Fundamentals of Calculus Lia Vas

# Review for Exam 2

- 1. Finding Derivative. Find the derivative  $\frac{dy}{dx}$  of the given function or relation.
  - (a)  $y = x e^{3x}$ (b)  $y = \log_2(x^2 + 7x)$ (c)  $x^2 + xy^4 = 6$ (d)  $y = x^2 \cos(x^2)$ (e)  $y = (6x^2 + 5)^{10}$ (f)  $y = \frac{x^2 + 3x}{5x - 2}$ (g)  $x^3 + 12xy = y^3$ (h)  $y = \sin(2x^2 + 4)$ (i)  $y = \sin 3x \cos 5x$ (j)  $y = \frac{1}{\sqrt[3]{3x^3 - 7}}$ (k)  $y = 2e^{x^3 + 2x - 5}$ (l)  $y = 3^{2x^2 + 5}$ (m)  $y = \ln(5x - x^2)$ (n)  $xe^y + x^2 = y^2$ (o)  $y = \log_2 x + 3 \sin x - xe^x$
- 2. **Tangent Line.** Find an equation of the line tangent to the graph of the given equation at the indicated point.
  - (a)  $x^2 + y^2 = 13$ , (3,2) (b)  $x \ln y = 2x^3 - 2y$ , (1,1) (c)  $y = \ln(2x - 1)$ , x = 1(d)  $x^2 + y^2 = e^y$ , (1,0)

# 3. Applications of Derivative.

- (a) The concentration of a certain medication in a patient's bloodstream (in mg per cm<sup>3</sup>) is given by  $C(t) = \frac{5t}{t^2+4}$ , where t is the number of hours after the medication has been administered. (i) Determine the concentration 3 hours after the medication is administered. (ii) Determine how fast the concentration changes 3 hours after the medication is administered. (iii) Determine how fast the concentration changes on average between 2nd and 4th hour.
- (b) The concentration of pollutants (in grams per liter) in a river is approximated by  $C(x) = .04e^{-4x}$  where x is the number of miles downstream from a place where the measurements are taken. (i) Determine the initial pollution and the pollution 2 miles downstream. (ii) Determine how much the concentration changes on average within the first two miles. (iii) Determine how fast the concentration changes 2 miles downstream.

### 4. Related Rates.

- (a) Water leaking onto a floor creates a circular puddle with an area that increases at the rate of 3 square centimeters per minute. Determine how fast the radius of the puddle increases when the radius is 10 cm. Recall that the formula for the area of a circle is  $A = r^2 \pi$ .
- (b) Assume that the number of bass in the pond is related to the level of polychlorinated biphenyls (PCBs, a group of industrial chemicals used in plasticizers, fire retardants and other materials) in the pond. The bass population is modeled by  $y = \frac{2500}{1+x}$  where x represents the PCB level in parts per million (ppm) and y represents the number of bass in the pond. If the level of PCBs is increasing at the rate of 40 ppm per year, find the rate at which is the number of bass changing when there are 100 bass in the pond.
- (c) In a physics experiment, the temperature T (in degrees Fahrenheit) and pressure P (in kilo Pascals) are related by  $P^3 = 400T^2$  and both T and P depend on time t in seconds. If the temperature is increasing by 0.2 degrees per second, find the rate of increase of pressure at the time when the temperature is 50 degrees Fahrenheit.
- (d) Water is leaking from a hole on the ceiling next to the wall. Sliding down the wall, the water is creating a semi-circular puddle on the floor next to the wall. The puddle is growing in surface area at a rate of 10 square inches per minute. Determine how rapidly the radius of the puddle is growing at the moment when the area of the puddle is 100 square inches.
- (e) A 20-foot ladder is leaning against the wall. If the base of the ladder is sliding away from the wall at the rate of 3 feet per second, find the rate at which the top of the ladder is sliding down when the top of the ladder is 8 feet from the ground.

## Solutions

Some solutions below contain only the final answer. See <u>the class handouts</u> for more step by step solutions.

1. Derivative.

(a) 
$$y' = e^{3x} + 3xe^{3x}$$
.

(b) 
$$y' = \frac{1}{\ln 2(x^2 + 7x)} \cdot (2x + 7) = \frac{2x + 7}{\ln 2(x^2 + 7x)}$$

(c) Implicit differentiation. 
$$2x + y^4 + 4xy^3y' = 0 \Rightarrow 4xy^3y' = -2x - y^4 \Rightarrow y' = \frac{-2x - y^4}{4xy^3}$$

- (d)  $y' = 2x \cos x^2 2x^3 \sin x^2$
- (e)  $y' = 10(6x^2 + 5)^9(12x) = 120x(6x^2 + 5)^9.$
- (f)  $y' = \frac{(2x+3)(5x-2)-5(x^2+3x)}{(5x-2)^2}$ .
- (g) Implicit differentiation.  $x^3 + 12xy = y^3 \Rightarrow 3x^2 + 12y + 12xy' = 3y^2y' \Rightarrow 3x^2 + 12y = 3y^2y' 12xy' \Rightarrow 3x^2 + 12y = (3y^2 12x)y' \Rightarrow y' = \frac{3x^2 + 12y}{3y^2 12x} = \frac{3(x^2 + 4y)}{3(y^2 4x)} = \frac{x^2 + 4y}{y^2 4x}$

(h) 
$$y' = \cos(2x^2 + 4) \cdot 4x = 4x\cos(2x^2 + 4)$$

(i)  $y' = \cos 3x \cdot 3 \cdot \cos 5x + (-\sin 5x) \cdot 5 \cdot \sin 3x = 3\cos 3x \cos 5x - 5\sin 5x \sin 3x$ 

(j) 
$$y = (3x^3 - 7)^{-1/3} \Rightarrow y' = \frac{-1}{3}(3x^3 - 7)^{-4/3}(9x^2) = \frac{-9x^2}{3\sqrt[3]{(3x^3 - 7)^4}} = \frac{-3x^2}{\sqrt[3]{(3x^3 - 7)^4}}$$
  
(k)  $y' = 2e^{x^3 + 2x - 5} \cdot (3x^2 + 2) = 2(3x^2 + 2)e^{x^3 + 2x - 5}$   
(l)  $y' = 3^{2x^2 + 5} \cdot \ln 3 \cdot 4x = 4 \ln 3x 3^{2x^2 + 5}$   
(m)  $y' = \frac{1}{5x - x^2} \cdot (5 - 2x) = \frac{5 - 2x}{5x - x^2}$   
(n) Implicit differentiation.  $xe^y + x^2 = y^2 \Rightarrow 1 \cdot e^y + e^y y' \cdot x + 2x = 2y \cdot y' \Rightarrow e^y + 2x = 2yy' - xy'e^y \Rightarrow e^y + 2x = (2y - xe^y)y' \ y' = \frac{e^y + 2x}{2y - xe^y}$   
(o)  $y' = \frac{1}{x \ln 2} + 3\cos x - e^x - xe^x$ .

2. Tangent line.

- (a)  $x^2 + y^2 = 13 \Rightarrow 2x + 2y \cdot y' = 0 \Rightarrow 2yy' = -2x \Rightarrow y' = \frac{-x}{y}$ . At point (3,2), the derivative is  $y' = \frac{-3}{2}$ . So, the tangent line is  $y 2 = \frac{-3}{2}(x 3) \Rightarrow y = \frac{-3}{2}x + \frac{13}{2}$ .
- (b)  $x \ln y = 2x^3 2y \Rightarrow \ln y + \frac{1}{y} \cdot y' \cdot x = 6x^2 2y' \Rightarrow 2y' + \frac{x}{y}y' = 6x^2 \ln y \Rightarrow (2 + \frac{x}{y})y' = 6x^2 \ln y \Rightarrow y' = \frac{6x^2 \ln y}{2 + \frac{x}{y}}$ . At point (1, 1),  $y' = \frac{6 0}{2 + 1} = 2$ , so  $y 1 = 2(x 1) \Rightarrow y = 2x 1$ .
- (c)  $y' = \frac{1}{2x-1} \cdot 2 = \frac{2}{2x-1}$ . When x = 1, the slope is  $y'(1) = \frac{2}{2(1)-1} = 2$ . Since  $f(1) = \ln(2(1)-1) = \ln 1 = 0$ , the tangent line is  $y 0 = 2(x 1) \Rightarrow y = 2x 2$ .
- (d)  $x^2 + y^2 = e^y \Rightarrow 2x + 2y \cdot y' = e^y y' \Rightarrow 2x = e^y y' 2yy' \Rightarrow 2x = (e^y 2y)y' \Rightarrow y' = \frac{2x}{e^y 2y}$ . At point (1,0), the derivative is  $y' = \frac{2}{1-0} = 2$ . The tangent line is  $y 0 = 2(x 1) \Rightarrow y = 2x 2$ .
- 3. Applications of Derivative.
  - (a) (i)  $C(3) = 1.15 \text{ mg/cm}^3$  (ii)  $C'(t) = \frac{5(t^2+4)-2t\cdot5t}{(t^2+4)^2} \Rightarrow C'(3) = \frac{65-90}{13^2} \approx -.148$  thus, the concentration is decreasing by .148 mg/cm<sup>3</sup> per hour. (iii)  $\frac{C(4)-C(2)}{4-2} = \frac{1-1.25}{4-2} = -.125$ , thus the concentration is decreasing on average by .125 mg/cm<sup>3</sup> per hour between hour 2 and hour 4.
  - (b) (i) C(0) = .04 and  $C(2) = 1.3 \cdot 10^{-5}$  grams per liter (ii)  $\frac{C(2)-C(0)}{2-0} \approx -.01999 \approx -.02$ . Thus the concentration is decreasing on average by .02 grams per liter per mile during the first two miles. (iii)  $C'(x) = .04e^{-4x}(-4) = -.16e^{-4x} \Rightarrow C'(2) = -5.37 \cdot 10^{-5}$ , thus the concentration is decreasing by .0000537 grams per liter per mile 2 miles downstream.

#### 4. Related Rates.

- (a)  $A = r^2 \pi \Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$ . As  $\frac{dA}{dt} = 3 \text{ cm}^2/\text{min}$  and r = 10 cm,  $3 = 2\pi (10) \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{3}{2\pi (10)} \approx 0.048 \text{ cm/min}$ .
- (b)  $y = \frac{2500}{1+x} = 2500(1+x)^{-1} \Rightarrow \frac{dy}{dt} = -2500(1+x)^{-2}\frac{dx}{dt} = \frac{-2500\frac{dx}{dt}}{(1+x)^2}$ . As  $y = 100, 100 = \frac{2500}{1+x} \Rightarrow 100(1+x) = 2500 \Rightarrow x+1 = 25 \Rightarrow x = 24$ . As  $\frac{dx}{dt} = 40$  ppm/ year,  $\frac{dy}{dt} = \frac{-2500\frac{dx}{dt}}{(1+x)^2} = \frac{-2500(40)}{(1+24)^2} = \frac{-4000}{25} = -160$  bass per year. Thus, the number of bass is decreasing by 160 each year.
- (c)  $P^3 = 400T^2 \Rightarrow 3P^2 \cdot \frac{dP}{dt} = 400 \cdot 2T \cdot \frac{dT}{dt} \Rightarrow 3P^2 \cdot \frac{dP}{dt} = 800 \cdot T \cdot \frac{dT}{dt}$ . As  $T = 50, P^3 = 400(50)^2 \Rightarrow P^3 = 1000000 \Rightarrow P = 100$ . Since  $\frac{dT}{dt} = 0.2, \ 3(100)^2 \frac{dP}{dt} = 800 \cdot 50 \cdot 0.2 \Rightarrow 30000 \frac{dP}{dt} = 8000 \Rightarrow \frac{dP}{dt} = \frac{8000}{30000} = \frac{8}{30} \approx 0.267$  kPa per second.
- (d)  $A = \frac{1}{2}r^2\pi \Rightarrow \frac{dA}{dt} = \frac{1}{2}2r\pi\frac{dr}{dt} = r\pi\frac{dr}{dt}$ . As  $\frac{dA}{dt} = 10$  in<sup>2</sup>/min and A = 100 in<sup>2</sup>,  $100 = \frac{1}{2}r^2\pi \Rightarrow 100 = \frac{1}{2}r^2\pi \Rightarrow r^2 = \frac{200}{\pi} \Rightarrow r \approx 7.98$ . Plug the given values and solve for  $\frac{dr}{dt}$  to get  $\frac{dr}{dt} \approx \frac{10}{7.98\pi} \approx 0.40$  inches per minute.
- (e) Decreasing by 6.87 ft per sec. See the handout for detailed solutions.