$. Thus the concentration is decreasing on average by .02 grams per liter per mile during the first two miles.

   (c) The derivative is $C'(x) = .04e^{-4x}(-4) = -.16e^{-4x}$ so that $C'(2) = -5.37\cdot 10^{-5}$, thus the concentration is decreasing by .0000537 grams per liter per mile 2 miles downstream.