First Order Differential Equations

Introduction to and Classifications of Differential Equations

A differential equation is an equation in an unknown function that contains one or more derivatives of the unknown function.

The **order** of a differential equation is the order of the highest derivative in the equation. Differential equations can be classified based on the order:

- First order just the first derivative appear in the equation. For example, $y'^2 + y = \sin x$.
- Higher order derivatives higher than the first appear in the equation. For example, $y'' + \sin(xy) = 0$ is the equation of the second order.

The first order differential equations have the general form F(y', y, x) = 0. When possible, solve the equation for y' to obtain the form

$$y' = f(x, y)$$

The general form of the n-th order differential equation is

$$F(y^{(n)}, y^{(n-1)}, \dots, y', y, x) = 0.$$

If the function F is a linear function of the variables $y, y', \ldots, y^{(n)}$, i.e. if the above equation is of the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_0(x)y = g(x)$$

then it is said to be **linear**. If it is not linear, it is said to be **nonlinear**. For example, the equation $xy'' + \sin xy = \ln x$ is linear while the equation $xy'' + \sin xy^2 = \ln x$ is nonlinear.

A linear differential equation of the first order has the form

$$a(x)y' + b(x)y = g(x)$$

Note: if a(x) is a nonzero function, when the equation is divided by it, one obtains $y' + \frac{b(x)}{a(x)}y = \frac{g(x)}{a(x)}$. Using $P = \frac{b}{a}$ and $Q = \frac{g}{a}$, we obtain the form y' + P(x)y = Q(x) which you may remember from Calculus 2.

Differential equations can be classified also based on the number of functions that are involved.

- A single differential equation there is a single unknown function. For example, $\frac{dy}{dt} + 4y = \ln t$.
- A system of differential equations there is more than one unknown function. For example, $\frac{dx}{dt} + 4y = \ln t$ together with $\frac{dy}{dt} + 4x = e^t$.

Based on the type of the solution, differential equations can be classified as follows.

- Ordinary differential equation is an equation in an unknown function of a single variable. For example, $\frac{dy}{dx} + \sin y = \ln x$, $\frac{d^2P}{dt^2} + P = te^t$, etc.
- **Partial** differential equation is an equation in an unknown function of more than one variable. For example $\frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} = \sin x + \ln t$, $y_{xx} + y_t = te^t$, etc.

The function y is a **solution** of the differential equation $F(y^{(n)}, y^{(n-1)}, \ldots, y', y, x) = 0$, if the equation is satisfied for every value of variable x when y and all its derivatives $y', \ldots, y^{(n)}$ are substituted into the equation. The function y is a solution on an interval (a, b) if y and its derivatives satisfy the equation for every value of x on the interval (a, b).

For example, the function $y = e^{2x}$ is a solution of the second order equation y'' + 2y' - 8y = 0 since the derivatives $y' = 2e^{2x}$ and $y'' = 4e^{2x}$ yield an identity $4e^{2x} + 4e^{2x} - 8e^{2x} = (4+4-8)e^{2x} = 0e^{2x} = 0$ when plugged in the equation. Note that this is identity does not depend on a specific value of x.

Convince yourself that functions of the form $y = c_1 e^{2x}$ are also solutions of the differential equation y'' + 2y' - 8y = 0 for every value of constant c_1 . This illustrates that the solution of a differential equation does not have to be unique. Moreover, the functions of the form $y = c_2 e^{-4x}$ are also solutions of the equation y'' + 2y' - 8y = 0.

The general solution of a differential equation is a family of all functions that satisfy the equation. We shall see later that the general solution of y'' + 2y' - 8y = 0 is of the form $y = c_1 e^{2x} + c_2 e^{-4x}$.

The general solution of a differential equation of the first order depends on a single constant and the general solution of a differential equation of the n-th degree depends on n constants.

In many applications, a solution passing a certain point or satisfying a certain condition may be more relevant than the general solution. For a first order differential equation, the condition $y(x_0) = y_0$ is called an **initial condition** and the differential equation

$$y' = f(x, y)$$
 together with the initial condition $y(x_0) = y_0$

is called an **initial value problem**. The solution that satisfies the equation and the condition $y(x_0) = y_0$ is called the **particular solution**.

For example, the function y = 2x + c is the general solution of the differential equation y' = 2. If the condition y(0) = 5 is considered together with the equation, then the solution y = 2x + c does not satisfy it for every, but only for one value of constant c. Plugging the initial condition values in the general solution, we obtain that 5 = 2(0) + c and so c = 5. Thus, y = 2x + 5 is the **particular solution** of this initial value problem.

Practice Problems.

- 1. Consider the equation $y' + 3x^2y = 6x^2$.
 - (a) Classify the equation based on the order and linearity.
 - (b) Check if $y = x^2$ and $y = 2 + e^{-x^3}$ are solutions of the equation.
- 2. Show that $y = \frac{1}{x+c}$ is a solution of differential equation $y' = -y^2$. Then, find a particular solution that satisfies the initial condition $y(0) = \frac{1}{4}$.

- 3. Classify the equation y'' 3y' + 2y = 0 based on the order, linearity and type of unknown function and show that $y = ce^{2x}$ is a solution of this differential equation for every constant c.
- 4. Show that $y = c_1 e^x + c_2 e^{2x}$ is a solution of differential equation y'' 3y' + 2y = 0 (it is the general solution in fact). Then, find the constants c_1 and c_2 such that the initial conditions y(0) = 2 and y'(0) = 5 are satisfied.
- 5. Show that $y = c_1 \cos 2x + c_2 \sin 2x$ is a solution of differential equation y'' + 4y = 0 (it is the general solution in fact). Then, find the constants c_1 and c_2 such that the boundary conditions y(0) = 2 and $y(\frac{\pi}{4}) = 5$ are satisfied.
- 6. Determine all values of r for which

$$6\frac{d^2y}{dt^2} - 7\frac{dy}{dt} - 3y = 0$$

has a solution of the form $y = e^{rt}$.

- 7. Find value of constants A, B and C for which the function $y = Ax^2 + Bx + C$ is the solution of the equation $y'' y' + 4y = 8x^2$.
- 8. Find value of constant A for which the function $y = Ae^{3x}$ is the solution of the equation $y'' 3y' + 2y = 6e^{3x}$.
- 9. Classify the following differential equations based on the order, linearity and type of unknown function.
 - a) The study of electrical circuits Kirchhoff's Laws (Physics):

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

where L, C, R are constants and E(t) is a given function.

b) Michaelis-Menten equation that describes the rate of change of plasma drug concentration C after an intravenous bolus injection (Pharmacy):

$$-\frac{dC}{dt} = \frac{v_{max}C}{k+C}$$

where v_{max} is the maximum velocity of reaction and k is the rate constant.

c) Wave equation – a model of the vibrating strings and propagation of waves (Physics):

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where c is a constant.

d) A model of the learning of a task (Psychology):

$$\frac{y'}{\sqrt{y^3(1-y)^3}} = \frac{2p}{\sqrt{n}}$$

where p and n are constants.

Solutions.

1. (a) Linear first order ordinary differential equation.

(b) $y = x^2 \Rightarrow y' = 2x$. Plug the function and its derivative into the equation $y' + 3x^2y = 6x^2 \Rightarrow 2x + 3x^2(x^2) = 6x^2 \Rightarrow 2x + 3x^4 = 6x^2$. This equation does not hold for every value of x (for example if x = 1 the equation false identity 2 + 3 = 6) so $y = x^2$ is not a solution of the given equation.

 $y = 2 + e^{-x^3} \Rightarrow y' = -3x^2e^{-x^3}$. Plug the function and its derivative into the equation $y' + 3x^2y = 6x^2 \Rightarrow -3x^2e^{-x^3} + 3x^2(2 + e^{-x^3}) = 6x^2 \Rightarrow -3x^2e^{-x^3} + 6x^2 + 3x^2e^{-x^3} = 6x^2 \Rightarrow 6x^2 = 6x^2$. This identity holds for every x so the given function is a solution of the equation.

2. $y = \frac{1}{x+c} \Rightarrow y' = \frac{-1}{(x+c)^2}$. Plug the function and its derivative into the equation $y' = -y^2 \Rightarrow \frac{-1}{(x+c)^2} = -\left(\frac{1}{x+c}\right) \Rightarrow \frac{-1}{(x+c)^2} = \frac{-1}{(x+c)^2}$. This identity holds for every x so the given function is a solution of the equation.

To find c, plug that x = 0 and $y = \frac{1}{4}$ into $y = \frac{1}{x+c} \Rightarrow \frac{1}{4} = \frac{1}{0+c} \Rightarrow c = 4$.

- 3. Linear, second order ordinary differential equation. $y = ce^{2x} \Rightarrow y' = 2ce^{2x} \Rightarrow y'' = 4ce^{2x}$. Plug into the equation $y'' 3y' + 2y = 0 \Rightarrow 4ce^{2x} 6ce^{2x} + 2ce^{2x} = 0 \Rightarrow (4 6 + 2)ce^{2x} = 0 \Rightarrow 0 = 0$. The given function is a solution of the equation.
- 4. First part is similar to the previous problem. Use the initial conditions to get $2 = c_1 e^0 + c_2 e^0$ and $5 = c_1 e^0 + 2c_2 e^0 \Rightarrow c_1 + c_2 = 2$ and $c_1 + 2c_2 = 5$. Solve for c_1 and c_2 and get $c_1 = -1$, $c_2 = 3$.
- 5. Using the boundary conditions get $2 = c_1 \cos 0 + c_2 \sin 0 = c_1 \Rightarrow c_1 = 2$, and $5 = c_1 \cos 2\frac{\pi}{4} + c_2 \sin 2\frac{\pi}{4} = c_2 \Rightarrow c_2 = 5$.
- 6. If $y = e^{rt}$, then $y' = re^{rt}$, and $y'' = r^2 e^{rt}$. Plugging that into the equation 6y'' 7y' 3y = 0 gives you $6r^2 e^{rt} 7re^{rt} 3e^{rt} = 0$. Factor e^{rt} . Get $e^{rt}(6r^2 7r 3) = 0$. Since e^{rt} is larger than zero for any value of t, $6r^2 7r 3$ has to be zero. This happens just when r = -1/3 and r = 3/2. Thus, $y = e^{rt}$ is a solution for r = -1/3 and r = 3/2.
- 7. Find the derivatives of $y = Ax^2 + Bx + C$ to be y' = 2Ax + B and y'' = 2A and plug them into the equation $y'' - y' + 4y = 8x^2$ to get $2A - 2Ax - B + 4Ax^2 + 4Bx + 4C = 8x^2$. Note that both sides are polynomial functions which need to be equal for *all* values of x. This is possible just if the coefficient of polynomials with each term are equal. Thus,
 - equating the terms with x^2 obtain that $4A = 8 \Rightarrow A = 2$.
 - Equating the terms with x obtain that -2A+4B = 0. Since $A = 2, -4+4B = 0 \Rightarrow B = 1$.
 - Equating the terms with no x obtain that $2A B + 4C = 0 \Rightarrow 4 1 + 4C = 0 \Rightarrow C = \frac{-3}{4}$.

Thus, $y = 2x^2 + x - \frac{3}{4}$ is a solution of differential equation.

8. Find the derivatives of $y = Ae^{3x}$ to be $y' = 3Ae^{3x}$ and $y'' = 9Ae^{3x}$ and substitute them into the equation $y'' - 3y' + 2y = 6e^{3x}$ to get

$$9Ae^{3x} - 9Ae^{3x} + 2Ae^{3x} = 6e^{3x} \Rightarrow 2Ae^{3x} = 6e^{3x} \Rightarrow 2A = 6 \Rightarrow A = 3$$

Thus, $y = 3e^{3x}$ is a solution of differential equation.

9. a) Linear, second order ordinary differential equation. b) Nonlinear, first order ordinary differential equation. c) Linear, second order partial differential equation. d) Nonlinear, first order ordinary differential equation.

Separable Differential Equations

The first order differential equation F(y', y, x) = 0 is **separable** if we can separate the variables x and y. Every separable differential equation can be written in a form

$$P(x) + Q(y)\frac{dy}{dx} = 0$$

or alternatively (if you solve for $\frac{dy}{dx}$ and rename the functions so that p = -P and $q = \frac{1}{Q}$), as

$$\frac{dy}{dx} = p(x)q(y)$$

To solve a separable differential equation,

- If the derivative is in the form y' write it as $\frac{dy}{dx}$.
- Rewrite the equation so that the left side has just one, and the right side just the other variable.

$$P(x)dx + Q(y)dy = 0$$
 giving you $Q(y)dy = -P(x)dx$

- Integrate both sides.
- Solve for the dependent variable y if possible.

We illustrate this method in the following examples.

Example 1. Find the general solution of the differential equation y' = 2x + 3. Then find the solution with y(0) = 5.

Solution. Separating the variables in $\frac{dy}{dx} = 2x + 3$ produces dy = (2x + 3)dx.

Integrating both sides, we have that
$$\int dy = \int (2x+3)dx \Rightarrow y = x^2 + 3x + c$$
.

Thus, the general solution is a family of parabolas of the form $y = x^2 + 3x + c$.

Considering the initial condition y(0) = 5, we have that $5 = 0^2 + 3(0) + c \Rightarrow 5 = c$. Hence, the parabola $y = x^2 + 3x + 5$ is the particular solution.

The next example has a bit more of "separating" present.

Example 2. Find the general solution of the differential equation y' = 2y. Discuss the nature of the general solution.

Solution. Writing y' as $\frac{dy}{dx}$ produces $\frac{dy}{dx} = 2y$. Divide by y and multiply by dx to obtain the variables separated and have that $\frac{dy}{y} = 2dx$.

Integrate both sides to have

$$\ln|y| = 2x + c$$

Solving for |y| produces $|y| = e^{2x+c}$ so that $y = \pm e^{2x+c}$. Note that e^{2x+c} is equal to $e^{2x}e^c$. Thus, denoting $\pm e^c$ by C eliminates the \pm (as well as the absolute value) so that we can write the general solution as

$$y = Ce^{2x}.$$



This is a family of exponential functions, increasing and positive if C > 0 and decreasing and negative if C < 0. If C = 0, the solution is y = 0.

The previous examples leads us to one of the most frequently occurring differential equations because it models the situation when the rate of change is proportional to the size of the quantity considered. If y denotes the size of the quantity considered at time x and k denotes the proportionality constant, the highlighted sentence translates to the following equation.

$$y' = ky$$

Example 3. Find the general solution of y' = ky where k is a parameter.

Solution. Follow the steps of the previous problem to have that

$$\frac{dy}{dx} = ky \Rightarrow \frac{dy}{y} = kdx \Rightarrow \ln|y| = kx + c \Rightarrow |y| = e^{kx+c} = e^{kx}e^c \Rightarrow y = \pm e^c e^{kx} \Rightarrow y = Ce^{kx}$$

where C again stands for $\pm e^c$. Thus, the solutions are exponential functions. Note that C corresponds to the initial size of y since $y(0) = Ce^0 = C$. If we denote it by y_0 , we have the familiar format

 $y = y_0 e^{kx}$

of the **exponential growth** for k > 0 or **exponential decay** for k < 0. If y_0 is positive, the solutions are increasing exponential functions for k > 0 and decreasing exponential functions for k < 0.



Practice Problems.

- 1. Find the general solution of the following differential equations. In parts (a) and (b), sketch a graph of the general solution.
 - (a) y'x = y(b) y'y = -x(c) $y' = 3x^2y$ (d) y' = x(y+1)(e) $y' = y^2xe^{2x}$
- 2. Find the solution of the differential equation that satisfies the given initial condition.
 - (a) y' = xy, y(0) = 5(b) $y' = \sqrt{4x+8}$, y(-2) = 3(c) $y' = \frac{xy}{x^2+1}$, y(0) = 2(d) $y' = 3y\sqrt{5-2x}$, $y(\frac{5}{2}) = 3$
- 3. Free fall, no friction. Recall that the velocity v is the derivative $\frac{dx}{dt}$ of the distance traveled x and that the acceleration a is the derivative $\frac{dv}{dt}$ of the velocity v. A differential equation describing free fall with no friction is obtained by equating the total force $F = ma = m\frac{d^2x}{dt^2}$ with the gravitational force mg. Thus,

$$m\frac{d^2x}{dt^2} = mg$$

Find the function describing the distance from the initial position x at time t if there is no initial velocity. Note that using that $v = \frac{dx}{dt}$, the second order equation above can be reduced to two separable first order equations:

$$\frac{dv}{dt} = g$$
, with $v(0) = 0$ and $\frac{dx}{dt} = v$ with $x(0) = 0$.

Note that here we treated a second order differential equation as a system of two first order differential equations. We shall later see that every differential equation of order n can be reduced to a system of n first order differential equations.

Solutions.

1. (a) The equation y'x = y is separable. Writing

y' as $\frac{dy}{dx}$ and separating the variables produces $\frac{dy}{y} = \frac{dx}{x}$. Integrate both sides $\int \frac{dy}{y} = \int \frac{dx}{x} \Rightarrow \ln |y| = \ln |x| + c \Rightarrow |y| = e^{\ln |x| + c} = e^{\ln |x|}e^c = |x|e^c \Rightarrow y = \pm e^c x$. Replacing the constant $\pm e^c$ by C enables you to get rid of the absolute values and obtain the general solution in the form y = Cx. Thus, the solutions are lines passing the origin.



(b) The equation y'y = -x is separable. Writing y' as $\frac{dy}{dx}$ and separating the variables gives you ydy = -xdx. Integrate both sides $\int ydy = \int -xdx \Rightarrow \frac{y^2}{2} = \frac{-x^2}{2} + c \Rightarrow y^2 = -x^2 + 2c$.

Since 2c is a constant, you can refer to it as c again. Thus, $y^2 = -x^2 + c$. Solve for y and get $y = \pm \sqrt{c - x^2}$. You can also note that the equation $y^2 = -x^2 + c$ is more telling in the form $x^2 + y^2 = c$. Thus, if c < 0 no (x, y) values satisfy the equation. If c > 0, say $c = C^2$, then the equation $x^2 + y^2 = C^2$ is an equation of the circle centered at the origin of radius C.



- (c) $y' = 3x^2y \Rightarrow \frac{dy}{dx} = 3x^2y \Rightarrow \frac{dy}{y} = 3x^2dx \Rightarrow \ln|y| = x^3 + c \Rightarrow |y| = e^{x^3 + c} = e^{x^3}e^c$. Putting $C = \pm e^c$, we have that $y = Ce^{x^3}$. Careful: $y = e^{x^3 + c}$ is not equal to $y = e^{x^3} + C$.
- (d) $y' = x(y+1) \Rightarrow \frac{dy}{dx} = x(y+1) \Rightarrow \frac{dy}{y+1} = xdx \Rightarrow \ln|y+1| = \frac{x^2}{2} + c \Rightarrow |y+1| = e^{x^2/2+c} = e^{x^2/2}e^c$. Putting $C = \pm e^c$, you have that $y+1 = Ce^{x^2/2} \Rightarrow y = Ce^{x^2/2} 1$.
- (e) $y' = y^2 x e^{2x} \Rightarrow \frac{dy}{dx} = y^2 x e^{2x} \Rightarrow \frac{dy}{y^2} = x e^{2x} dx$. Integrate the equation. Get $\frac{-1}{y} = \int x e^{2x} dx$. Use the integration by parts with u = x and $dv = e^{2x} dx$ for this integral so that $v = \frac{1}{2} e^{2x}$. Thus,

$$\frac{-1}{y} = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}dx \implies \frac{-1}{y} = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c \implies y = \frac{-1}{\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c}$$

The final answer can also be written as $y = \frac{1}{\frac{-1}{2}xe^{2x} + \frac{1}{4}e^{2x} + c}$.

- 2. (a) $y' = xy \Rightarrow \frac{dy}{dx} = xy \Rightarrow \frac{dy}{y} = xdx \Rightarrow \int \frac{dy}{y} = \int xdx \Rightarrow \ln|y| = \frac{x^2}{2} + c \Rightarrow |y| = e^{x^2/2+c} = e^{x^2/2}e^c$. Thus $y = \pm e^c e^{x^2/2} = Ce^{x^2/2}$. Careful: $y = e^{x^2/2+c}$ is not equal to $y = e^{x^2/2} + C$. Using the initial condition x = 0, y = 5, in the general solution $y = Ce^{x^2/2}$, obtain that $5 = Ce^0 \Rightarrow C = 5$. So, the particular solution is $y = 5e^{x^2/2}$.
 - (b) $y' = \sqrt{4x+8} \Rightarrow dy = \sqrt{4x+8}dx \Rightarrow y = \int \sqrt{4x+8}dx$. Use the substitution u = 4x+8 to get $y = \frac{1}{6}(4x+8)^{3/2} + c$. Using the initial condition x = -2, y = 3, in the general solution, obtain that $3 = 0 + c \Rightarrow c = 3$. So, the particular solution is $y = \frac{1}{6}(4x+8)^{3/2} + 3$.
 - (c) $y' = \frac{xy}{x^2+1} \Rightarrow \frac{dy}{y} = \frac{xdx}{x^2+1} \Rightarrow \ln|y| = \int \frac{xdx}{x^2+1}$. Use the substitution $u = x^2 + 1$ for this last integral. Obtain that $\ln|y| = \frac{1}{2}\ln(x^2+1) + c$. Note that $x^2 + 1$ is positive, so no absolute value is needed on the right side. Thus, $|y| = e^{\frac{1}{2}\ln(x^2+1)+c} \Rightarrow y = \pm e^{\ln(x^2+1)^{1/2}}e^c = Ce^{\ln(x^2+1)^{1/2}} = C(x^2+1)^{1/2} = C\sqrt{x^2+1}$. Using that y = 2 when x = 0, obtain that $2 = C\sqrt{1} \Rightarrow C = 2$. So, the particular solution is $y = 2\sqrt{x^2+1}$.
 - (d) $y' = 3y\sqrt{5-2x} \Rightarrow \frac{dy}{y} = 3\sqrt{5-2x}dx$. Use substitution with u = 5 2x for the antiderivative of the function on the right side. After integrating both sides obtain that $\ln|y| = \frac{-3}{2}\frac{2}{3}(5-2x)^{3/2} + c = -(5-2x)^{3/2} + c \Rightarrow y = \pm e^{-(5-2x)^{3/2}+c} = \pm e^{-(5-2x)^{3/2}}e^c = Ce^{-(5-2x)^{3/2}}$. Using that $y(\frac{5}{2}) = 3$, we have that $3 = Ce^0 \Rightarrow C = 3$. So, the particular solution is $y = 3e^{-(5-2x)^{3/2}}$ or $y = 3e^{-\sqrt{(5-2x)^3}}$.
- 3. Solve $\frac{dv}{dt} = g$ first. Separate to have dv = gdt so that v = gt + c. Using the initial condition v(0) = 0, get v = gt. Then solve $\frac{dx}{dt} = gt$. Separate to have dx = gtdt so that $x = \frac{g}{2}t^2 + c$. Use the initial condition x(0) = 0 to get $x = \frac{g}{2}t^2$.

Linear Differential Equation

A first order differential equation is **linear** if it can be written in the form a(x)y' + b(x)y = c(x). Note that if a(x) = 0, the equation is not differential. So, let us assume that a(x) is not zero. In this case we can **divide the equation with** a(x) and obtain the form $y' + \frac{b(x)}{a(x)}y = \frac{c(x)}{a(x)}$. If we denote $\frac{b(x)}{a(x)}$ by P(x) and $\frac{c(x)}{a(x)}$ by Q(x), the equation becomes

$$y' + P(x)y = Q(x).$$

This differential equation can be solved following the steps below.

- 1. Write the equation in the form y' + P(x)y = Q(x).
- 2. Find the integrating factor $I(x) = e^{\int P(x)dx}$ and multiply both sides of the equation with it.
- 3. Note that the left side is the derivative of the product $I(x) \cdot y$.
- 4. Integrate both sides. On the left side you will have the product $I(x) \cdot y$.
- 5. Solve for y.

Practice Problems. Solve the following equations.

- 1. $y' + 2y = 2e^x$, y(0) = 1. 2. y' - 2y = x.
- 3. $xy' + 2y = x^3$.

4.
$$x^2y' + xy = 1$$
, $y(1) = 2$.

5.
$$xy' + 2y = \cos x$$
, $y(\pi) = 0$.

Solutions.

1. For the equation $y' + 2y = 2e^x$, you have that P = 2. Determine the integrating factor as $I = e^{\int 2dx} = e^{2x}$. Multiply the equation by it to get $y'e^{2x} + 2e^{2x}y = 2e^xe^{2x}$. Note that the left side is the derivative of the product ye^{2x} (check: the product rule for ye^{2x} gives you $y'e^{2x} + 2e^{2x}y$ which is exactly the left side). So, the equation becomes $(ye^{2x})' = 2e^{3x}$. Integrate both sides to get $ye^{2x} = \int 2e^{3x}dx \Rightarrow ye^{2x} = \frac{2}{3}e^{3x} + c$. Finally, divide by e^{2x} to get the general solution $y = \frac{\frac{2}{3}e^{3x}+c}{e^{2x}} = \frac{2}{3}e^x + ce^{-2x}$.

Using the initial condition y(0) = 1, you have $1 = \frac{2}{3}e^0 + ce^0 = \frac{2}{3} + c \Rightarrow c = \frac{1}{3}$. Thus the solution is $y = \frac{2}{3}e^x + \frac{1}{3}e^{-2x}$.

2. For the equation y' - 2y = x, you have that P = -2. Careful: don't forget the negative sign. The integrating factor is $I = e^{\int -2dx} = e^{-2x}$. Multiply the equation by it to get $y'e^{-2x} - 2e^{-2x}y = xe^{-2x}$. Note that the left side is the derivative of the product ye^{-2x} . So, the equation becomes $(ye^{-2x})' = xe^{-2x}$. Integrate both sides to get $ye^{-2x} = \int xe^{-2x}dx$. Using the integration by parts with u = x and $dv = e^{-2x}dx$ for the right side, obtain that $ye^{-2x} = \frac{-x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + c$. Divide by e^{-2x} to get the general solution $y = \frac{\frac{-x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + c}{e^{-2x}} = \frac{-x}{2} - \frac{1}{4} + ce^{2x}$. 3. Careful: before determining P, you have to write the equation in the form y' + Py = Q. So, you need to divide by x first. Obtain $y' + \frac{2}{x}y = x^2$. This gives you that $P = \frac{2}{x}$. The integrating factor is $I = e^{\int \frac{2}{x} dx} = e^{2\ln x} = e^{\ln x^2} = x^2$. Careful: don't cancel $e^{2\ln x}$ as 2x.

Multiply the equation by x^2 to get $y'x^2 + 2xy = x^4$. Note that the left side is the derivative of the product yx^2 . So, the equation becomes $(yx^2)' = x^4$. Integrate both sides to get $yx^2 = \int x^4 dx = \frac{x^5}{5} + c \Rightarrow y = \frac{\frac{x^5}{5} + c}{x^2} = \frac{x^3}{5} + \frac{c}{x^2}$.

4. To write the equation in the form y' + Py = Q, you need to divide by x^2 first. Obtain $y' + \frac{1}{x}y = \frac{1}{x^2}$. This gives you that $P = \frac{1}{x}$. Determine the integrating factor now. $I = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$. Multiply the equation by x to get $y'x + y = \frac{1}{x}$. Note that the left side is the derivative of the product yx. So, the equation becomes $(yx)' = \frac{1}{x}$. Integrate both sides to get $yx = \int \frac{1}{x} dx = \ln x + c \Rightarrow y = \frac{\ln x + c}{x}$.

Using the initial condition y(1) = 2, you have $2 = \frac{0+c}{1} \Rightarrow c = 2$. Thus the solution is $y = \frac{\ln x+2}{x}$.

5. Divide by x first to get $y' + \frac{2}{x}y = \frac{\cos x}{x}$. $P = \frac{2}{x} \Rightarrow I = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$. Multiply by I to get $y'x^2 + 2xy = x\cos x \Rightarrow (yx^2)' = x\cos x \Rightarrow yx^2 = \int x\cos x dx$. Using the integration by parts with u = x and $dv = \cos x dx$, obtain that $yx^2 = x\sin x + \cos x + c$. Divide by x^2 to get the general solution $y = \frac{x\sin x + \cos x + c}{x^2} = \frac{1}{x}\sin x + \frac{1}{x^2}\cos x + \frac{c}{x^2}$.

With $y(\pi) = 0$ you have that $0 = \frac{-1}{\pi^2} + \frac{c}{\pi^2} \Rightarrow 0 = -1 + c \Rightarrow c = 1$. Thus, the particular solution is $y = \frac{1}{x} \sin x + \frac{1}{x^2} \cos x + \frac{1}{x^2}$.

Bernoulli Equation

A first order differential equation is called **Bernoulli equation** if it can be written in the form

$$y' + P(x)y = Q(x)y^n.$$

Note that for n = 0 and n = 1 this is a linear differential equation (for n = 1 it is also separable).

If $n \neq 0$ or 1, the substitution $u = y^{1-n}$ reduces Bernoulli's equation to a linear equation. Note that if $u = y^{1-n}$ then

$$y = u^{1/(1-n)}$$
, so that $y' = \frac{1}{1-n}u^{1/(1-n)-1}u' = \frac{1}{1-n}u^{n/(1-n)}u'$ and $y^n = u^{n/(1-n)}u'$

Thus, the power of u of the first term on the left is the same as the power of u of the term on the right. Dividing the equation by $u^{n/(1-n)}$ make the power of u of the second term on the left be $\frac{1}{1-n} - \frac{n}{1-n} = \frac{1-n}{1-n} = 1$. Because of this, the *u*-equation one obtains in this way is *linear*.

Examples of Bernoulli equations can be found in the study of the stability of the fluid flow and in population dynamics.

Practice Problems. Solve the following Bernoulli equations.

1. $y' - 2y + 4y^2 = 0$

- 2. $y' y + 2y^3 = 0$
- 3. $x^2y' + 2xy = y^3$

Solutions.

- 1. The equation $y' 2y + 4y^2 = 0$ is a Bernoulli's equation with n = 2. Use the substitution $u = y^{1-2} = y^{-1}$. Thus $y = u^{-1}$ and so $y' = -u^{-2}u'$. Substitute that into the equation. Get $-u^{-2}u' 2u^{-1} + 4u^{-2} = 0$. Multiply by $-u^2$. Get u' + 2u = 4. This is a linear equation that can be solved using the integrating factor $I = e^{\int 2dx} = e^{2x}$. After multiplying by I, get $ue^{2x} = \int 4e^{2x}dx = 2e^{2x} + c$. Solve for u. Get $u = 2 + ce^{-2x}$. Solve for y and get $y = \frac{1}{2+ce^{-2x}}$.
- 2. $y' y + 2y^3 = 0$ is a Bernoulli's equation with n = 3. Use the substitution $u = y^{1-3} = y^{-2}$. Thus $y = u^{-1/2}$ and so $y' = \frac{-1}{2}u^{-3/2}u'$. Substitute that into the equation. Get $\frac{-1}{2}u^{-3/2}u' - u^{-1/2} + 2u^{-3/2} = 0$. Multiply by $-2u^{3/2}$. Get u' + 2u = 4. This is a linear equation with $I = e^{2x}$ and solution $u = 2 + ce^{-2x}$. Thus $y = \frac{1}{\sqrt{2+ce^{-2x}}}$.
- 3. The equation $x^2y' + 2xy = y^3$ is a Bernoulli's equation with n = 3. Use the substitution $u = y^{1-3} = y^{-2}$. Thus $y = u^{-1/2}$ and so $y' = \frac{-1}{2}u^{-3/2}u'$. Substitute that into the equation. Get $\frac{-1}{2}x^2u^{-3/2}u' + 2xu^{-1/2} = u^{-3/2}$. Multiply by $u^{3/2}$. Get $\frac{-1}{2}x^2u' + 2xu = 1$. Divide by $\frac{-x^2}{2}$ to make the first term be u'. Get $u' \frac{4}{x}u = \frac{-2}{x^2}$. This is a linear equation that can be solved using the integrating factor $I = e^{\int -4/xdx} = e^{-4\ln x} = x^{-4}$. After multiplying by I, get $ux^{-4} = \int \frac{-2}{x^6}dx = \frac{-2}{-5x^5} + c$. Solve for u. Get $u = \frac{2}{5x} + cx^4$. Solve for y and get $y = (\frac{2}{5x} + cx^4)^{-1/2} = \frac{1}{\sqrt{\frac{2}{5x} + cx^4}}$.

Exact Equation

Partial Derivatives. If a function F depends on two variables x and y, then you can differentiate it with respect to x and with respect to y. The derivative of F with respect to x, obtained by treating y as a constant, is called the *partial derivative with respect to* x and it is denoted by $\frac{\partial F}{\partial x}$ or by F_x . Analogously, the derivative of F with respect to y, obtained by treating x as a constant, is called the *partial derivative with respect to* y and it is denoted by $\frac{\partial F}{\partial y}$ or by F_y .

For example, partial derivatives F_x and F_y of $F(x, y) = 3x^2 + 2xy - 5y^2$ are $F_x = 6x + 2y$ and $F_y = 2x - 10y$.

Exact equations. A first order differential equation $M(x, y) + N(x, y)\frac{dy}{dx} = 0$ is **exact** if there is a function F(x, y) such that

$$\frac{\partial F}{\partial x} = M$$
 and $\frac{\partial F}{\partial y} = N$

In this case, $Mdx + Ndy = F_x dx + F_y dy$ is the differential dF. When the given equation is multiplied by dx, it becomes $Mdx + Ndy = 0 \Rightarrow dF = 0$. Hence, the solution is F = c where c is any constant. Thus, the formula F(x, y) = c is the implicit form of the general solution.

Test for exactness. If M and N are continuous functions on a region in xy-plane, the differential equation M(x, y)dx + N(x, y)dy = 0 is exact if and only if

$$M_y = N_x$$

on the entire region.

Example. Test if differential equations $xe^y + ye^xy' = 0$ and $x^3y^4 + (x^4y^3 + 2y)y' = 0$ are exact.

Solution. For the first equation $M = xe^y$ and $N = ye^x$. Since $M_y = xe^y$ and $N_x = ye^x$, the equation is not exact.

For the second equation, $M = x^3y^4$ and $N = x^4y^3 + 2y$. Since $M_y = 4x^3y^3$ and $N_x = 4x^3y^3$, the equation is exact.

Finding the solution. To find the solution F(x, y) = 0,

- 1. Integrate M(x, y) with respect to x. After integrating, the undetermined part is not a constant but a function of y. Let use denote it by g(y). Thus $F(x, y) = \int M(x, y)dx + g(y)$.
- 2. To find the unknown function g(y), differentiate $F(x, y) = \int M(x, y)dx + g(y)$ with respect to y and equate it with N(x, y). Denote the derivative of g(y) by g'(y). This will give you an equation which you can solve for g'(y). Integrating that with respect to y gives you the unknown function g(y) and, finally, the solution F(x, y) = 0.

Alternatively, you can:

- 1. Integrate N(x, y) with respect to y. After integrating, the undetermined part is not a constant but a function of x. Let use denote it by h(x). Thus $F(x, y) = \int N(x, y) dy + h(x)$.
- 2. To find the unknown function h(x), differentiate $F(x,y) = \int N(x,y)dy + h(x)$ with respect to x and equate it with M(x,y). Denote the derivative of h(x) by h'(x). This will give you an equation which you can solve for h'(x). Integrating that with respect to x gives you the unknown function h(x) and, finally, the solution F(x,y) = 0.

Example. Solve the differential equation $2x + y^2 + 2xyy' = 0$.

Solution. Note that this equation is neither linear (because of y^2 term) nor separable (because it has three terms, one just with x, one just with y and a mixed term). Here $M = 2x + y^2$ and N = 2xy.

Check if the equation is exact:

$$M_y = 2y$$
 and $N_x = 2y$

So, the equation is exact. Let us integrate M with respect to x.

$$F(x,y) = \int M dx = \int (2x + y^2) dx = x^2 + xy^2 + g(y).$$

Then go to the second step: differentiate with respect to y and equate the result with N.

$$F_y = 2xy + g'(y) = N = 2xy.$$

From this, we obtain that g'(y) = 0 and so g(y) is a constant c. Hence, the solution is

$$F(x,y) = x^{2} + xy^{2} + c = 0$$
 or $x^{2} + xy^{2} = C$.

To illustrate the alternative approach, integrate N with respect to y. Obtain that $F(x, y) = \int N dy = \int 2xy dy = xy^2 + h(x)$. Differentiate with respect to x and equate it with M. So, we have

that $y^2 + h'(x) = 2x + y^2$. From here, h'(x) = 2x and so $h(x) = x^2 + c$. Thus $F(x, y) = xy^2 + x^2 + c = 0$ which is the same answer that we got using the first approach.¹

Example. Find the value of *a* for which the equation

$$(ae^{x^2} + 2y)y' - 2x^{-3} + 2xe^{x^2}y = 0$$

is exact. Solve the equation using that value of a.

Solution. Let $M = -2x^{-3} + 2xe^{x^2}y$ and $N = ae^{x^2} + 2y$. For the equation to be exact, M_y should

be equal to N_x . $M_y = 2xe^{x^2}$ and $N_x = 2axe^{x^2}$. Thus $2 = 2a \Rightarrow a = 1$. In this case, $F = \int (-2x^{-3} + 2xe^{x^2}y)dx = x^{-2} + e^{x^2}y + g(y)$. $F_y = e^{x^2} + g'(y) = N = e^{x^2} + 2y$ So g'(y) = 2y, giving you that $g(y) = \int 2ydy = y^2 + c$. Thus, the solution is $F = x^{-2} + e^{x^2}y + y^2 + c = 0$. or $e^{x^2}y + x^{-2} + y^2 = C$.

Practice Problems. Check if the equations (1)-(4) are exact and, if they are, find the solution.

1.
$$x^3y^4 + (x^4y^3 + 2y)y' = 0$$

- 2. $3xy + y^2 + (x^2 + xy)y' = 0$
- 3. 2x + y + (x 2y)y' = 0
- 4. $e^{x}(y-x) + (1+e^{x})y' = 0$
- 5. Find the value of parameters a and b for which the equation $2x \sin ay + (x^2 \cos y by^2)y' = 0$ is exact and solve the equation using those values.
- 6. Find the value of a for which the equation

$$ay^{2}e^{3x} + 2x^{2}y + (4ye^{3x} + \frac{2}{3}x^{3} + 12e^{4y})y' = 0$$

is exact. Solve the equation using that value of a.

Solutions. Following the same steps as in two solved examples above, obtain the following solutions.

- 1. The equation is exact. $F = \int x^3 y^4 dx = \frac{1}{4}x^4 y^4 + g(y)$. $F_y = x^4 y^3 + g' = N = x^4 y^3 + 2y \Rightarrow g' = 2y \Rightarrow g = y^2 + c$. So, $F = \frac{1}{4}x^4 y^4 + y^2 + c$ and the solution is $\frac{1}{4}x^4 y^4 + y^2 + c = 0$.
- 2. The equation is not exact.
- 3. The equation is exact. $F = \int (2x+y)dx = x^2 + xy + g(y)$. $F_y = x + g' = N = x 2y \Rightarrow g' = -2y \Rightarrow g = -y^2 + c$. $F = x^2 + xy y^2 + c$ and the solution is $x^2 + xy y^2 + c = 0$.
- 4. The equation is exact. It may be easier to integrate N with respect to y than M with respect to x, so you can find F as $\int N dy = \int (1 + e^x) dy = y + y e^x + h(x)$. Then $F_x = y e^x + h' = M =$ $ye^x - xe^x \Rightarrow h' = -xe^x \Rightarrow h = \int -xe^x dx = -xe^x + e^x + c$ the solution is $y + ye^x - xe^x + e^x + c = 0$.

¹For students with Calculus 3. Note that the equation Mdx + Ndy = 0 is exact precisely when the vector function f = (M, N) is conservative. Thus, checking if an equation is exact uses the same method as checking if a given vector field is conservative and finding the solution F(x,y) = c of the exact equation uses the same procedure as finding a **potential** (a function F(x, y) such that $\nabla F = \vec{f}$).

- 5. Let $M = 2x \sin ay$ and $N = x^2 \cos y by^2$. Then $M_y = 2ax \cos ay$ and $N_x = 2x \cos y$. If $M_y = N_x$, then a has to be 1 and b can take any real value. $F = \int 2x \sin y dx = x^2 \sin y + g(y)$. $F_y = x^2 \cos y + g'(y) = N = x^2 \cos y - by^2 \Rightarrow g' = -by^2 \Rightarrow g = -\frac{b}{3}y^3 + c$. Thus, the solution is $x^2 \sin y - \frac{b}{3}y^3 + c = 0$ or $x^2 \sin y - \frac{b}{3}y^3 = C$.
- 6. Let $M = ay^2 e^{3x} + 2x^2 y$ and $N = 4ye^{3x} + \frac{2}{3}x^3 + 12e^{4y}$. For the equation to be exact, M_y should be equal to N_x . $M_y = 2aye^{3x} + 2x^2$ and $N_x = 12ye^{3x} + 2x^2$. Thus 2a = 12 and so a = 6.

In this case, $F = \int (6y^2 e^{3x} + 2x^2 y) dx = 2y^2 e^{3x} + \frac{2}{3}x^3 y + g(y)$. $F_y = 4y e^{3x} + \frac{2}{3}x^3 + g'(y) = N = 4y e^{3x} + \frac{2}{3}x^3 + 12e^{4y}$ So $g'(y) = 12e^{4y}$, giving you that $g(y) = 3e^{4y} + c$. Thus, the solution is $F = 2y^2 e^{3x} + \frac{2}{3}x^3 y + 3e^{4y} + c = 0$ or $2y^2 e^{3x} + \frac{2}{3}x^3 y + 3e^{4y} = C$.

Homogeneous Equation

A first order differential equation is **homogeneous** if it can be written in the form

$$y' = f(\frac{y}{x}).$$

The substitution $u = \frac{y}{x}$ reduces homogeneous equation to a separable equation.

Since $u = \frac{y}{x}$, we have that y = ux so that y' = u'x + u.

Thus, the equation becomes u'x + u = f(u) so that $\frac{du}{dx}x = f(u) - u$. This is a separable equation since $\frac{du}{f(u)-u} = \frac{dx}{x}$.

Practice Problems. Solve the following homogeneous differential equations.

y' = \$\frac{x^2 + xy + y^2}{x^2}\$ Rewrite the right side as \$1 + \frac{y}{x} + (\frac{y}{x})^2\$ to see that the equation is homogeneous.
 y' = \$\frac{4y - 3x}{2x - y}\$ Divide the numerator and the denominator of the right side by \$x\$ to have \$\frac{4\frac{y}{x} - 3}{2 - \frac{y}{x}}\$.
 y' = \$\frac{x + 3y}{x - y}\$ Start with the same step as in the previous problem.

Solutions.

- 1. Using the hint obtain $y' = 1 + \frac{y}{x} + (\frac{y}{x})^2$. Use the substitution $u = \frac{y}{x} \Rightarrow y = ux \Rightarrow y' = u'x + u$ to get $u'x + u = 1 + u + u^2 \Rightarrow u'x = 1 + u^2$. Separate the variables. $\frac{du}{1+u^2} = \frac{dx}{x}$. Integrate $\tan^{-1} u = \ln |x| + c \Rightarrow u = \tan(\ln |x| + c)$. Substitute back and solve for y to get $y = x \tan(\ln |x| + c)$.
- 2. Using the hint get $y' = \frac{4(y/x) 3}{2 (y/x)} \Rightarrow u'x + u = \frac{4u 3}{2 u} \Rightarrow u'x = \frac{4u 3}{2 u} u = \frac{4u 3 2u + u^2}{2 u} = \frac{u^2 + 2u 3}{2 u} \Rightarrow \frac{(2 u)du}{u^2 + 2u 3} = \frac{dx}{x} \Rightarrow \frac{(2 u)du}{(u + 3)(u 1)} = \frac{dx}{x}$. Use the partial fractions for the integral of the left side. Obtain $\frac{-5/4}{u + 3} + \frac{1/4}{u 1} = \frac{dx}{x} \Rightarrow \frac{-5}{u + 3} + \frac{1}{u 1} = \frac{4dx}{x} \Rightarrow -5 \ln |u + 3| + \ln |u 1| = 4 \ln |x| + c \Rightarrow |u + 3|^{-5} |u 1| = C|x|^4 \Rightarrow |u 1| = C|x|^4 |u + 3|^5$. Substitute back $|\frac{y}{x} 1| = C|x|^4 |\frac{y}{x} + 3|^5$. Multiply by x to get $|y x| = C|y + 3x|^5$.

Note that at the step when we multiplied both sides by $|u+3|^5$ we assumed this term is nonzero. In case when this is zero, the line $u = -3 \Rightarrow y = -3x$ is also a solution. So, the general solution is the family of curves either of the form $|y-x| = c|y+3x|^5$ or y = -3x. 3. Rewrite the right side as $\frac{x+3y}{x-y} = \frac{1+3(y/x)}{1-(y/x)} \Rightarrow u'x + u = \frac{1+3u}{1-u} \Rightarrow u'x = \frac{1+3u}{1-u} - u = \frac{1+3u-u+u^2}{1-u} = \frac{u^2+2u+1}{1-u} \Rightarrow \frac{(1-u)du}{u^2+2u+1} = \frac{dx}{x} \Rightarrow \frac{(1-u)du}{(u+1)^2} = \frac{dx}{x}$. Use the partial fractions for the integral of the left side. Obtain $\frac{-1}{u+1} + \frac{2}{(u+1)^2} = \frac{dx}{x} \Rightarrow -\ln|u+1| - \frac{2}{u+1} = \ln|x| + c \Rightarrow \ln|u+1| + \frac{2}{u+1} = -\ln|x| + c \Rightarrow \ln(|u+1||x|) + \frac{2}{u+1} = c$. Substitute back to get $\ln|x+y| + \frac{2x}{x+y} = c$.

This describes all solutions except when |x + y| = 0. So, the general solution are curves of the form $\ln |x + y| + \frac{2x}{x+y} = c$ or y = -x.

Euler's Method

In some cases, it is impossible to find analytical solution of an equation (a formula for a solution even in the implicit form). In those cases, one looks for a numerical solution, a list of (x, y) points that represents an approximation of the analytical solution. Thus, the difference between an analytical and numerical solution is that the first is given by an exact formula y = y(x) of the solution, while the second is a list of points that approximate the points on the exact solution. Numerical methods of solving differential equations are important because many relevant differential equations cannot be solved exactly.

One of the simplest numerical methods for solving a first order differential equation y' = f(x, y) with the initial condition $y(x_0) = y_0$, is Euler's method.

Euler's method approximates the values of the solution at equally spaced numbers x_0 , $x_1 = x_0 + h$, $x_2 = x_1 + h$,... where h is the step size.

We treat x_0 as the *x*-initial value and y_0 as the *y*-initial value. At the point (x_0, y_0) , the slope of the solution is given by $y' = f(x_0, y_0)$ so the tangent line to the solution curve at the initial point is

$$\frac{y - y_0}{x - x_0} = f(x_0, y_0)$$

or, in point-slope form,

$$y - y_0 = f(x_0, y_0)(x - x_0)$$
 or $y = y_0 + f(x_0, y_0)(x - x_0)$

For the next point $x_1 = x_0 + h$, we can compute the y-value of the approximate solution by

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0) = y_0 + f(x_0, y_0)h.$$

Having obtained the point (x_1, y_1) , we note that the slope of the solution is given by $y' = f(x_1, y_1)$. So, the tangent line to the solution curve at (x_1, y_1) is

$$y - y_1 = f(x_1, y_1)(x - x_1)$$

For the point $x_2 = x_1 + h$, the *y*-value computed using the tangent line is

$$y_2 = y_1 + f(x_1, y_1)(x_2 - x_1) = y_1 + f(x_1, y_1)h.$$

Continuing in this way, we obtain a sequence of (x, y) values

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + f(x_n, y_n)h$$

The accuracy of the Euler's method can be increased by decreasing the step size h.

Example. Let us find the first three approximations of y' = y + 1, y(0) = 1 with step size 0.1. We have that

$$y_1 = y_0 + (y_0 + 1)0.1 = 1 + (1 + 1)0.1 = 1.2$$

$$y_2 = y_1 + (y_1 + 1)0.1 = 1.2 + (1.2 + 1)0.1 = 1.42$$

$$y_3 = y_2 + (y_2 + 1)0.1 = 1.42 + (1.42 + 1)0.1 = 1.662$$

Continuing on this way, we can approximate the value of solution at x = 1 to be $y_{10} = 4.187$.

Below is a Matlab script which uses the Euler's method to approximate a solution of the initial value problem y' = f(x, y), $y(x_0) = y_0$ on the interval $[x_0, x_n]$ using *n* steps. The input consists of the function f, *x*-initial x_0 , *y*-initial y_0 , *x*-final x_n and the number of steps *n*.

function [x, y] = euler(f, xinit, yinit, xfinal, n) h = (xfinal - xinit)/n; (calculates the step size) x = zeros(n+1, 1); y = zeros(n+1, 1); (initialize x and y as column vectors of size n + 1) x(1) = xinit; y(1) = yinit; (the first entry in the vectors x and y is x_0 and y_0 respectively) for i = 1:n x(i + 1) = x(i) + h; (every entry in vector x is the previous entry plus the step size h) $y(i + 1) = y(i) + h^*f(x(i), y(i));$ (Euler's Method formula) end

Practice Problems.

- 1. Use the Matlab script **euler** with the step size 0.1 to approximate y(1) where y(x) is the solution of the initial-value problem y' = x + y, y(0) = 1. Sketch the solution.
- 2. Use the Matlab script **euler** with the step size 0.2 to approximate y(2) where y(x) is the solution of the initial-value problem $y' = y e^{-x}$, y(0) = 1. Sketch the solution.
- 3. Use the Matlab script **euler** with the step size .5 to approximate the size of a fish population P at time t = 5 where t is measured in weeks if there are 4 population members initially and the size of the population is changing according to the equation $\frac{dP}{dt} = -0.045P(P-20)$. Sketch the solution.

Solutions.

1. Note that x-initial is 0, y-initial is 1, x-final is 1 and the step size is given to be 0.1 so $n = \frac{1-0}{.1} = 10$. Represent the right side of the equation by $\mathbf{f}=@(\mathbf{x},\mathbf{y})\mathbf{x}+\mathbf{y}$, execute the script by $[\mathbf{x},\mathbf{y}]=(\mathbf{f}, \mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{10})$, and obtain that y(1) = 3.187. Graph by $\mathbf{plot}(\mathbf{x},\mathbf{y})$.

- 2. Note that x-initial is 0, y-initial is 1, x-final is 2 and the step size is given to be 0.2 so $n = \frac{2-0}{.2} = 10$. Represent the right side of the equation by $\mathbf{f}=@(\mathbf{x},\mathbf{y})\mathbf{y}-\exp(-\mathbf{x})$, execute the script by $[\mathbf{x},\mathbf{y}]=(\mathbf{f}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{10})$, and obtain that y(2) = 3.014. Graph by $\mathbf{plot}(\mathbf{x},\mathbf{y})$.
- You can use y for the dependent variable P and x for the independent variable t. Thus the equation is dy/dx = -0.045y(y 20) (Careful: the right side is not -0.045x(x 20)). Represent the right side of the equation as a function by f=@(x,y)-0.045*y*(y-20). The initial time is 0, thus x-initial is 0. The initial population is 4, thus y-initial is 4. The final time is 5 weeks, thus x-final is 5. With the step size of 0.5, n = 5-0/0.5 = 10. Execute the script by [x,y]=(f, 0, 5, 4, 10) and obtain that P(5) = 19.42. So, the population size is approximately 19 members after 5 weeks.

Autonomous Differential Equations

In general, if a first order differential equation is solved for y', it has the form y' = f(x, y). If the function on the right side does not depend of the independent variable x, i.e. if the equation is of the form

$$\frac{dy}{dx} = f(y),$$

the equation said to be **autonomous**. Note that an autonomous equation is separable.

If f(y) = 0 is zero at y = a, then the horizontal line y = a is a solution because both sides of the equation y' = f(y) become zero when y = a. This solution is called the **equilibrium solution** and a is called a critical point. After finding the equilibrium solutions, check the sign of f. On the intervals of y with y' = f(y) positive, the solutions y are increasing and on the intervals of y with y' = f(y) negative, the solutions y are decreasing. Thus, the analysis of the sign of f(y) can tell us a lot about the graph of the solutions. Getting a graph of solutions may provide valuable information about the solutions especially in cases when it is difficult to obtain an explicit formula of the general solution.

If the solutions asymptotically approach an equilibrium solution y = a for $x \to \infty$, regardless of whether the values of the initial conditions are smaller or larger than a, then the solution y = a is said to be **asymptotically stable**.

If small initial differences in the initial conditions produce large differences of the solutions in the long run so that the solutions diverge from an equilibrium solution y = a, the solution y = a is said to be **unstable**.

In case that the solutions with the initial conditions larger than the equilibrium solution y = a are converging towards it and the solutions with the initial conditions smaller than the equilibrium are diverging from it (or vice versa), the solution y = a is said to be **semistable**.



If f(y) = 0 has multiple solution, one can have several types of equilibrium solution present as the next two examples illustrate.

Example. Sketch a graph of the general solution of the equation $y' = 2y - y^2$. Solution. Find the equilibrium solutions by solving $2y - y^2 = 0$. Factor to get y(2 - y) = 0

which produces y = 0 and y = 2 as the equilibrium solutions. Examine the sign of $y' = 2y - y^2$ using the number line $\frac{- + -}{0}$. Use the number line to sketch the graph of all solutions and conclude that the solutions with initial conditions above y = 2, and below y = 0 are decreasing, and the solutions with initial conditions between y = 0 and y = 2 are increasing. Conclude that y = 2 is asymptotically stable and y = 0 is unstable.



Example. Sketch a graph of the general solution of the equation $y' = (y+1)(y-2)^2$. Solution. $y' = (y+1)(y-2)^2 = 0 \Rightarrow y = -1$ and y = 2.

So, y = -1 and y = 2 are the equilibrium solutions. Examine the sign of $y' = (y + 1)(y - 2)^2$ using the number line $\frac{-}{-1} + \frac{+}{2}$. Use the number line to sketch the graph of all solutions and conclude that the solutions with initial conditions above y = 2, and between y = -1 and y = 2 are increasing, and the solutions with initial conditions below y = -1 and y = 2 are decreasing. Conclude that y = 2 is semistable and y = -1 is unstable.



Differential equations with parameters

In many cases, a mathematical model leading to a differential equation which depends on a parameter. For example, in section on Separable Equations, we have seen that if the rate of change is proportional to the size of the quantity considered, then one has that y' = ky where k is a nonzero proportionality constant. Analyzing this equation as an autonomous equation, we can see that the sign of y' = ky depends on the sign of k also, not just the sign of y. Hence, we distinguish two different cases: k > 0 and k < 0. In both cases, y = 0 is an equilibrium solution.

If k > 0, the number line for y' is $\frac{-}{0}$. Thus, in this case, the solution y = 0 is unstable.

If k < 0, the number line for y' is $\frac{+}{0}$. Thus, in this case, the solution y = 0 is stable.

Thus, the stability of the equilibrium solutions and the graphs of solutions are very different in these two cases.



This analysis agrees with the form of the general solution $y = ce^{kt}$ we obtained by solving the equation as a separable equation: if k > 0, the term e^{kt} diverges for $t \to \infty$ and, if k < 0, the term e^{kt} converges to zero for $t \to \infty$.

The equation y' = ky models the population size if the population changes in time at a rate proportional to its size. We encounter this situation in examples when the percent birth rate is band the percent death rate is c so that $\frac{dy}{dt} = by - cy$. Here we have that k = b - c. If k > 0 (so b > c) the population is increasing for any positive initial value y(0). If k < 0 (so b < c) the population is decreasing to zero for any positive initial value y(0)

For another example, consider the modes in which the rate of change is proportional to the product y(A - y) where A is a positive constant. This leads us to the equation

$$y' = ky(A - y)$$

where k is a nonzero proportionality constant. The right side is zero when $ky = 0 \Rightarrow y = 0$ and when $A - y = 0 \Rightarrow y = A$. Hence, there are two equilibrium solutions y = 0 and y = A. The sign of k again impacts the sign of y' so we distinguish two cases again: k > 0 and k < 0.

Case k > **0**. If k > 0, the number line for y' is $\frac{-}{0} + \frac{-}{A}$. If unsure how to obtain this number line, take some specific values for k and A, for example, k = 2 and A = 5 (or take your two favorite positive numbers for k and A) and analyze the sign of y' = 2y(5 - y). Obtain the sign distribution $\frac{-}{0} + \frac{-}{5}$. The sign distribution is the same for any positive k and A values producing the first number line above.

Thus, if k > 0 the solution y = A is stable and y = 0 unstable.

If y represents the population size, this case corresponds to the **limited growth** of the population and the stable solution y = A corresponds to the **carrying capacity** of population. A growth can be limited because of factors such as limited resources like food or internal competitiveness.



Hence, the population size increases to A if the initial size y(0) is smaller than A If y(0) is larger than A, the population size decreases to A as the population is too large to grow due to some constraints like the limiting resources of the environment, for example. If y(0) = A, the solution is constant y = A. Hence, the equilibrium solution y = A is stable since $\lim_{t\to\infty} y = A$ regardless of the initial size.

Case k < **0**. If k < 0, the number line for y' is $\frac{+}{0} - \frac{+}{A}$. If unsure how to obtain this number line, you can take some specific values for k and A, for example, k = -2 and A = 5 (or take your favorite positive number for A and your two favorite negative number for k) and analyze the sign of y' = -2y(5-y). Obtain the sign distribution $\frac{+}{0} - \frac{-}{5}$. The sign distribution is the same for any negative k and positive A values.

Thus, if k < 0 the solution y = 0 is stable and y = A unstable.

If y represents the population size, this case corresponds to the **growth with the threshold level** and the unstable solution y = A corresponds to the **threshold level** of population. This situation appears if the population is critically vulnerable to predators if it is small enough and can survive just if the initial size is above the threshold level.



Thus, the population size decreases to 0 if the initial size y(0) is smaller than the threshold A. If the initial size y(0) is larger than A, the population size increases without a bound. If y(0) = A, the solution is constant y = A.

Practice Problems.

1. Sketch the graph of solutions of the following equations.

a)
$$y' = y^2 - 2y$$
 b) $y' = y(y+1)(y-2)$ c) $y' = y(2-y)^2(5-y)^3$

2. Determine the stability of the equilibrium solutions of following equations

a)
$$y' = (y - a)(y - b)$$

b) $y' = y(ay^2 - b)$

where a and b are constants. For part b), you can assume that a > 0.

- 3. The size of a population of rabbits is modeled by differential equation P' = -kP(100 P) where k is a positive parameter.
 - a) Estimate the number of rabbits after a long period of time if the initial size of the population is 103 rabbits.
 - b) Estimate the number of rabbits after long period of time if the initial size of the population is 99 rabbits.

- c) If k = 0.02 per year, use the program **euler** to estimate the size of the population after 4 years if the population was 99 initially. Use 0.5 for the step size.
- 4. The Pacific halibut fishery is modeled by differential equation B' = kB(K-B) where B is the biomass (total mass of the members of the population) in kilograms at time $t, K = 8 \cdot 10^7$ kg and $k = 8.7 \cdot 10^{-9}$ per year.
 - a) Estimate the biomass after many years if the initial biomass is $3 \cdot 10^6$.
 - b) Estimate the biomass after many years if the initial biomass is $9 \cdot 10^7$.
 - c) If the biomass is $2 \cdot 10^7$ kg initially, use the program **euler** to estimate the biomass 5 years later. Use 0.5 for the step size.

Solutions.

1. a) To find equilibrium solution solve $y^2 - 2y = y(y - 2) = 0 \Rightarrow y = 0$ and y = 2. Then analyze the sign of y'. $\frac{+ - +}{0}$ the solution to sketch the graph of the general solution: above y = 2, and below y = 0, the solutions are increasing, and between y = 0 and y = 2 the solutions are decreasing. From the graph, you can see that y = 0 is asymptotically stable and y = 2 is unstable.

b) Equilibrium solutions:
$$y = -1$$
, $y = 0$ and $y = 2$. Sign of $y' : \frac{-}{-1} + \frac{-}{0} + \frac{-}{2}$
Conclude that $y = 0$ is stable and $y = -1$ and $y = 2$ are unstable.

c) $y' = y(2-y)^2(5-y)^3 = 0 \Rightarrow y = 0, (2-y)^2 = 0 \text{ or } (5-y)^3 = 0 \Rightarrow y = 0, 2-y = 0 \text{ or } 5-y = 0.$ So, the equilibrium solutions are y = 0, y = 2 and y = 5. Sign of $y' : \frac{-+++-}{0}$. Conclude that y = 0 is unstable, y = 2 is semistable and y = 5 is stable.

2. a) You can consider the following cases: a > b, a < b and a = b. In the first case, the sign of y' is $\frac{+ - + b}{b - a}$ So, y = b is stable and y = a is unstable. In the second case, a and b are interchanged in the previous number line so y = a is stable and y = b is unstable. If a = b, there is just one equilibrium solution and it is semistable.

b) Note that $y' = y(ay^2 - b) = 0 \Rightarrow y = 0$ or $y^2 = \frac{b}{a}$. This brings you to the following cases. Case 1. $\frac{b}{a} \leq 0$. In this case, y = 0 is the only solution, so y = 0 is the only equilibrium solution. The sign of y' is given by $\frac{-}{0}$ so y = 0 is unstable solution. Thus, the solutions with positive initial value increase to infinity and those with negative initial value decrease to negative infinity.

Case 2. $\frac{b}{a} > 0$. In this case, the equation $y(ay^2 - b) = 0$ has three solutions: y = 0 and $y = \pm \sqrt{\frac{b}{a}}$. The sign of y' can be obtained using the number line $\frac{-}{-\sqrt{b/a}} + \frac{-}{0} + \frac{+}{\sqrt{b/a}}$ So, y = 0 is stable and $y = \pm \sqrt{\frac{b}{a}}$ are unstable. Thus, the solutions with initial values in $(\frac{-b}{a}, \frac{b}{a})$, converge towards 0, the solutions with initial values larger than $\frac{b}{a}$ increase without bounds and the solutions with initial values smaller than $\frac{-b}{a}$ decrease without bounds.

3. Parts a) and b) can be obtained by analyzing the graph and stability of the equilibrium solutions. $-kP(100 - P) = 0 \Rightarrow P = 0$ and P = 100. Sign of $P' : \frac{+ - - +}{0}$. Thus, with initial condition above P = 100 (and below P = 0 but that is not relevant in this case) the solutions are increasing. In particular, if the initial population size is 103, the population will be increasing. Thus $\lim_{t\to\infty} P = \infty$. So, the population size increases without bounds. The solutions with initial conditions between P = 0 and P = 100 are decreasing. In particular, if the initial population size is 99, the population will be decreasing to 0. Thus $\lim_{t\to\infty} P = 0$. So, the population size decreases to 0 in this case.

c) Represent the right side as a function $\mathbf{f}=@(\mathbf{x},\mathbf{y})-0.02^*\mathbf{y}^*(100-\mathbf{y})$, use 0 for x-initial, 99 for y-initial, 4 for x-final and 0.5 for the step size (so that $n = \frac{4-0}{0.5} = 8$). Obtain that the population size decreased to about 7.63 (can round to 8) four years after.

4. Parts a) and b) can be obtained by analyzing the graph and stability of the equilibrium solu-

tions.
$$B' = kB(8 \cdot 10^7 - B) = 0 \Rightarrow B = 0$$
 and $B = 8 \cdot 10^7$. Sign of $B' : \frac{-}{0} + \frac{-}{8 \cdot 10^7}$

Thus, with initial condition above $B = 8 \cdot 10^7$ (and below B = 0 but that is not relevant in this case) the solutions are decreasing. In particular, if the initial biomass is $9 \cdot 10^7$, the biomass will be decreasing to $8 \cdot 10^7$ so $\lim_{t\to\infty} B = 8 \cdot 10^7$. The solutions with initial conditions between B = 0 and $B = 8 \cdot 10^7$ are increasing. In particular, if the initial biomass is $3 \cdot 10^7$, the biomass will be increasing to $8 \cdot 10^7$. Thus $\lim_{t\to\infty} B = 8 \cdot 10^7$ as well.

c) Represent the right side as a function $\mathbf{f}=@(\mathbf{x},\mathbf{y})\mathbf{8.7*10}^{(-9)*\mathbf{y}*(\mathbf{8*10}^{7}-\mathbf{y})}$, and use 0 for x-initial, $2 \cdot 10^{7}$ for y-initial, 5 for x-final and 0.5 for the step size (so $n = \frac{5-0}{0.5} = 10$). Obtain that the biomass increased to $74242300 \approx 7.4 \cdot 10^{7}$ kg in 5 years.

Modeling with First Order Differential Equations

In application problems or real life scenarios, one may need to come up with a differential equation that accurately describes certain scenario *first*, before solving it. So, being able to model the problem using an equation is equally important as being able to solve the equation. The process of writing an equation describing the given situation is referred to as *mathematical modeling*. In order to successfully model a problem by a differential equation, it might be helpful to ask yourself the questions listed below.

- 1. Identify the real problem. Identify the problem variables. What do we need to describe or find out? What is the problem asking for?
- 2. Construct an appropriate relation between the variables a differential equation. Determine how the dependent variable, the independent variable and the rate of change are connected. Figuring this out results in a differential equation that models the problem.

- 3. Obtain a mathematical solution. Recognize the type of the equation. Decide if you can solve it analytically ("by hand") or if you need to find a numerical solution using technology. In both cases, decide on the method that you will use (e.g. determine if the equation is separable, linear, exact, Bernoulli, homogeneous or some other type; determine if it is appropriate to use Euler's method, Matlab's ode45 command or other numerical method).
- 4. Interpret the solution. After solving the equation, check if the mathematical answer agrees with the context of the original problem. Check the validity: Does your answer make sense? Do the predictions agree with real data? Do the values have correct sign? Correct units? Correct size? Check effectiveness: Could a simpler model be used? Have I found a right balance between greater precision (i.e. greater complexity) and simplicity?

Example 1. A bacteria culture starts with 500 bacteria and grows at a rate proportional to its size. After 3 hours there are 8000 bacteria. Find the number of bacteria after 4 hours.

Solution. Identifying variables: let y stands for the bacteria culture and t stands for time passed. The first part of the problem "A bacteria culture starts with 500 bacteria.." tells us that y(0) = 500. The second part "... and grows at a rate proportional to its size" is the key for getting the mathematical model. Recall that the rate is the derivative and that "...is proportional to.." corresponds to "equal to constant multiple of..." So, the equation relating the variables is $\frac{dy}{dt} = ky$. The solution of this differential equation is $y = y_0 e^{kt}$. Since $y_0 = 500$, it remains to determine the proportionality constant k. From the condition "After 3 hours there are 8000 bacteria" we obtain that $8000 = 500e^{3k}$ which gives us that $k = \frac{1}{3} \ln 16 = .924$. Thus, the number of bacteria after t hours can be described by $y = 500e^{.924t}$. Using the function we have obtained, we find the number of bacteria after 4 hours to be y(4) = 20159 bacteria.

In the previous example, we have seen that the equation y' = ky models the situation when the rate is proportional to the size. Another often encountered scenario is when the total rate $\frac{dy}{dt}$ is computed as the difference of the rate causing an increase and the rate causing a decrease:

the total rate = rate in - rate out.

The following example illustrates this situation.

Example 2. A population of field mice inhabits a certain rural area. In the absence of predators, the mice population increases continuously by 50%. However, several owls live in the same area and they kill 450 mice per month. Find an equation describing the population size and use it to predict the long term behavior of the population.

Solution. Identify the variables first. Let y stands for the size of mice population and t be the time in months. In this case, the total change in mice population $\frac{dy}{dt}$ can be describe as the difference of the rate at which the population is increasing and the rate at which the population is decreasing: $\frac{dy}{dt}$ = rate in - rate out.

The population increases by 50% so the rate in is 0.5y. Incorporating the information about the owls, we have that the rate out is 450 mice per month. Thus, the differential equation is obtained by equating the total rate $\frac{dy}{dt}$ with the difference of rate in and rate out so that

$$\frac{dy}{dt} = .5y - 450.$$

Note that this is an autonomous equation with the equilibrium solution of $.5y - 450 = 0 \Rightarrow$ $.5y = 450 \Rightarrow y = 900$. Analyzing the sign of the solution, we conclude that the sign changes at 900 from negative to positive.

Thus, we conclude that it is an unstable solution and that the number of mice will

- drop to 0 if the initial number is smaller than 900,
- stay constant at 900 if the initial number is equal to 900, and
- keep increasing without bounds if the initial number is larger than 900.



The equation can be solved by separating the variables to get $\frac{dy}{.5y-450} = dt$ and integrating both sides to obtain $2\ln|.5y-450| = t+c \Rightarrow \ln|.5y-450| = \frac{t}{2} + \frac{c}{2} \Rightarrow |.5y-450| = e^{t/2+c/2} \Rightarrow .5y-450 = \pm e^{t/2+c/2} = \pm e^{c/2}e^{t/2} = Ce^{t/2} \Rightarrow .5y = Ce^{t/2} + 450 \Rightarrow y = Ce^{t/2} + 900$ (in this last step, we denoted 2C by C again). Note that the positive coefficient with t in the exponent causes this term to increase indefinitely when $t \to \infty$ and reflects the fact that the equilibrium solution 900 is unstable.

Some mathematical models of physical phenomena can be obtained by the rule that

the total force is a sum of all acting forces.

You can assume that the total force F is given by the formula F = ma where a = v' is the acceleration from the Newton's second law. The next example illustrates this scenario.

Example 3. Suppose that an object is falling in the atmosphere near the sea level. Assume that the drag is proportional to the velocity with the drag coefficient of 2 kg/sec and that the mass of the object is 10 kg.

- (a) Formulate a differential equation describing the velocity of the object. Find the limiting velocity by analyzing the equation that models this situation.
- (b) Find the general solution of the equation and its limit of the solution for $t \to \infty$. Compare the answer with the part (a).
- (c) Assuming that the object is *dropped* from a height of 300 m, determine how long it will take for the object to hit the ground and how fast it will go at the time of the impact.

Solution. (a) Identify the variables first. Let v denotes the velocity and t denotes the time. Our goal is to get the solution of a differential equation in unknown velocity as a function of time. As $a = \frac{dv}{dt}$ and F = ma, we have that the total force is $F = m\frac{dv}{dt}$. We have two forces acting on this object: the gravitational force equaling mg where g = 9.8 m/sec² and the drag force which is, by assumption of the problem equal to 2v. Since these two forces act in the opposite directions, the total force is equal to the difference of these two forces. Since the force mg is dominant (because the object does fall down in the direction of this force), you can choose to have mg as a positive and 2v as a negative term. Thus, we have that

$$m\frac{dv}{dt} = mg - 2v.$$

The mass of the object in question is 10 kg, so we have that $\frac{dv}{dt} = 9.8 - \frac{v}{5}$.

This is an autonomous equation with the equilibrium solution when $9.8 - \frac{v}{5} = 0 \Rightarrow v = 49$. Thus, we can sketch the solutions by analyzing the sign of $9.8 - \frac{v}{5}$. Since the sign is changing from positive

to negative, v = 49 m/sec is a stable solution. Thus, the velocity

- increases to 49 m/sec if the initial velocity is between 0 and 49,
- stays constant at 49 m/sec if the initial velocity is 49, and
- decreases to 49 m/sec if the initial velocity is larger than 49.



(b) The equation $\frac{dv}{dt} = 9.8 - \frac{v}{5}$ is separable. Multiply by 5 (for simplicity) and separate the variables. Get $\frac{5dv}{49-v} = dt \Rightarrow -5 \ln |49-v| = t+c \Rightarrow \ln |49-v| = \frac{-t}{5} - \frac{c}{5} \Rightarrow |49-v| = e^{-t/5-c/5} \Rightarrow 49-v = \pm e^{-t/5-c/5} = \pm e^{-c/5}e^{-t/5} = Ce^{-t/5} \Rightarrow -v = -49 + Ce^{-t/5} \Rightarrow v = 49 + Ce^{-t/5}$. Note that in this last step we denoted -C by C again.

When $t \to \infty$, the term $e^{-t/5}$ approaches 0. Thus, $v = 49 + Ce^{-t/5} \to 49 - 0 = 49$. This agrees with the analysis from (a) and also illustrates that the terminal velocity will be 49 m/sec regardless of the initial velocity.

(c) As the object is dropped, the initial velocity is 0. Find the particular solution with v(0) = 0 from $v = 49 + Ce^{-t/5}$. Get 0 = 49 + C. So, C = -49 and thus $v = 49 - 49e^{-t/5}$.

In order to determine the velocity and time of the impact, we need to find a formula describing the distance s as a function of time t. As $v = \frac{ds}{dt}$, $s = 49t + 245e^{-t/5} + c$. Note that here the coordinate system is chosen so that gravity acts in a positive direction (downward) so, s measures the distance of the object from the initial height to the current position. Thus, the initial position of the object corresponds to s(0) = 0. This gives us that 0 = 0 + 245 + c. Thus, c = -245 and so $s = 49t + 245e^{-t/5} - 245$.

So, if the initial height is 300 meters, the time the object hits the ground can be obtained from the equation

$$300 = 49t + 245e^{-t/5} - 245 \Rightarrow 49t + 245e^{-t/5} - 545 = 0$$

This equation requires you to use some technology (Matlab, your calculator, or anything else). Obtain that t = 10.51 second which means that the object will hit the ground 0.51 seconds after it is dropped. The velocity at that time is $v(10.51) = 49 - 49e^{-10.51/5} = 43.01$ m/sec.

Let us also discuss a different choice of the coordinate system in this problem: if s denotes the distance from the current position to the ground, then s is decreasing as time passes by so velocity has negative values. The equation that corresponds to this choice of this coordinate system is $mv' = -mg + 2|v| \Rightarrow mv' = -mg - 2v$. This equation also has a stable equilibrium solution with the same absolute value as the one previously discussed, just the opposite sign. With m = 10, the equation is $v' = -9.8 - \frac{1}{5}v \Rightarrow \frac{-5dv}{49+v} = dt \Rightarrow \ln(49+v) = \frac{-t}{5} + c \Rightarrow v = Ce^{-t/5} - 49$.

With the initial condition v(0) = 0, we have that $v = 49e^{-t/5} - 49$. In this case $s = -245e^{-t/5} - 49t + c$ and with the initial height of 300 m, the initial condition should be s(0) = 300 in this case, not s(0) = 0. This gives us $300 = -245 + c \Rightarrow c = 545$.

To find the time the object hits the ground, set the particular solution $s = -245e^{-t/5} - 49t + 545$ to 0. When multiplied by -1, this gives us the same equation as before $245e^{-t/5} + 49t - 545 = 0$ and the same *t*-value 10.51 second. The velocity at that time is $49e^{-10.51/5} - 49 = -43.01$ m/sec.

In the following example, we also obtain a differential equation using the argument that the total rate is equal to the difference of the rate in and the rate out.

Example 4. A tank initially contains 15 thousands gallons of pure water. A mixture of water and dye enters the tank at the rate of 3 thousands gallons per day and the mixture flows out at the same rate. The concentration of dye in the incoming water is increasing in time t according to the expression 0.5t grams per gallon. Determine the differential equation and an appropriate initial condition that model this situation. Find the corresponding particular solution and use it to determine the amount of dye in the tank after 3 days.

Solution. If Q denotes the amount of dye measured in grams and t the time measured in days, the rate of change of Q (in grams per day) is equal to the difference of rate of flow in and rate of flow out of the pond, $\frac{dQ}{dt}$ = rate in - rate out. The rate in is the product $3 \cdot 10^3 \frac{\text{gal}}{\text{day}} 0.5t \frac{\text{g}}{\text{gal}}$ which results in 1500t $\frac{\text{g}}{\text{day}}$. The rate out is the product of $3 \cdot 10^3 \frac{\text{gal}}{\text{day}}$ and the quantity $\frac{Q}{15 \cdot 10^3} \frac{\text{g}}{\text{gal}}$ describing the ratio of the amount of grams of the dye at time t and the total number of gallons of the mixture. Thus, the rate out is $\frac{Q}{5} \frac{\text{g}}{\text{day}}$. So, the equation

$$\frac{dQ}{dt} = 1500t - \frac{Q}{5}$$

models this situation. Since the tank initially contains no dye, the initial condition corresponding to this situation is Q(0) = 0.

The equation $Q' = 1500t - \frac{Q}{5}$ is linear. Write it in the form $Q' + \frac{1}{5}Q = 1500t$ and find the integrating factor to be $e^{t/5}$. So, $Qe^{t/5} = 1500 \int te^{t/5} dt$. Use the integration by parts for the integral on the right and obtain that $Qe^{t/5} = 1500(5te^{t/5} - 25e^{t/5}) + c$. Thus $Q = 7500(t-5) + ce^{-t/5}$ is the general solution.

 $Q(0) = 0 \Rightarrow 0 = 0 - 37500 + c \Rightarrow c = 37500$. The particular solution is $Q = 7500(t-5) + 37500e^{-t/5} = 7500(t-5+5e^{-t/5})$. After three days, t = 3 and so Q = 5580.44 grams or 5.58 kg.

Practice Problems.

- 1. A population of bacteria grows at a rate proportional to the size of population with the proportionality constant 0.7. Initially, the population consist of two members. Find the population size after six days.
- 2. A glucose solution is administered intravenously into the bloodstream at a constant rate of r mg per minute. As the glucose is added, it is converted into other substances and removed from the bloodstream at a rate proportional to the amount of glucose at that time.

- (a) Set up a differential equation that models this situation.
- (b) If r = 4 and the proportionality constant is 2, sketch the graphs of the general solution and examine the stability. Determine the amount of glucose present after a long period of time.
- (c) Suppose that the initial amount is 1 mg. Solve the equation with this initial condition and sketch the graph of this solution.
- 3. The mass of a bacteria colony has been monitored. Let M(t) denote the mass (in mg) of the colony at time t (in days). The mass M is increasing by 20%. On the other hand, extreme dryness to which the colony is exposed makes the mass M decrease at a rate of 3 mg per day.
 - (a) Write a differential equation whose solution describes the mass M at any given time t using the assumptions from the paragraph above.
 - (b) Sketch the graphs of the solutions, find the equilibrium solution(s) and examine the stability. Explain what happens with the mass of the bacteria colony in the long run for any initial value of the mass.
 - (c) Find the general solution of the differential equation from part (a).
 - (d) Determine the bacteria mass after 5 days if the initial mass is 10 mg.
- 4. Let T(t) represents the temperature of an object at time t. If the current temperature of the object is T_0 and the room temperature is T_r , the Newton's Law of Cooling states that the rate of cooling $\frac{dT}{dt}$ is proportional to the temperature difference between the room temperature and the temperature of the object at time t.
 - (a) Write a differential equation and the initial condition whose solution would provide a formula for T(t). Make a sketch of the general solution. Use the graph to conclude what happens to the temperature as time goes by for all possible values of the initial temperature.
 - (b) Let us consider a 95° C coffee cup that is in a 20° C room. Assume that the proportionality constant is 0.1. Solve the differential equation from part (a) to find the function describing the temperature of the coffee as a function of time (in minutes). Use the solution to estimate the temperature of the coffee after 20 minutes.
- 5. Let A(t) be the area of tissue culture at time t (in days). Let the final area of the tissue when the growth is complete be 10 cm². Most cell divisions occur on the periphery of the tissue and the number of cells on the periphery is proportional to \sqrt{A} . So, a reasonable model for the growth of tissue is obtained by assuming that the rate of growth is jointly proportional to \sqrt{A} and 10 - A (i.e. proportional to the product of \sqrt{A} and 10 - A).
 - (a) Formulate the differential equation that models this situation.
 - (b) If the proportionality constant is 1/4 and the area of tissue culture initially is 1 cm², use the Euler's method (you can use Matlab) to approximate the area of the culture after 3, 5 and 10 days using the step size 0.2 for each approximation. Sketch the graph of the solution.

Solutions.

- 1. The equation is $\frac{dy}{dt} = 0.7y$. Separating the variables and solving gives you $y = Ce^{0.7t}$. Using the initial condition y(0) = 2, the solution becomes $y = 2e^{0.7t}$. Plugging 6 for t produces y(6) = 133 protozoa.
- 2. (a) Let y denotes the amount of glucose at time t. The amount is increasing at a constant rate r and is decreasing at a rate proportional to y. Let k denote the proportionality constant. Then the rate in is r and the rate out is ky. So, the differential equation y' = r - ky models this situation.
 - (b) With the given values, the equation becomes y' = 4 2y.

The equilibrium solution is $4 - 2y = 0 \Rightarrow 4 = 2y \Rightarrow y = 2$. y' = 4 - 2y > 0 for y < 2and y' = 4 - 2y < 0 for y > 2. Thus, the solutions are decreasing toward the equilibrium solution if y(0) > 2 and increasing towards 2 if y(0) < 2. So, y = 2 is stable. This means that regardless of the initial amount present, the amount of glucose present becomes 2 mg after sufficiently long time period.



(c) Separate the variables $\frac{dy}{dt} = 4 - 2y \Rightarrow \frac{dy}{4-2y} = dt$. Integrate both sides. Use u = 4 - 2y for the left side. Obtain $\frac{1}{-2} \ln |4 - 2y| = t + c \Rightarrow \ln |4 - 2y| = -2t - 2c \Rightarrow |4 - 2y| = e^{-2t-2c} \Rightarrow 4 - 2y = \pm e^{-2t-2c} = \pm e^{-2c}e^{-2t} = Ce^{-2t} \Rightarrow -2y = -4 + Ce^{-2t} \Rightarrow y = 2 - \frac{1}{2}Ce^{-2t}$. Replacing $\frac{-1}{2}C$ by c, we obtain the general solution $y = 2 + ce^{-2t}$. Note that the term ce^{-2t} converges to 0 and so $y \to 2$ for $t \to \infty$ regardless of the value of the constant c. This agrees with the conclusion from part b).

If y(0) = 1, then $1 = 2 + ce^0 \Rightarrow 1 = 2 + c \Rightarrow c = -1$. Thus $y = 2 - e^{-2t}$. This function passes (0, 1) and it is increasing to 2 when $t \to \infty$. Hence, it agrees with the graph above.

- 3. (a) Since the rate in is 0.2M and the rate out is 3 the total rate $\frac{dM}{dt}$ = rate in rate out = 0.2M 3.
 - (b) The equilibrium solution is $0.2M 3 = 0 \Rightarrow M = 15$ mg. Examining the sign of the derivative M' = 0.2M 3 obtain $\frac{-}{15}$ and conclude that M = 15 is an unstable solution. Thus, if the initial mass is below 15 mg, the population will eventually die out. If the initial size is above 15 mg, the population will increase without a bound.



(c) The equation is separable. Separate the variables. Get $\frac{dM}{0.2M-3} = dt$. Integrate both sides. Get $\frac{1}{0.2} \ln |0.2M-3| = t+c \Rightarrow \ln |0.2M-3| = .2t+.2c \Rightarrow |0.2M-3| = e^{.2t+.2c} \Rightarrow 0.2M-3 = 0.2M-3 = 0.2M-3$

 $\pm e^{\cdot 2t + \cdot 2c} = \pm e^{\cdot 2c} e^{\cdot 2t} = C e^{\cdot 2t} \Rightarrow \cdot 2M = C e^{\cdot 2t} + 3 \Rightarrow M = \frac{1}{0.2} (C e^{0.2t} + 3) \Rightarrow M = C e^{0.2t} + 15.$ Note that in this last step, we use C for $\frac{C}{\cdot 2}$ from the previous step.

The equation M' - 0.2M = -3 is also linear and you can solve by using $P = -0.2 \Rightarrow I = e^{-0.2t} \Rightarrow Me^{-0.2t} = \int -3e^{-0.2t}dt \Rightarrow Me^{-0.2t} = 15e^{-0.2t} + c \Rightarrow M = 15 + ce^{0.2t}$.

- (d) The initial condition is M(0) = 10. Thus, with solution $M = 15 + ce^{0.2t}$, we have that $10 = 15 + c \Rightarrow c = -5$. Hence, the particular solution is $M = 15 5e^{0.2t}$. When t = 5 days, $M = 15 5e \approx 1.41$ mg.
- 4. (a) Let T denote the temperature at time t, T_r denote the room temperature, and T_0 denote the initial temperature of the object. The equation describing the rate of change of the object temperature T is given by $\frac{dT}{dt} = k(T_r - T)$ with the initial condition $T(0) = T_0$. Note that the equation is not $\frac{dT}{dt} = k(T_r - T_0)$ because the right side of the equation is constant and the object is not cooling at a constant rate. If so, the solution would be a linear, not an exponential function.

The equation $\frac{dT}{dt} = k(T_r - T)$ has one equilibrium solution $T = T_r$.

Since the solutions are decreasing if temperature is higher than T_r , and increasing if the temperature is lower than T_r , the solution $T = T_r$ is stable. So, if the object has initial temperature higher than that of the room, it will cool down to T_r , and if the object has initial temperature lower than that of the room, it will warm up to T_r . If the initial temperature is the room temperature, the temperature stays constant at T_r .



(b) The equation is separable. Separating the variables get $\frac{dT}{T_r-T} = kdt$. Integrating both sides get $-\ln |T_r - T| = kt + c$. Solving for T get $\ln |T_r - T| = -kt - c \Rightarrow T_r - T = \pm e^{-kt-c} = \pm e^{-c}e^{-kt} = Ce^{-kt} \Rightarrow T = T_r - Ce^{-kt}$. Alternatively, you can have $T = T_r + Ce^{-kt}$ if you use C for -C of the previous version.

With the initial condition $T(0) = T_0$ and $T = T_r + Ce^{-kt}$, you have that $T_0 = T_r + C$, thus $C = T_0 - T_r$ and so $T = T_r + (T_0 - T_r)e^{-kt}$. With $T_0 = 95$, $T_r = 20$, and k = 0.1, the solution is $T = 20 + 75e^{-0.1t}$. To estimate the temperature of the coffee after 20 minutes, plug t = 20 into the equation for T. Get $T = 20 + 75e^{-2} = 30.15$ degrees Centigrade.

- 5. (a) The differential equation is $\frac{dA}{dt} = k\sqrt{A}(10 A)$.
 - (b) Represent the right side $\frac{1}{4}\sqrt{y}(10-y)$ of the equation as a function f in Matlab and use 0 for x-initial, 1 for y-initial, 10 for x-final, and 0.2 for the step size so that $n = \frac{10-0}{0.2} = 50$. Obtain that the area is 8.3 cm² after 3 days, it is 9.67 cm² after 5 days and it is 9.995 cm² after 10 days. The graph is a logistic curve with the horizontal asymptote y = 10 and y-intercept 1.