Differential Equations
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First Order Differential Equations

Introduction to Differential Equations; Classifications of Differential Equation

A differential equation is an equation in unknown function that contains one or more derivatives of the unknown function.

The **order** of differential equation is the order of the highest derivative in the equation. Differential equations can be classified based on the order:

- **First order** – just the first derivative appear in the equation. For example, $y'^2 + y = \sin x$.
- **Higher order** – derivatives higher than the first appear in the equation. For example, $y'' + \sin(xy) = 0$.

First order differential equations. General form:

$$F(y', y, x) = 0$$

When possible, solve the equation for y' to obtain the form

$$y' = f(x, y)$$

Higher order differential equations. General form of n-th order differential equation:

$$F(y^{(n)}, y^{(n-1)}, \ldots, y', y, x) = 0$$

If the function F is a linear function of the variables $y, y', \ldots, y^{(n)}$, i.e. if it is of the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_0(x)y = g(x),$$

the differential equation is said to be **linear**. If it is not linear, it is said to be **nonlinear**.

A linear differential equation of the first order has the form

$$a(x)y' + b(x)y = g(x)$$

Note: if $a(x)$ is a nonzero function so that you can divide by it, we arrive to the form of linear equation that you may remember from Calculus 2: $y' + P(x)y = Q(x)$.

The function y is a **solution** of differential equation $F(y^{(n)}, y^{(n-1)}, \ldots, y', y, x) = 0$, if the equation is satisfied when y and all its derivatives $y', \ldots, y^{(n)}$ are substituted into the equation for every value of variable x. y is a solution on an interval (a, b) if y and its derivatives satisfy the equation for every value of x on interval (a, b).

For example, the function $y = e^{2x}$ is a solution of the second order equation $y'' + 2y' - 8y = 0$ since the derivatives $y' = 2e^{2x}$ and $y'' = 4e^{2x}$ yield an identity $4e^{2x} + 4e^{2x} - 8e^{2x} = (4+4-8)e^{2x} = 0e^{2x} = 0$ when plugged in the equation. Note that this is identity does not depend on a specific value of x.

Convince yourself that functions of the form $y = c_1 e^{2x}$ are also solutions of the differential equation $y'' + 2y' - 8y = 0$ for every value of constant c_1. This illustrates that the solution of differential equation does not have to be unique. Moreover, you should convince yourself that functions of the form $y = c_2 e^{-4x}$ are also solutions of the equation $y'' + 2y' - 8y = 0$.

The general solution of a differential equation is a family of all functions that satisfy the equation. We shall see later that general solution of $y'' + 2y' - 8y = 0$ is of the form $y = c_1 e^{2x} + c_2 e^{-4x}$.

In most cases, a general solution of differential equation of first order depends on a single constant. A general solution of differential equation of n-th degree depends on n constants.

In many applications however, one is not interested in general solution but in a solution passing a certain point or satisfying a certain condition. For a first order differential equation the condition $y(x_0) = y_0$ is called an **initial condition** and the differential equation

$$y' = f(x, y) \quad \text{together with the initial condition} \quad y(x_0) = y_0$$

is called an **initial value problem**.

For example, the function $y = ce^{2x}$ is general solution of differential equation $y' = 2y$. If the condition $y(0) = 5$ is added to the equation, then the solution $y = ce^{2x}$ does not satisfy it for every, but for a single value of constant c. Plugging the initial condition values in the general solution, we obtain a **particular solution** of the equation. In this case, $5 = ce^{2(0)}$, gives us the value of $c = 5$. Thus the particular solution is $y = 5e^{2x}$.

Differential equations can be classified also based on the number of functions that are involved.

- A **single** differential equation – there is a single unknown function. For example, $\frac{dy}{dt} + 4y = \ln t$.
- A **system** of differential equations – there is more than one unknown function. For example, $\frac{dx}{dt} + 4y = \ln t$ together with $\frac{dy}{dt} + 4x = e^t$.

Next, differential equations can be classified also based on the type of unknown function:

- **Ordinary** – unknown function is a function in a single variable. For example, $\frac{dy}{dx} + \sin y = \ln x$, $\frac{d^2P}{dt^2} + P = te^t$, etc.
- **Partial** – unknown function is a function in more than one variable. For example $\frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} = \sin x + \ln t$, $y_{xx} + y_t = te^t$, etc.

Practice Problems.

1. Consider the equation $y' + 3x^2 y = 6x^2$.
   (a) Classify the equation based on the order and linearity.
   (b) Check if $y = x^2$ and $y = 2 + e^{-x^3}$ are solutions of the equation.

2. Show that $y = \frac{1}{x+c}$ is a solution of differential equation $y' = -y^2$. Then, find a particular solution that satisfies the initial condition $y(0) = \frac{1}{4}$.

3. Classify the equation $y'' - 3y' + 2y = 0$ based on the order, linearity and type of unknown function and show that $y = ce^{2x}$ is a solution of this differential equation for every constant c.

4. Show that $y = c_1e^x + c_2e^{2x}$ is a solution of differential equation $y'' - 3y' + 2y = 0$ (it is a general solution in fact). Then, find the constants c_1 and c_2 such that the initial conditions $y(0) = 2$ and $y'(0) = 5$ are satisfied.

5. Show that $y = c_1 \cos 2x + c_2 \sin 2x$ is a solution of differential equation $y'' + 4y = 0$ (it is a general solution in fact). Then, find the constants c_1 and c_2 such that the boundary conditions $y(0) = 2$ and $y(\frac{\pi}{4}) = 5$ are satisfied.

6. Determine all values of r for which

$$6\frac{d^2y}{dt^2} - 7\frac{dy}{dt} - 3y = 0$$

has a solution of the form $y = e^{rt}$.

7. Find value of constants A, B and C for which the function $y = Ax^2 + Bx + C$ is the solution of the equation $y'' - y' + 4y = 8x^2$.

8. Find value of constant A for which the function $y = Ae^{3x}$ is the solution of the equation $y'' - 3y' + 2y = 6e^{3x}$.

9. Classify the following differential equations based on the order, linearity and type of unknown function.

   a) The study of electrical circuits Kirchhoffs Laws (Physics):

   $$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

   where L, C, R are constants and $E(t)$ is a given function.

   b) Michaelis-Menten equation that describes the rate of change of plasma drug concentration C after an intravenous bolus injection (Pharmacy):

   $$-\frac{dC}{dt} = \frac{v_{max}C}{k + C}$$

   where v_{max} is the maximum velocity of reaction and k is the rate constant.

   c) Wave equation – a model of the vibrating strings and propagation of waves (Physics):

   $$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

   where c is a constant.

d) A model of the learning of a task (Psychology):

$$\frac{y'}{\sqrt{y^3(1-y)^3}} = \frac{2p}{\sqrt{n}}$$

where p and n are constants.

Solutions.

1. (a) Linear first order ordinary differential equation.

   (b) $y = x^2 \Rightarrow y' = 2x$. Plug the function and its derivative into the equation $y' + 3x^2 y = 6x^2 \Rightarrow 2x + 3x^2(x^2) = 6x^2 \Rightarrow 2x + 3x^4 = 6x^2$. This equation does not hold for every value of x (for example if $x = 1$ the equation false identity $2 + 3 = 6$) so $y = x^2$ is not a solution of the given equation.

   $y = 2 + e^{-x^3} \Rightarrow y' = -3x^2 e^{-x^3}$. Plug the function and its derivative into the equation $y' + 3x^2 y = 6x^2 \Rightarrow -3x^2 e^{-x^3} + 3x^2(2 + e^{-x^3}) = 6x^2 \Rightarrow -3x^2 e^{-x^3} + 6x^2 + 3x^2 e^{-x^3} = 6x^2 \Rightarrow 6x^2 = 6x^2$. This identity holds for every x so the given function is a solution of the equation.

2. $y = \frac{1}{x+c} \Rightarrow y' = \frac{-1}{(x+c)^2}$. Plug the function and its derivative into the equation $y' = -y^2 \Rightarrow \frac{-1}{(x+c)^2} = -\left(\frac{1}{x+c}\right) \Rightarrow \frac{-1}{(x+c)^2} = \frac{-1}{(x+c)^2}$. This identity holds for every x so the given function is a solution of the equation.

   To find c, plug that $x = 0$ and $y = \frac{1}{4}$ into $y = \frac{1}{x+c} \Rightarrow \frac{1}{4} = \frac{1}{0+c} \Rightarrow c = 4$.

3. Linear, second order ordinary differential equation. $y = ce^{2x} \Rightarrow y' = 2ce^{2x} \Rightarrow y'' = 4ce^{2x}$. Plug into the equation $y'' - 3y' + 2y = 0 \Rightarrow 4ce^{2x} - 6ce^{2x} + 2ce^{2x} = 0 \Rightarrow (4 - 6 + 2)ce^{2x} = 0 \Rightarrow 0 = 0$. The given function is a solution of the equation.

4. First part is similar to the previous problem. Use the initial conditions to get $2 = c_1 e^0 + c_2 e^0$ and $5 = c_1 e^0 + 2c_2 e^0 \Rightarrow c_1 + c_2 = 2$ and $c_1 + 2c_2 = 5$. Solve for c_1 and c_2 and get $c_1 = -1, c_2 = 3$.

5. Using the boundary conditions get $2 = c_1 \cos 0 + c_2 \sin 0 = c_1 \Rightarrow c_1 = 2$, and $5 = c_1 \cos 2\frac{\pi}{4} + c_2 \sin 2\frac{\pi}{4} = c_2 \Rightarrow c_2 = 5$.

6. If $y = e^{rt}$, then $y' = re^{rt}$, and $y'' = r^2 e^{rt}$. Plugging that into the equation $6y'' - 7y' - 3y = 0$ gives you $6r^2 e^{rt} - 7re^{rt} - 3e^{rt} = 0$. Factor e^{rt}. Get $e^{rt}(6r^2 - 7r - 3) = 0$. Since e^{rt} is larger than zero for any value of t, $6r^2 - 7r - 3$ has to be zero. This happens just when $r = -1/3$ and $r = 3/2$. Thus, $y = e^{rt}$ is a solution for $r = -1/3$ and $r = 3/2$.

7. Find the derivatives of $y = Ax^2 + Bx + C$ to be $y' = 2Ax + B$ and $y'' = 2A$ and plug them into the equation $y'' - y' + 4y = 8x^2$ to get $2A - 2Ax - B + 4Ax^2 + 4Bx + 4C = 8x^2$. Note that both sides are polynomial functions which need to be equal for *all* values of x. This is possible just if the coefficient of polynomials with each term are equal. Thus,

   - equating the terms with x^2 obtain that $4A = 8 \Rightarrow A = 2$.
   - Equating the terms with x obtain that $-2A + 4B = 0$. Since $A = 2$, $-4 + 4B = 0 \Rightarrow B = 1$.
   - Equating the terms with no x obtain that $2A - B + 4C = 0 \Rightarrow 4 - 1 + 4C = 0 \Rightarrow C = \frac{-3}{4}$.

Thus, $y = 2x^2 + x - \frac{3}{4}$ is a solution of differential equation.

8. Find the derivatives of $y = Ae^{3x}$ to be $y' = 3Ae^{3x}$ and $y'' = 9Ae^{3x}$ and substitute them into the equation $y'' - 3y' + 2y = 6e^{3x}$ to get

$$9Ae^{3x} - 9Ae^{3x} + 2Ae^{3x} = 6e^{3x} \Rightarrow 2Ae^{3x} = 6e^{3x} \Rightarrow 2A = 6 \Rightarrow A = 3$$

Thus, $y = 3e^{3x}$ is a solution of differential equation.

9. a) Linear, second order ordinary differential equation. b) Nonlinear, first order ordinary differential equation. c) Linear, second order partial differential equation. d) Nonlinear, first order ordinary differential equation.

Separable Differential Equations

The first order differential equation $F(y', y, x) = 0$ is **separable** if we can separate the variables x and y. Every separable differential equation can be written in a form

$$P(x) + Q(y)\frac{dy}{dx} = 0$$

or alternatively (if you solve for $\frac{dy}{dx}$ and rename the functions so that $p = -P$ and $q = \frac{1}{Q}$), as

$$\frac{dy}{dx} = p(x)q(y)$$

To solve a separable differential equation,

- Rewrite the equation so that the left side has just one, and the right side just the other variable.

$$P(x)dx + Q(y)dy = 0 \quad \text{giving you} \quad P(x)dx = -Q(y)dy$$

- Integrate both sides.
- If possible, solve for the dependent variable.

Practice Problems.

1. Solve the differential equations and sketch the general solutions.

   a) $y' = 2y$
   b) $y'x = y$
   c) $y'y = -x$

   Then, solve the initial problems

   a) $y' = 2y, \quad y(0) = 3$

b) $y'x = y, \quad y(2) = 4$

c) $y'y = -x, \quad y(0) = 2$

2. Find the general solution of the following differential equations.

   a) $y' = 3x^2y$

   b) $y' = x(y+1)$

   c) $y' = y^2xe^{2x}$

3. Find the solution of the differential equation that satisfies the given initial condition.

   a) $y' = \sqrt{4x+8}, \quad y(-2) = 3$

   b) $y' = xy, \quad y(0) = 5$

   c) $y' = \frac{xy}{x^2+1}, \quad y(0) = 2$

4. **Free fall, no friction.** A differential equation describing free fall with no friction is obtained by equating the total force with the opposite of the gravitational force. Thus,

$$m\frac{d^2x}{dt^2} = -mg$$

   Find the function describing the height passed at time t if there is no initial velocity and the initial hight is x_0. Note that this is a second order differential equation. However, using that $v = \frac{dx}{dt}$, this can be reduced to two separable first order equations:

$$\frac{dv}{dt} = -g, \text{ with } v(0) = 0 \text{ and } \frac{dx}{dt} = v \text{ with } x(0) = x_0.$$

   Note that here we treated a second order differential equation as a system of two first order differential equations. We shall later see that *every differential equation of order n can be reduced to a system of n first order differential equations.*

5. Exponential growth and decay. If the rate of growth of a quantity is proportional to the quantity size y at any time, this situation can be described by a differential equation

$$\frac{dy}{dt} = ky$$

   The constant k is called the proportionality constant. If k is positive, the rate is positive (so that the quantity size is increasing) and if k is negative, the rate is negative (so that the quantity size is decreasing). Find the solution if $y(0) = y_0$.

Solutions.

1. a) Exponential functions $y = ce^{2x}$. b) Lines passing the origin $y = cx$ c) Circles centered at the origin for $c \geq 0$ $x^2 + y^2 = c$. No solutions for $c < 0$. For the second part of the problem a) Function $y = 3e^{2x}$. b) Line $y = 2x$. c) Circle $x^2 + y^2 = 4$.

2. a) $y' = 3x^2y \Rightarrow \frac{dy}{dx} = 3x^2y \Rightarrow \frac{dy}{y} = 3x^2dx \Rightarrow \ln y = x^3 + c \Rightarrow y = e^{x^3+c} = e^{x^3}e^c$ or $y = Ce^{x^3}$. *Careful* not to say that $y = e^{x^3+c}$ is equal to $y = e^{x^3} + C$.

   b) $y' = x(y+1) \Rightarrow \frac{dy}{dx} = x(y+1) \Rightarrow \frac{dy}{y+1} = xdx \Rightarrow \ln(y+1) = \frac{x^2}{2} + c \Rightarrow y + 1 = e^{x^2/2+c} \Rightarrow y = Cex^2/2 - 1$.

   c) $y' = y^2xe^{2x} \Rightarrow \frac{dy}{dx} = y^2xe^{2x} \Rightarrow \frac{dy}{y^2} = xe^{2x}dx$. Integrate the equation. Get $\frac{-1}{y} = \int xe^{2x}dx$. Use the integration by parts with $u = x$ and $dv = e^{2x}dx$ for this integral and obtain $\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$. Thus $y = \frac{-1}{\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c}$ or $y = \frac{1}{\frac{-1}{2}xe^{2x} + \frac{1}{4}e^{2x} + c}$.

3. a) $y' = \sqrt{4x+8} \Rightarrow dy = \sqrt{4x+8}dx \Rightarrow y = \int \sqrt{4x+8}dx$. Use the substitution $u = 4x + 8$ to get $y = \frac{1}{6}(4x+8)^{3/2} + c$. Using the initial condition $x = -2, y = 3$, in the general solution, obtain that $3 = 0 + c \Rightarrow c = 3$. So the particular solution is $y = \frac{1}{6}(4x+8)^{3/2} + 3$.

   b) $y' = xy \Rightarrow \frac{dy}{dx} = xy \Rightarrow \frac{dy}{y} = xdx \Rightarrow \int \frac{dy}{y} = \int xdx \Rightarrow \ln y = \frac{x^2}{2} \Rightarrow y = e^{x^2/2+c}$ or $y = e^ce^{x^2/2} = Ce^{x^2/2}$. *Careful* not to say that $y = e^{x^2/2+c}$ is equal to $y = e^{x^2/2} + C$.

   Using the initial condition $x = 0, y = 5$, in the general solution $y = Ce^{x^2/2}$, obtain that $5 = Ce^0 \Rightarrow C = 5$. So the particular solution is $y = 5e^{x^2/2}$.

   c) $y' = \frac{xy}{x^2+1} \Rightarrow \frac{dy}{y} = \frac{xdx}{x^2+1} \Rightarrow \ln y = \int \frac{xdx}{x^2+1}$. Use the substitution $u = x^2 + 1$ for this last integral. Obtain that $\ln y = \frac{1}{2}\ln(x^2+1) + c \Rightarrow y = e^{\frac{1}{2}\ln(x^2+1)+c}$. Note that this simplifies as $y = e^{\frac{1}{2}\ln(x^2+1)+c} = e^{\ln(x^2+1)^{1/2}}e^c = C(x^2+1)^{1/2} = C\sqrt{x^2+1}$.

   Using that $y = 2$ when $x = 0$, obtain that $2 = C\sqrt{1} \Rightarrow C = 2$. So the particular solution is $y = 2\sqrt{x^2+1}$.

4. Solve for velocity first. Get $v = -gt + c$. Using the initial condition, get $v = -gt$. Then solve the differential equation $\frac{dx}{dt} = -gt$. Obtain $x = \frac{-g}{2}x^2 + c$. Use the initial condition to get $x = x_0 - \frac{g}{2}t^2$.

5. Separate the variables $\frac{dy}{y} = kdt \Rightarrow \ln y = kt + c \Rightarrow y = e^{kt+c} = Ce^{kt}$. Using the initial condition get $y = y_0e^{kt}$.

Linear Differential Equation

A first order differential equation is **linear** if it can be written in the form $a_1(x)y' + a_0(x)y = b(x)$.

Note that if $a_1(x) = 0$, the equation is not differential. So, let us assume that the function $a_1(x)$ is not zero. In this case we can **divide the equation with** $a_1(x)$ and obtain the form

$$y' + P(x)y = Q(x).$$

Note that here $P = a_0/a_1$ and $Q = b/a_1$.

To solve this differential equation you should:

1. Write the equation in the form $y' + P(x)y = Q(x)$.

2. Find the **integrating factor** $I(x) = e^{\int P(x)dx}$ and multiply both sides of the equation with it.

3. Note that the left side is the derivative of the product $I(x) \cdot y$.

4. Integrate both sides. On the left side you will have the product $I(x) \cdot y$.

5. Solve for y.

Practice Problems. Solve the following equations.

1. $y' + 2y = 2e^x, \quad y(0) = 1.$

2. $y' - 2y = x.$

3. $xy' + 2y = x^3.$

4. $x^2y' + xy = 1, \quad y(1) = 2.$

5. $xy' + 2y = \cos x, \quad y(\pi) = 0.$

Solutions.

1. For the equation $y' + 2y = 2e^x$, you have that $P = 2$. Determine the integrating factor as $I = e^{\int 2dx} = e^{2x}$. Multiply the equation by it to get $y'e^{2x} + 2e^{2x}y = 2e^x e^{2x}$. Note that the left side is the derivative of the product ye^{2x} (check: the product rule for ye^{2x} gives you $y'e^{2x} + 2e^{2x}y$ which is exactly the left side). So, the equation becomes $(ye^{2x})' = 2e^{3x}$. Integrate both sides to get $ye^{2x} = \int 2e^{3x}dx \Rightarrow ye^{2x} = \frac{2}{3}e^{3x} + c$. Finally, divide by e^{2x} to get the general solution $y = \frac{\frac{2}{3}e^{3x}+c}{e^{2x}} = \frac{2}{3}e^x + ce^{-2x}$.

   Using the initial condition $y(0) = 1$, you have $1 = \frac{2}{3}e^0 + ce^0 = \frac{2}{3} + c \Rightarrow c = \frac{1}{3}$. Thus the solution is $y = \frac{2}{3}e^x + \frac{1}{3}e^{-2x}$.

2. For the equation $y' - 2y = x$, you have that $P = -2$. *Careful:* don't forget the negative sign. The integrating factor is $I = e^{\int -2dx} = e^{-2x}$. Multiply the equation by it to get $y'e^{-2x} - 2e^{-2x}y = xe^{-2x}$. Note that the left side is the derivative of the product ye^{-2x}. So, the equation becomes $(ye^{-2x})' = xe^{-2x}$. Integrate both sides to get $ye^{-2x} = \int xe^{-2x}dx$. Using the integration by parts with $u = x$ and $dv = e^{-2x}dx$ for the right side, obtain that $ye^{-2x} = \frac{-x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + c$. Divide by e^{-2x} to get the general solution $y = \frac{\frac{-x}{2}e^{-2x} - \frac{1}{4}e^{-2x}+c}{e^{-2x}} = \frac{-x}{2} - \frac{1}{4} + ce^{2x}$.

3. *Careful*: before determining P, you have to write the equation in the form $y' + Py = Q$. So, you need to divide by x first. Obtain $y' + \frac{2}{x}y = x^2$. This gives you that $P = \frac{2}{x}$. The integrating factor is $I = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$. *Careful*: don't cancel $e^{2\ln x}$ as $2x$.

   Multiply the equation by x^2 to get $y'x^2 + 2xy = x^4$. Note that the left side is the derivative of the product yx^2. So, the equation becomes $(yx^2)' = x^4$. Integrate both sides to get $yx^2 = \int x^4 dx = \frac{x^5}{5} + c \Rightarrow y = \frac{\frac{x^5}{5}+c}{x^2} = \frac{x^3}{5} + \frac{c}{x^2}$.

4. To write the equation in the form $y' + Py = Q$, you need to divide by x^2 first. Obtain $y' + \frac{1}{x}y = \frac{1}{x^2}$. This gives you that $P = \frac{1}{x}$. Determine the integrating factor now. $I = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$. Multiply the equation by x to get $y'x + y = \frac{1}{x}$. Note that the left side is the

derivative of the product yx. So, the equation becomes $(yx)' = \frac{1}{x}$. Integrate both sides to get $yx = \int \frac{1}{x} dx = \ln x + c \Rightarrow y = \frac{\ln x + c}{x}$.

Using the initial condition $y(1) = 2$, you have $2 = \frac{0+c}{1} \Rightarrow c = 2$. Thus the solution is $y = \frac{\ln x + 2}{x}$.

5. Divide by x first to get $y' + \frac{2}{x}y = \frac{\cos x}{x}$. $P = \frac{2}{x} \Rightarrow I = e^{\int \frac{2}{x} dx} = e^{2\ln x} = e^{\ln x^2} = x^2$. Multiply by I to get $y'x^2 + 2xy = x\cos x \Rightarrow (yx^2)' = x\cos x \Rightarrow yx^2 = \int x\cos x dx$. Using the integration by parts with $u = x$ and $dv = \cos x dx$, obtain that $yx^2 = x\sin x + \cos x + c$. Divide by x^2 to get the general solution $y = \frac{x\sin x + \cos x + c}{x^2} = \frac{1}{x}\sin x + \frac{1}{x^2}\cos x + \frac{c}{x^2}$.

   With $y(\pi) = 0$ you have that $0 = \frac{-1}{\pi^2} + \frac{c}{\pi^2} \Rightarrow 0 = -1 + c \Rightarrow c = 1$. Thus, the particular solution is $y = \frac{1}{x}\sin x + \frac{1}{x^2}\cos x + \frac{1}{x^2}$.

Homogeneous and Bernoulli Equations

A first order differential equation is **homogeneous** if it can be written in the form

$$y' = f(\frac{y}{x}).$$

The **substitution** $u = \frac{y}{x}$ **reduces homogeneous equation to a separable equation.** Note that then $y = ux$, so $y' = u'x + u$.

A first order differential equation is called **Bernoulli equation** if it can be written in the form

$$y' + P(x)y = Q(x)y^n.$$

Note that for $n = 0$, this is a linear differential equation and for $n = 1$, this is a separable differential equation.

If $n \neq 0$ or 1, the **substitution** $u = y^{1-n}$ **reduces Bernoulli's equation to a linear equation.** Note that then $y = u^{1/(1-n)}$, so $y' = \frac{1}{1-n}u^{1/(1-n)-1}u'$.

Examples of Bernoulli equations can be found in the study of the stability of the fluid flow, in population dynamics.

Practice Problems.

1. Solve the following homogeneous differential equations.
   (a) $y' = \frac{x^2+xy+y^2}{x^2}$ (Hint: rewrite right side as $1 + \frac{y}{x} + (\frac{y}{x})^2$)
   (b) $y' = \frac{4y-3x}{2x-y}$ (Hint: rewrite right side as $\frac{4(y/x)-3}{2-(y/x)}$)
   (c) $y' = \frac{x+3y}{x-y}$ (do it similarly as the previous problem)
2. Solve the following Bernoulli equations.
   (a) $y' - 2y + 4y^2 = 0$

(b) $y' - y + 2y^3 = 0$

(c) $x^2y' + 2xy = y^3$

Solutions.

1. (a) Using the hint obtain $y' = 1 + \frac{y}{x} + (\frac{y}{x})^2$. Use the substitution $u = \frac{y}{x} \Rightarrow y = ux \Rightarrow y' = u'x + u$ to get $u'x + u = 1 + u + u^2 \Rightarrow u'x = 1 + u^2$. Separate the variables. $\frac{du}{1+u^2} = \frac{dx}{x}$. Integrate $\tan^{-1} u = \ln|x| + c \Rightarrow u = \tan(\ln|x| + c)$. Substitute back and solve for y to get $y = x\tan(\ln|x| + c)$.

(b) Using the hint get $y' = \frac{4(y/x)-3}{2-(y/x)} \Rightarrow u'x + u = \frac{4u-3}{2-u} \Rightarrow u'x = \frac{4u-3}{2-u} - u = \frac{4u-3-2u+u^2}{2-u} = \frac{u^2+2u-3}{2-u} \Rightarrow \frac{(2-u)du}{u^2+2u-3} = \frac{dx}{x} \Rightarrow \frac{(2-u)du}{(u+3)(u-1)} = \frac{dx}{x}$. Use the partial fractions for the integral of the left side. Obtain $\frac{-5/4}{u+3} + \frac{1/4}{u-1} = \frac{dx}{x} \Rightarrow \frac{-5}{u+3} + \frac{1}{u-1} = \frac{4dx}{x} \Rightarrow -5\ln|u+3| + \ln|u-1| = 4\ln|x| + c \Rightarrow |u+3|^{-5}|u-1| = C|x|^4 \Rightarrow |u-1| = C|x|^4|u+3|^5$. Substitute back $|\frac{y}{x} - 1| = C|x|^4|\frac{y}{x} + 3|^5$. Multiply by x to get $|y - x| = C|y + 3x|^5$.

Note that at the step when we multiplied both sides by $|u + 3|^5$ we assumed this term is nonzero. In case when this is zero, the line $u = -3 \Rightarrow y = -3x$ is also a solution. So, the general solutions have the form $|y - x| = c|y + 3x|^5$ or $y = -3x$.

(c) Rewrite the right side as $\frac{x+3y}{x-y} = \frac{1+3(y/x)}{1-(y/x)} \Rightarrow u'x+u = \frac{1+3u}{1-u} \Rightarrow u'x = \frac{1+3u}{1-u} - u = \frac{1+3u-u+u^2}{1-u} = \frac{u^2+2u+1}{1-u} \Rightarrow \frac{(1-u)du}{u^2+2u+1} = \frac{dx}{x} \Rightarrow \frac{(1-u)du}{(u+1)^2} = \frac{dx}{x}$. Use the partial fractions for the integral of the left side. Obtain $\frac{-1}{u+1} + \frac{2}{(u+1)^2} = \frac{dx}{x} \Rightarrow -\ln|u+1| - \frac{2}{u+1} = \ln|x| + c \Rightarrow \ln|u+1| + \frac{2}{u+1} = -\ln|x| + c \Rightarrow \ln(|u+1||x|) + \frac{2}{u+1} = c$. Substitute back to get $\ln|x+y| + \frac{2x}{x+y} = c$.

This describes all the general solutions except when $|x + y| = 0$. So, the general solutions have the form $\ln|x+y| + \frac{2x}{x+y} = c$ or $y = -x$.

2. (a) The equation $y' - 2y + 4y^2 = 0$ is a Bernoulli's equation with $n = 2$. Use the substitution $u = y^{1-2} = y^{-1}$. Thus $y = u^{-1}$ and so $y' = -u^{-2}u'$. Substitute that into the equation. Get $-u^{-2}u' - 2u^{-1} + 4u^{-2} = 0$. Multiply by $-u^2$. Get $u' + 2u = 4$. This is a linear equation that can be solved using the integrating factor $I = e^{\int 2dx} = e^{2x}$. After multiplying by I, get $ue^{2x} = \int 4e^{2x}dx = 2e^{2x} + c$. Solve for u. Get $u = 2 + ce^{-2x}$. Solve for y and get $y = \frac{1}{2+ce^{-2x}}$.

(b) $y' - y + 2y^3 = 0$ is a Bernoulli's equation with $n = 3$. Use the substitution $u = y^{1-3} = y^{-2}$. Thus $y = u^{-1/2}$ and so $y' = \frac{-1}{2}u^{-3/2}u'$. Substitute that into the equation. Get $\frac{-1}{2}u^{-3/2}u' - u^{-1/2} + 2u^{-3/2} = 0$. Multiply by $-2u^{3/2}$. Get $u' + 2u = 4$. This is a linear equation with $I = e^{2x}$ and solution $u = 2 + ce^{-2x}$. Thus $y = \frac{1}{\sqrt{2+ce^{-2x}}}$.

(c) The equation $x^2y' + 2xy = y^3$ is a Bernoulli's equation with $n = 3$. Use the substitution $u = y^{1-3} = y^{-2}$. Thus $y = u^{-1/2}$ and so $y' = \frac{-1}{2}u^{-3/2}u'$. Substitute that into the equation. Get $\frac{-1}{2}x^2u^{-3/2}u' + 2xu^{-1/2} = u^{-3/2}$. Multiply by $u^{3/2}$. Get $\frac{-1}{2}x^2u' + 2xu = 1$. Divide by $\frac{-x^2}{2}$ to make the first term be u'. Get $u' - \frac{4}{x}u = \frac{-2}{x^2}$. This is a linear equation that can be solved using the integrating factor $I = e^{\int -4/xdx} = e^{-4\ln x} = x^{-4}$. After multiplying by I, get $ux^{-4} = \int \frac{-2}{x^6}dx = \frac{-2}{-5x^5} + c$. Solve for u. Get $u = \frac{2}{5x} + cx^4$. Solve for y and get $y = (\frac{2}{5x} + cx^4)^{-1/2} = \frac{1}{\sqrt{\frac{2}{5x} + cx^4}}$.

Exact Equations

A first order differential equation $M(x,y)+N(x,y)\frac{dy}{dx}=0$ is **exact** if there is a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x}=M \quad \text{and} \quad \frac{\partial F}{\partial y}=N$$

In this case, $F(x,y)=c$, represents an implicit, general solution of the differential equation. This is because the equation $M(x,y)+N(x,y)\frac{dy}{dx}=0$ i.e. $M(x,y)dx+N(x,y)dy=0$ becomes $dF=F_x(x,y)dx+F_y(x,y)dy=0$.

Test for exactness. If M and N are continuous functions on a region in xy-plane, the differential equation $M(x,y)dx+N(x,y)dy=0$ is exact if and only if

$$M_y=N_x$$

on the entire region.

Example. Test if differential equations $xe^y+ye^xy'=0$ and $x^3y^4+(x^4y^3+2y)y'=0$ are exact.

Solution. For the first equation $M=xe^y$ and $N=ye^x$. Since $M_y=xe^y$ and $N_x=ye^x$, the equation is not exact.

For the second equation, $M=x^3y^4$ and $N=x^4y^3+2y$. Since $M_y=4x^3y^3$ and $N_x=4x^3y^3$, the equation is exact.

Finding the solution. To find the solution $F(x,y)=0$,

1. Integrate $M(x,y)$ with respect to x. After integrating, the undetermined part is not a constant but a function of y. Let use denote it by $g(y)$. Thus $F(x,y)=\int M(x,y)dx+g(y)$.

2. To find the unknown function $g(y)$, differentiate $F(x,y)=\int M(x,y)dx+g(y)$ with respect to y and equate it with $N(x,y)$. Denote the derivative of $g(y)$ by $g'(y)$. This will give you an equation which you can solve for $g'(y)$. Integrating that with respect to y gives you the unknown function $g(y)$ and, finally, the solution $F(x,y)=0$.

Alternatively, you can:

1. Integrate $N(x,y)$ with respect to y. After integrating, the undetermined part is not a constant but a function of x. Let use denote it by $h(x)$. Thus $F(x,y)=\int N(x,y)dy+h(x)$.

2. To find the unknown function $h(x)$, differentiate $F(x,y)=\int N(x,y)dy+h(x)$ with respect to x and equate it with $M(x,y)$. Denote the derivative of $h(x)$ by $h'(x)$. This will give you an equation which you can solve for $h'(x)$. Integrating that with respect to x gives you the unknown function $h(x)$ and, finally, the solution $F(x,y)=0$.

Example. Solve the differential equation $2x+y^2+2xyy'=0$.

Solution. Note that this equation is neither linear (because of y^2 term) nor separable (because it has three terms, one just with x, one just with y and a mixed term). Here $M=2x+y^2$ and $N=2xy$.

Check if the equation is exact:

$$M_y=2y \quad \text{and} \quad N_x=2y$$

So, the equation is exact. Let us integrate M with respect to x.

$$F(x,y) = \int M dx = \int (2x + y^2)dx = x^2 + xy^2 + g(y).$$

Then go to the second step: differentiate with respect to y and equate the result with N.

$$F_y = 2xy + g'(y) = N = 2xy.$$

From this, we obtain that $g'(y) = 0$ and so $g(y)$ is a constant c. Hence, the solution is

$$F(x,y) = x^2 + xy^2 + c = 0 \quad \text{or} \quad x^2 + xy^2 = C.$$

To illustrate the alternative approach, integrate N with respect to y. Obtain that $F(x,y) = \int N dy = \int 2xydy = xy^2 + h(x)$. Differentiate with respect to x and equate it with M. So, we have that $y^2 + h'(x) = 2x + y^2$. From here, $h'(x) = 2x$ and so $h(x) = x^2 + c$. Thus $F(x,y) = xy^2 + x^2 + c = 0$ which is the same answer that we got using the first approach.

For students with Calculus 3. Note that the equation $Mdx + Ndy = 0$ is exact if the vector function $\vec{f} = (M,N)$ is **conservative**. Finding the solution $F(x,y) = c$ uses the same method as finding the **potential** $F(x,y)$, that is, a function with $\nabla F = \vec{f}$. Note that checking if an equation is exact uses the same method as checking if a given vector field is conservative. And finding the solution of the exact equation is the same procedure as finding the potential.

Example. Find the value of a for which the equation

$$(ae^{x^2} + 2y)y' - 2x^{-3} + 2xe^{x^2}y = 0$$

is exact. Solve the equation using that value of a.

Solution. Let $M = -2x^{-3} + 2xe^{x^2}y$ and $N = ae^{x^2} + 2y$. For the equation to be exact, M_y should be equal to N_x. $M_y = 2xe^{x^2}$ and $N_x = 2axe^{x^2}$. Thus $2 = 2a \Rightarrow a = 1$.

In this case, $F = \int(-2x^{-3} + 2xe^{x^2}y)dx = x^{-2} + e^{x^2}y + g(y)$. $F_y = e^{x^2} + g'(y) = N = e^{x^2} + 2y$ So $g'(y) = 2y$, giving you that $g(y) = \int 2ydy = y^2 + c$. Thus, the solution is $F = x^{-2} + e^{x^2}y + y^2 + c = 0$. or $e^{x^2}y + x^{-2} + y^2 = C$.

Practice Problems. Check if the equations (1)–(4) are exact and, if they are, find the solution.

1. $x^3y^4 + (x^4y^3 + 2y)y' = 0$
2. $3xy + y^2 + (x^2 + xy)y' = 0$
3. $2x + y + (x - 2y)y' = 0$
4. $e^x(y - x) + (1 + e^x)y' = 0$
5. Find the value of parameters a and b for which the equation $2x \sin ay + (x^2 \cos y - by^2)y' = 0$ is exact and solve the equation using those values.
6. Find the value of a for which the equation

$$ay^2e^{3x} + 2x^2y + (4ye^{3x} + \frac{2}{3}x^3 + 12e^{4y})y' = 0$$

is exact. Solve the equation using that value of a.

Solutions. Following the same steps as in two solved examples above, obtain the following solutions.

1. The equation is exact. $F = \int x^3y^4dx = \frac{1}{4}x^4y^4 + g(y)$. $F_y = x^4y^3 + g' = N = x^4y^3 + 2y \Rightarrow g' = 2y \Rightarrow g = y^2 + c$. So, $F = \frac{1}{4}x^4y^4 + y^2 + c$ and the solution is $\frac{1}{4}x^4y^4 + y^2 + c = 0$.

2. The equation is not exact.

3. The equation is exact. $F = \int(2x + y)dx = x^2 + xy + g(y)$. $F_y = x + g' = N = x - 2y \Rightarrow g' = -2y \Rightarrow g = -y^2 + c$. $F = x^2 + xy - y^2 + c$ and the solution is $x^2 + xy - y^2 + c = 0$.

4. The equation is exact. It may be easier to integrate N with respect to y than M with respect to x, so you can find F as $\int Ndy = \int(1 + e^x)dy = y + ye^x + h(x)$. Then $F_x = ye^x + h' = M = ye^x - xe^x \Rightarrow h' = -xe^x \Rightarrow h = \int -xe^x dx = -xe^x + e^x + c$ the solution is $y + ye^x - xe^x + e^x + c = 0$.

5. Let $M = 2x \sin ay$ and $N = x^2 \cos y - by^2$. Then $M_y = 2ax \cos ay$ and $N_x = 2x \cos y$. If $M_y = N_x$, then a has to be 1 and b can take any real value.

   $F = \int 2x \sin y dx = x^2 \sin y + g(y)$. $F_y = x^2 \cos y + g'(y) = N = x^2 \cos y - by^2 \Rightarrow g' = -by^2 \Rightarrow g = -\frac{b}{3}y^3 + c$. Thus, the solution is $x^2 \sin y - \frac{b}{3}y^3 + c = 0$ or $x^2 \sin y - \frac{b}{3}y^3 = C$.

6. Let $M = ay^2e^{3x} + 2x^2y$ and $N = 4ye^{3x} + \frac{2}{3}x^3 + 12e^{4y}$. For the equation to be exact, M_y should be equal to N_x. $M_y = 2aye^{3x} + 2x^2$ and $N_x = 12ye^{3x} + 2x^2$. Thus $2a = 12$ and so $a = 6$.

   In this case, $F = \int(6y^2e^{3x} + 2x^2y)dx = 2y^2e^{3x} + \frac{2}{3}x^3y + g(y)$. $F_y = 4ye^{3x} + \frac{2}{3}x^3 + g'(y) = N = 4ye^{3x} + \frac{2}{3}x^3 + 12e^{4y}$ So $g'(y) = 12e^{4y}$, giving you that $g(y) = 3e^{4y} + c$. Thus, the solution is $F = 2y^2e^{3x} + \frac{2}{3}x^3y + 3e^{4y} + c = 0$ or $2y^2e^{3x} + \frac{2}{3}x^3y + 3e^{4y} = C$.

Slope (Direction) Field

The **slope field** (or direction field) is a collection of line segments at a number of points (x, y) with the slope equal to the derivative $y' = f(x, y)$. These line segments indicate the direction in which solution is heading and help us visualize the general shape of the solution curves. Sketching the direction field of the equation $y' = f(x, y)$ is especially useful in cases when the analytical (exact) solution cannot be found. In most cases, it is better to use Matlab to sketch the direction field. See examples in "First Order Differential Equations in Matlab" handout.

Practice Problems.

1. Sketch the direction field of the differential equation $y' = -\frac{x}{y}$. Sketch the graphs of the solutions satisfying the initial conditions $y(1) = 0$ and $y(0) = 2$.

2. Sketch the direction field of the differential equation $y' = \frac{y}{x}$. Sketch the graphs of the solutions satisfying the initial conditions $y(1) = 1$, $y(1) = -1$ and $y(0) = 2$.

3. Use Matlab to sketch the direction field of the differential equation $y' = x + y$.

Solutions. 1. Circles centered at the origin. 2. Lines passing the origin.

Euler's Method

A numerical solution of a differential equation is a list of (x, y) points that represents an approximation of the exact solution. Note that a difference between an analytical and numerical solution is that the first is given by an exact formula $y = y(x)$ of the solution, while the second is a list of points that approximate the points on the exact solution. Numerical methods of solving differential equations are important because many differential equations cannot be solved exactly. For example, $y' = e^{x^2}$, $y' = \frac{\sin x}{x}$, and many more.

One of the simplest numerical methods for solving a first order differential equation $y' = f(x, y)$ with the initial condition $y(x_0) = y_0$, is the Euler's method.

Euler's method approximates the values of the solution at equally spaced numbers x_0, $x_1 = x_0 + h$, $x_2 = x_1 + h, \ldots$ where h is the **step size.**

We start at x**-initial** value x_0 and y**-initial** y_0. At the point (x_0, y_0), the slope of the solution is given by $y' = f(x_0, y_0)$ so the tangent line to the solution curve at the initial point is

$$\frac{y - y_0}{x - x_0} = f(x_0, y_0)$$

or, in point-slope form

$$y - y_0 = f(x_0, y_0)(x - x_0) \text{ or } y = y_0 + f(x_0, y_0)(x - x_0).$$

For the point $x_1 = x_0 + h$, we can compute the y-value of the approximate solution by

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0) = y_0 + f(x_0, y_0)h.$$

Next, start at the point (x_1, y_1) and note that the slope of the solution is given by $y' = f(x_1, y_1)$. So, the tangent line to the solution curve at (x_1, y_1) is

$$y - y_1 = f(x_1, y_1)(x - x_1)$$

For the point $x_2 = x_1 + h$, the y-value computed using the tangent line is

$$y_2 = y_1 + f(x_1, y_1)(x_2 - x_1) = y_1 + f(x_1, y_1)h.$$

Continuing on this way, we obtain a sequence of (x, y) values

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + f(x_n, y_n)h$$

The accuracy of the Euler's method can be increased by decreasing the step size h.

Example. Let us find the first three approximations of $y' = y + 1$, $y(0) = 1$ with step size 0.1. We have that

$$y_1 = y_0 + (y_0 + 1)0.1 = 1 + (1 + 1)0.1 = 1.2$$

$$y_2 = y_1 + (y_1 + 1)0.1 = 1.2 + (1.2 + 1)0.1 = 1.42$$

$$y_3 = y_2 + (y_2 + 1)0.1 = 1.42 + (1.42 + 1)0.1 = 1.662$$

Continuing on this way, we can approximate the value of solution at $x = 1$ to be $y_{10} = 4.187$.

Below is Matlab function file that uses the Euler method to approximate a solution of the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ on the interval $[x_0, x_n]$ using n steps. The input is the inline function f, x-initial x_0, y-initial y_0, x-final x_n and number of steps n.

function [x, y] = euler(f, xinit, yinit, xfinal, n)
h = (xfinal - xinit)/n; (calculates the step size)
x = zeros(n+1, 1);
y = zeros(n+1, 1); (initialize x and y as column vectors of size $n + 1$)
x(1) = xinit;
y(1) = yinit; (the first entry in the vectors x and y is x_0 and y_0 respectively)
for i = 1:n
x(i + 1) = x(i) + h; (every entry in vector x is the previous entry plus the step size h)
y(i + 1) = y(i) + h*f(x(i), y(i)); (Euler Method formula)
end

Practice Problems.

1. Use Euler's method with the step size 0.1 to approximate $y(1)$ where $y(x)$ is the solution of the initial-value problem $y' = x + y, \quad y(0) = 1$. Sketch the solution.

2. Use Euler's method with the step size 0.2 to approximate $y(2)$ where $y(x)$ is the solution of the initial-value problem $y' = y - e^{-x}, \quad y(0) = 1$. Sketch the solution.

3. Use Euler's method with the step size 0.1 to approximate $y(1)$ where $y(x)$ is the solution of the initial-value problem $y' = \sin(x + y), \quad y(0) = 0$. Sketch the solution.

4. Use Euler's method with the step size .5 to approximate the size of a fish population P at time $t = 5$ where t is measured in weeks if there are 4 population members initially and the size of the population is changing according to the equation $\frac{dP}{dt} = -0.045P(P - 20)$. Sketch the solution.

Solutions.

1. x-initial is 0, y-initial is 1, x-final is 1 and the step size is given to be 0.1 so $n = \frac{1-0}{.1} = 01$. Obtain that $y(1) = 3.187$.

2. x-initial is 0, y-initial is 1, x-final is 2 and the step size is given to be 0.2 so $n = \frac{2-0}{.2} = 10$. Obtain that $y(2) = 3.014$.

3. x-initial is 0, y-initial is 0, x-final is 1 and the step size is given to be 0.1 so $n = \frac{1-0}{.1} = 10$. Obtain that $y(1) = .501$.

4. Use y for the dependent variable P and x for the independent variable t. Thus the equation is $\frac{dy}{dx} = -0.045y(y - 20)$. *Careful:* don't enter the right side as $-0.045x(x - 20)$ The initial time

is 0, thus x-initial is 0. The initial population is 4, thus y-initial is 4. The final time is 5 weeks, thus x-final is 5. With the step size of 0.5, obtain that $P(5) = 19.42$ so, the population size is approximately 19 after 5 weeks.

Autonomous Differential Equations

If a differential equation is of the form

$$\frac{dy}{dt} = f(y),$$

it is called **autonomous**. Note that an autonomous equation is a separable differential equation.

If $f(y) = 0$ is zero at $y = a$, then the horizontal line $y = a$ is a solution. This solution is called the **equilibrium solution** and a is called a critical point. After finding the equilibrium solutions, check the sign of f. On intervals of y with $y' = f(y)$ positive, the solutions y are increasing and on intervals of y with $y' = f(y)$ negative, the solutions y are decreasing. Thus, the analysis of the sign of $f(y)$ can tell us a lot about the graph of the solutions.

If the solutions asymptotically approach the equilibrium solution for $t \to \infty$, regardless of the values of initial conditions, then the equilibrium solution is called **asymptotically stable** solution. If the solutions are not converging towards the equilibrium solution for all values of initial conditions, then the equilibrium solution is called **unstable** solution. In case that on one side the solutions are converging towards the equilibrium solution and on the other they do not, we say that the equilibrium solution is **semistable**.

In some cases, it may be difficult to obtain an explicit formula of the solution. In those cases, getting a graph of an autonomous equation may provide valuable information about the solution.

Applications - Models of population growth

1. **Unlimited Growth**. The simplest model of population growth is one describing the population that the population changes at a rate proportional to its size. The differential equation model for that situation is
$$\frac{dP}{dt} = kP$$
We encounter this situation in examples when the percent birth rate is b and the percent death rate is c so that $\frac{dP}{dt} = bP - cP$. Let $k = b - c$. If $b > c$ (so $k > 0$) the population will be increasing, if $b < c$ (so $k < 0$) the population will be decreasing and if $b = c$ (so $k = 0$) the population will remain constant.

   If the initial size is P_0, separating the variables you obtain $\frac{dP}{P} = kdt \Rightarrow lnP = kt + c \Rightarrow P = e^{kt+c} = Ce^{kt}$. Using the initial condition, you obtain that $C = P_0$. So, the solution is $P = P_0 e^{kt}$.

   For a given set of data, a good way to **test** if the unlimited growth is a suitable modes is to check if the dependence of $\ln y$ and x is linear.

2. **Limited Growth**. If a given environment has limited resources to support the population growth, the population might not increase indefinitely. Suppose that the rate of increase is proportional to the population size and the difference between a constant K (called **carrying capacity**) limiting the growth. The differential equation model for that situation is

   $$\frac{dP}{dt} = kP(K - P)$$

   Graphing the solution of this autonomous equation, we can see that the population size increases to P if the initial size P_0 is smaller than K but the growth does not increase the capacity K. If $P_0 > K$, the population size decreases to K as the population is too large to grow due to the limiting resources of the environment. If $P_0 = K$, the solution is constant $P = K$.

   Note that in this case, the equilibrium solution $P = K$ is stable since $\lim_{t\to\infty} P = K$ regardless of the initial size.

   The differential equation is separable and can be solved analytically by using partial fractions. The general solution is

   $$P = \frac{K}{1 + (K/P_0 - 1)e^{-kKt}}$$

   This function is called the **logistic curve** and the growth (if $P_0 < K$) is referred to as logistic growth. Note that the maximal growth can be obtained by considering the zero of the second derivative. Since $P'' = k(KP' - 2PP') = kP'(K - 2P)$, $P'' = 0$ when $P = K/2$. Thus, the increase of population keeps increasing until half of the carrying capacity is reached. At the time when $P = K/2$ the growth is the largest. After that time, the growth decreases towards zero.

   For a given set of data, a good way to **test** if the limited growth is a suitable modes is to check if the dependence of $\ln \frac{y}{K-y}$ and x is linear.

3. **A model with a threshold level.** Suppose that a given population can keep increasing just if the initial size is large enough and it dies out otherwise. The level that allows the increase of population is called a **threshold level**. In this case the rate of increase is proportional to the population size and the difference of the present population size and the threshold level T. The differential equation model for that situation is

   $$\frac{dP}{dt} = kP(P - T)$$

   Graphing the solution of this autonomous equation, we can see that the population size decreases to 0 if the initial size P_0 is smaller than T. If $P_0 > T$, the population size increases without a bound. If $P_0 = T$, the solution is constant $P = T$. Note that in this case, the equilibrium solution $P = T$ is unstable.

Example. Assume that a whale population is such that

1. The population becomes extinct if the number of whales falls below a minimum survival level m.

2. If the population is above the minimum survival level, then the growth is limited by the carrying capacity M.

3. If the population is above M, then it will decline due to the environment that cannot sustain such high population level.

Write a differential equation that models this situation.

Solution. Let $P = P(t)$ denote the whale population at time t. We can use an autonomous differential equation with equilibrium solutions $P = m$ and $P = M$. The conditions determine that

1. If $P_0 < m$, P is to decrease towards 0. Thus $P' < 0$.

2. If $m < P_0 < M$, P is to increase towards M. Thus $P' > 0$.

3. If $P_0 > M$, P is to decrease towards M. Thus $P' < 0$.

These conditions are satisfied for the product $(M - P)(P - m)$. Thus the differential equations

$$\frac{dP}{dt} = k(M - P)(P - m)$$

where k is a positive constant, models the given situation.

Practice Problems.

1. Sketch the graph of solutions of the following equations.

   a) $y' = y^2 - 2y$ b) $y' = -y^2 + 2y$ c) $y' = (y + 1)(y - 2)^2$

   d) $y' = y(y + 1)(y - 2)$ e) $y' = y(2 - y)^2(5 - y)^3$

2. Determine the stability of the equilibrium solutions of following equations

   a) $y' = (y - a)(y - b)$ b) $y' = y(ay^2 - b)$

   where a and b are constants. For part b), you can assume that $a > 0$.

3. The size of a population of rabbits is modeled by differential equation $P' = -kP(100 - P)$ where k is a positive parameter.

   a) Estimate the number of rabbits after a long period of time if the initial size of the population is 103 rabbits.

   b) Estimate the number of rabbits after long period of time if the initial size of the population is 99 rabbits.

   c) If $k = 0.02$ per year, use the Euler program to estimate the size of the population after 4 years if the population was 99 initially. Use 0.5 for the step size.

4. The Pacific halibut fishery is modeled by differential equation $B' = kB(K - B)$ where B is the biomass (total mass of the members of the population) in kilograms at time t, $K = 8 \cdot 10^7$ kg and $k = 8.7 \cdot 10^{-9}$ per year.

a) Estimate the biomass after many years if the initial biomass is $3 \cdot 10^6$.

b) Estimate the biomass after many years if the initial biomass is $9 \cdot 10^7$.

c) If the biomass is $2 \cdot 10^7$ kg initially, use the Euler program to estimate the biomass 5 years later. Use 0.5 for the step size.

5. In a seasonal-growth model for population growth, a periodic function of time is introduced to account for seasonal variations in the rate of growth. Such variations could, for example, be caused by seasonal changes in the availability of food. The rate of change of the population is proportional to the size of population multiplied with a periodic function. Thus, this situation can be modeled by the differential equation

$$\frac{dP}{dt} = kP\cos(rt - \phi) \quad P(0) = P_0,$$

where k, r, ϕ and P_0 are positive constants. Find the solution of this differential equation.

Solutions.

1. a) To find equilibrium solution solve $y^2 - 2y = y(y-2) = 0 \Rightarrow y = 0$ and $y = 2$. Then analyze the sign of y'. $\frac{+ \quad\quad - \quad\quad +}{\quad 0 \quad\quad 2 \quad}$. Use this information to sketch the graph of general solutions: above $y = 2$, and below $y = 0$, the solutions are increasing, and between $y = 0$ and $y = 2$ the solutions are decreasing. From the graph, you can see that $y = 0$ is asymptotically stable and $y = 2$ is unstable.

b) $-y^2 + 2y = -y(y-2) = 0 \Rightarrow y = 0$ and $y = 2$. $\frac{- \quad\quad + \quad\quad -}{\quad 0 \quad\quad 2 \quad}$. Thus, above $y = 2$, and below $y = 0$, the solutions are decreasing, and between $y = 0$ and $y = 2$ the solutions are increasing. Conclude that $y = 2$ is asymptotically stable and $y = 0$ is unstable.

c) Equilibrium solutions: $y = -1$ and $y = 2$. Sign of y' : $\frac{- \quad\quad + \quad\quad +}{\quad -1 \quad\quad 2 \quad}$. Thus, below $y = -1$, the solutions are decreasing. Between $y = -1$ and $y = 2$ and above $y = 2$ the solutions are increasing. Conclude that $y = 2$ is semistable and $y = -1$ is unstable.

d) Equilibrium solutions: $y = -1$, $y = 0$ and $y = 2$. Sign of y' : $\frac{- \quad\quad + \quad\quad - \quad\quad +}{\quad -1 \quad\quad 0 \quad\quad 2 \quad}$. Conclude that $y = 0$ is stable and $y = -1$ and $y = 2$ are unstable.

e) $y' = y(2-y)^2(5-y)^3 = 0 \Rightarrow y = 0$, $(2-y)^2 = 0$ or $(5-y)^3 = 0 \Rightarrow y = 0$, $2-y = 0$ or $5-y = 0$. So, the equilibrium solutions are $y = 0$, $y = 2$ and $y = 5$. Sign of y' : $\frac{- \quad\quad + \quad\quad + \quad\quad -}{\quad 0 \quad\quad 2 \quad\quad 5 \quad}$. Conclude that $y = 0$ is unstable, $y = 2$ is semistable and $y = 5$ is stable.

2. a) You can consider the following cases: $a > b$, $a < b$ and $a = b$. In the first case, the sign of y' is $\frac{+ \quad\quad - \quad\quad +}{\quad b \quad\quad a \quad}$ So, $y = b$ is stable and $y = a$ is unstable. In the second case, a and b are interchanged in the previous number line so $y = a$ is stable and $y = b$ is unstable. If $a = b$, there is just one equilibrium solution and it is semistable.

b) Note that $y' = y(ay^2 - b) = 0 \Rightarrow y = 0$ or $y^2 = \frac{b}{a}$. This brings you to the following cases.

Case 1. $\frac{b}{a} \leq 0$. In this case, $y = 0$ is the only solution, so $y = 0$ is the only equilibrium solution. The sign of y' is given by $\frac{-\qquad +}{0}$ so $y = 0$ is unstable solution. Thus, the solutions with positive initial value increase to infinity and those with negative initial value decrease to negative infinity.

Case 2. $\frac{b}{a} > 0$. In this case, the equation $y(ay^2 - b) = 0$ has three solutions: $y = 0$ and $y = \pm\sqrt{\frac{b}{a}}$. The sign of y' can be obtained using the number line $\frac{-\qquad\qquad +\qquad -\qquad\qquad +}{-\sqrt{b/a}\qquad 0\qquad \sqrt{b/a}}$ So, $y = 0$ is stable and $y = \pm\sqrt{\frac{b}{a}}$ are unstable. Thus, the solutions with initial values in $(\frac{-b}{a}, \frac{b}{a})$, converge towards 0, the solutions with initial values larger than $\frac{b}{a}$ increase without bounds and the solutions with initial values smaller than $\frac{-b}{a}$ decrease without bounds.

3. Parts a) and b) can be obtained by analyzing the graph and stability of the equilibrium solutions. $-kP(100 - P) = 0 \Rightarrow P = 0$ and $P = 100$. Sign of P' : $\frac{+\qquad -\qquad +}{0\qquad 100}$. Thus, with initial condition above $P = 100$ (and below $P = 0$ but that is not relevant in this case) the solutions are increasing. In particular, if the initial population size is 103, the population will be increasing. Thus $\lim_{t\to\infty} P = \infty$. So, the population size increases without bounds. The solutions with initial conditions between $P = 0$ and $P = 100$ are decreasing. In particular, if the initial population size is 99, the population will be decreasing to 0. Thus $\lim_{t\to\infty} P = 0$. So, the population size decreases to 0 in this case.

   c) Inline the equation $y' = -0.02y(100 - y)$, 0 for x-initial, 99 for y-initial, 4 for x-final and 0.5 for the step size (so that $n = \frac{4-0}{0.5} = 8$) and obtain that the population size decreased to about 7.63 (can round to 8) four years after.

4. Parts a) and b) can be obtained by analyzing the graph and stability of the equilibrium solutions. $B' = kB(8 \cdot 10^7 - B) = 0 \Rightarrow B = 0$ and $B = 8 \cdot 10^7$. Sign of B' : $\frac{-\qquad +\qquad -}{0\qquad 8\cdot 10^7}$. Thus, with initial condition above $B = 8 \cdot 10^7$ (and below $B = 0$ but that is not relevant in this case) the solutions are decreasing. In particular, if the initial biomass is $9 \cdot 10^7$, the biomass will be decreasing to $8 \cdot 10^7$ so $\lim_{t\to\infty} B = 8 \cdot 10^7$. The solutions with initial conditions between $B = 0$ and $B = 8 \cdot 10^7$ are increasing. In particular, if the initial biomass is $3 \cdot 10^7$, the biomass will be increasing to $8 \cdot 10^7$. Thus $\lim_{t\to\infty} B = 8 \cdot 10^7$ as well.

   c) Inline the equation $y' = 8.7 \cdot 10^{-9} y(8 \cdot 10^7 - y)$, 0 for x-initial, $2 \cdot 10^7$ for y-initial, 5 for x-final and 0.5 for the step size (so $n = \frac{5-0}{0.5} = 10$) and obtain that the biomass increased to $74242300 \approx 7.4 \cdot 10^7$ kg in 5 years.

5. Separate the variables $\frac{dP}{dt} = kP\cos(rt - \phi) \Rightarrow \frac{dP}{P} = k\cos(rt - \phi)dt \Rightarrow \ln P = \int k\cos(rt - \phi)dt \Rightarrow \ln P = \frac{k}{r}\sin(rt - \phi) + c \Rightarrow P = e^{\frac{k}{r}\sin(rt-\phi)+c} = Ce^{\frac{k}{r}\sin(rt-\phi)}$. Using the initial condition, obtain that $C = P_0$. Thus $P = P_0 e^{\frac{k}{r}\sin(rt-\phi)}$.

Modeling with First Order Differential Equations

In most cases, it is equally important to be able to come up with a differential equation that accurately describes the problem you need to solve as to being able to solve the equation. The process of writing an equation describing the situation is referred to as mathematical modeling. In order to successfully model a problem with and solve a differential equation, it might be helpful to ask yourself the questions listed below.

1. **Identify the real problem. Identify the problem variables.** What do we need to find out? What is the problem asking for?

2. **Construct appropriate relation between the variables – a differential equation** What is dependent, what independent variable and what is the rate of change? Figuring out how these quantities are related will result in a differential equation that models the problem.

3. **Obtain the mathematical solution.** Recognize the type of the equation. Decide if you can solve it analytically ('by hand') or if you need to use the technology. In both cases, decide on the method that you will use (e.g. is the equation separable, linear or some other type; could Euler's method, ODE45 or other numerical solution be found).

4. **Interpret the mathematical solution.** After solving the equation, check if the mathematical answer agrees with the context of the original problem. **Check the validity**: Does your answer make sense? Do the predictions agree with real data? Do the values have correct sign? Correct units? Correct size? **Check effectiveness**: Could a simpler model be used? Have I found a right balance between greater precision (i.e. greater complexity) and simplicity?

Example 1. A bacteria culture starts with 500 bacteria and grows at a rate proportional to its size. After 3 hours there are 8000 bacteria. Find the number of bacteria after 4 hours.

Discussion. Identifying variables: let y stands for the bacteria culture and t stands for time passed. The first part of the problem "A bacteria culture starts with 500 bacteria.." tells us that $y(0) = 500$. The second part "... and grows at a rate proportional to its size." is the key for getting the mathematical model. Recall that the rate is the derivative and that "...is proportional to.." corresponds to " equal to constant multiple of.." So, the equation relating the variables is $\frac{dy}{dt} = ky$. The solution of this differential equation is $y = y_0 e^{kt}$. Since $y_0 = 500$, it remains to determine the proportionality constant k. From the condition "After 3 hours there are 8000 bacteria." we obtain that $8000 = 500e^{3k}$ which gives us that $k = \frac{1}{3}\ln 16 = .924$. Thus, the number of bacteria after t hours can be described by $y = 500e^{.924t}$.

Solution. Using the function we have obtained, we find the number of bacteria after 4 hours to be $y(4) = 20159$ bacteria.

Example 2. Suppose that an object is falling in the atmosphere near the sea level. Assume that the drag is proportional to the velocity with the drag coefficient of 2 kg/sec and that the mass of the object is 10 kg.

(a) Formulate a differential equation describing the velocity of the object. Find the limiting velocity by analyzing the equation that models this situation.

(b) Find the general solution of the equation and its limit of the solution for $t \to \infty$. Compare the answer with the part (a).

(c) Assuming that the object is *dropped* from a height of 300 m, determine how long it will take for the object to hit the ground and how fast it will go at the time of the impact.

Solution. (a) Identifying variables: Let v denotes the velocity and t denotes the time. Our goal is to get the differential equation describing the fall. The physical law that governs the motion is the Newton's second law $F = ma$. As $a = \frac{dv}{dt}$, we have that $F = m\frac{dv}{dt}$. We have two forces acting on this object: gravitational force equaling mg where $g = 9.8$ m/sec^2 and drag force which is, by assumption of the problem equal to $2v$. Since these two forces act in the opposite directions, the total force is equal to the difference of these two forces. Thus, we have that

$$m\frac{dv}{dt} = mg - 2v.$$

The mass of the object in question is 10 kg, so we have that $\frac{dv}{dt} = 9.8 - \frac{v}{5}$.

This is an autonomous equation. Thus, we can sketch the solutions by analyzing the sign of $9.8 - \frac{v}{5}$. Since $9.8 - \frac{v}{5} = 0$ for $v = 49$, and the sign is changing from positive to negative, this means that the velocity will

- increase to 49 m/sec if the initial velocity is between 0 and 49;
- stay constant at 49 m/sec if the initial velocity is 49; and
- drop to 49 m/sec if the initial velocity is larger than 49.

This also gives us that the equilibrium solution $v = 49$ is stable.

(b) The equation $\frac{dv}{dt} = 9.8 - \frac{v}{5}$ is separable. Multiply by 5 (for simplicity) and separate the variables. Get $\frac{5dv}{49-v} = dt \Rightarrow -5\ln(49 - v) = t + c \Rightarrow \ln(49 - v) = \frac{-t}{5} + c \Rightarrow 49 - v = e^{-t/5}C$. Solve for v and get the general solution $v = 49 - Ce^{-t/5}$.

When $t \to \infty$, the term $e^{-t/5}$ approaches 0. Thus, $v = 49 - Ce^{-t/5} \to 49 - 0 = 49$. This agrees with the analysis from (a) and also illustrates that the terminal velocity will be 49 m/sec regardless of the initial velocity.

(c) As the object is dropped, the initial velocity is 0. Find the particular solution with $v(0) = 0$ from $v = 49 - Ce^{-t/5}$. Get $0 = 49 - C$. So, $C = 49$ and thus $v = 49 - 49e^{-t/5}$.

In order to determine the velocity and time of the impact, we need to find a formula describing the distance x as a function of time t. As $v = \frac{dx}{dt}$, $x = 49t + 245e^{-t/5} + c$. Note that here the coordinate system is chosen so that gravity acts in a positive direction (downward) so, x measures the distance of the object from the initial hight to the current position. Thus, the initial position of the object corresponds to $x(0) = 0$. This gives us that $0 = 0 + 245 + c$. Thus, $c = -245$ and so $x = 49t + 245e^{-t/5} - 245$.

So, if the initial height is 300 meters, the time the object hits the ground can be obtained from the equation

$$300 = 49t + 245e^{-t/5} - 245 \Rightarrow 49t + 245e^{-t/5} - 545 = 0$$

Using Matlab, your calculator or any other technology, we obtain that the object will hit the ground after $t = 10.51$ second. The velocity at that time is $v(10.51) = 49 - 49e^{-10.51/5} = 43.01$ m/sec.

Let us also discuss a different choice of the coordinate system in this problem: if x denotes the distance from the current position to the ground, then x is decreasing as time passes by so velocity has negative values. The equation that corresponds to this choice of the coordinate system is $mv' = -mg + 2|v| \Rightarrow mv' = -mg - 2v$. This equation also has a stable equilibrium solution with same absolute value as the one previously discussed, just the sign is the opposite. With $m = 10$, the equation is $v' = -9.8 - \frac{1}{5}v \Rightarrow \frac{-5dv}{49+v} = dt \Rightarrow \ln(49+v) = \frac{-t}{5} + c \Rightarrow v = Ce^{-t/5} - 49$.

With the initial condition $v(0) = 0$, we have that $v = 49e^{-t/5} - 49$. In this case $x = -245e^{-t/5} - 49t + c$ and with the initial height of 300 m, the initial condition should be $x(0) = 300$ in this case, not $x(0) = 0$. This gives us $300 = -245 + c \Rightarrow c = 545$.

To find the time the object hits the ground, set the particular solution $x = -245e^{-t/5} - 49t + 545$ to 0. When multiplied by -1, this gives us the same equation as before $245e^{-t/5} + 49t - 545 = 0$ and the same t-value 10.51 second. The velocity at that time is $49e^{-10.51/5} - 49 = -43.01$ m/sec.

Example 3. A population of field mice inhabits a certain rural area. In the absence of predators, the mice population increases so that each month, the population increases by 50%. However, several owls live in the same area and they kill 15 mice per day. Find an equation describing the population size and use it to predict the long term behavior of the population.

Solution. Identifying variables: Let y stands for the size of mice population and t be the time in months. In this case, the total change in mice population $\frac{dy}{dt}$ can be describe as the difference of the rate at which the population is increasing and the rate at which the population is decreasing:

$$\frac{dy}{dt} = \text{ rate in - rate out.}$$

The population increases by 50% so the rate in is $0.5y$. Incorporating the information about the owls, we must subtract the monthly loss in the number of mice. As 15 are killed daily, $15 \cdot 30 = 450$ is killed monthly and so the rate out is 450. Thus, the equation is $\frac{dy}{dt} = .5y - 450$.

Solving the equation we obtain, $y = ce^{t/2} + 900$. Graphing the solutions with different initial conditions, we can see that the number of mice will

- drop to 0 if the initial number is smaller than 900,
- stay constant at 900 if the initial number is equal to 900, and
- keep increasing if the initial number is larger than 900.

Thus $y = 900$ is an unstable equilibrium solution in this case.

Example 4. If the current temperature of an object is T_o and the room temperature is T_r, the Newton's Law of Cooling states that the rate of cooling of the object is proportional to the temperature difference between the room temperature the temperature of the object.

a) Write a differential equation and initial condition whose solution would describe the temperature of the object as a function of time. Make a rough sketch of general solutions. When does the object cool most quickly? What happens to the rate of cooling as the time goes by?

b) Let us consider a 95° C coffee cup that is in a 20° C room. Assume that the proportionality constant is 0.1. Solve the differential equation in part a) to find the function describing the temperature of the coffee as a function of time (in minutes). Use the solution to estimate the temperature of the coffee after 20 minutes.

Solution. a) If T denotes the temperature at time t, the differential equation $\frac{dT}{dt} = k(T_r - T)$ and the initial condition $T(0) = T_o$ model the situation. Note that a common mistake is to write the differential equation as $\frac{dT}{dt} = k(T_r - T_o)$ because the right side of the equation is constant and the object is not cooling at a constant rate. If so, the solution would be a linear, not an exponential function.

The initial value problem $\frac{dT}{dt} = k(T_r - T)$ and $T(0) = T_o$ is a homogeneous equation with one equilibrium solution $T = T_r$. Since the solutions are decreasing if temperature is higher than T_r, and increasing if the temperature is lower than T_r, the solution T= T_r is stable (this conclusion could be reached by considering the sign of the right side without any physical interpretation of it). From the graph one can also see that the object cools down more quickly initially. The rate decreases as the time goes by.

b) This is a separable differential equation. Separate the variables and integrate to obtain the general solution $T = T_r - ce^{-kt}$. Substituting the initial condition, we obtain $T = T_r - (T_r - T_o)e^{-kt}$. Thus, if $k = .1$, $T_r = 20$ and $T_o = 95$, and $t = 20$, then $T = 30.15$ degrees Centigrade.

Example 5. A tank initially contains 15 thousands gallons of pure water. A mixture of water and dye enters the tank at the rate of 3 thousands gallons per day and the mixture flows out at the same rate. The concentration of dye in the incoming water is increasing in time t according to the expression $0.5t$ grams per gallon. Determine the differential equation and an appropriate initial condition that model this situation. Find the corresponding particular solution and use it to determine the amount of dye in the tank after 3 days.

Solution. If Q denotes the amount of dye measured in grams and t the time measured in days, the rate of change of Q (in grams per day) is equal to the difference of rate of flow in and rate of flow out of the pond

$$\frac{dQ}{dt} = \text{ rate in - rate out.}$$

Since the rate in is $3 \cdot 10^3 \frac{\text{gal}}{\text{day}} 0.5t \frac{\text{g}}{\text{gal}}$ and the rate out is $3 \cdot 10^3 \frac{\text{gal}}{\text{day}} \frac{Q}{15 \cdot 10^3} \frac{\text{g}}{\text{gal}}$, the equation

$$\frac{dQ}{dt} = 1500t - 0.2Q$$

models this situation. Since the tank initially contains no dye, the initial condition corresponding to this situation is $Q(0) = 0$.

The equation $Q' = 1500t - 0.2Q$ is linear. Write it in the form $Q' + 0.2Q = 1500t$ and find integrating factor to be $e^{0.2t}$. So, $Qe^{.2t} = 1500 \int te^{.2t} dt = 1500(\frac{1}{0.2} te^{0.2t} - \frac{1}{0.2^2} e^{0.2t}) + c$. Thus $Q = 7500(t-5) + ce^{-0.2t}$ is the general solution.

$Q(0) = 0 \Rightarrow 0 = 0 - 37500 + c \Rightarrow c = 37500$. The particular solution is $Q = 7500(t-5) + 37500e^{-0.2t} = 7500(t - 5 + 5e^{-0.2t})$. After three days, $t = 3$ and so $Q = 5580.44$ grams or 5.58 kg.

Practice Problems.

1. A population of bacteria grows at a rate proportional to the size of population. The proportionality constant is 0.7. Initially, the population consist of two members. Find the population size after six days.

2. A glucose solution is administered intravenously into the bloodstream at a constant rate r. As the glucose is added, it is converted into other substances and removed from the bloodstream at a rate proportional to the concentration at that time.

a) Set up a differential equation that models this situation.

b) If $r = 4$ and the proportionality constant is 2, sketch the graphs of general solutions and examine their stability. Determine the concentration of the glucose after a long period of time.

c) Suppose that the initial amount is 1 mg/cm^3. Solve the equation with this initial condition and sketch the graph of this solution.

3. Experiments show that if the chemical reaction $N_2O_5 \to 2NO_2 + \frac{1}{2}O_2$ takes place at 45 degrees Celsius, the rate of reaction of dinitrogen pentoxide is proportional to its concentration $C(t)$ with proportionality constant equal to -.0005.

   a) Find an expression for the concentration of dinitrogen pentoxide if the initial concentration is C_0.

   b) Determine how long it takes for the concentration of N_2O_5 to be reduced to 90% of its original value.

4. Let $A(t)$ be the area of tissue culture at time t (in days). Let the final area of the tissue when the growth is complete be 10 cm^2. Most cell divisions occur on the periphery of the tissue and the number of cells on the periphery is proportional to $\sqrt{A}$. So, a reasonable model for the growth of tissue is obtained by assuming that the rate of growth is jointly proportional to $\sqrt{A}$ and $10 - A$ (i.e. proportional to the product of $\sqrt{A}$ and $10 - A$).

   a) Formulate the differential equation that models this situation.

   b) If the proportionality constant is 1/4 and the area of tissue culture initially is 1 cm^2, use the Euler's method (you can use Matlab) to approximate the area of the culture after 3, 5 and 10 days using the step size 0.2 for each approximation. Sketch the graph of the solution.

   c) Display the list of x and y values you obtained. Suppose that we can consider the growth complete if the area is within 0.01 cm^2 from 10 cm^2. Determine the approximate time at which we can consider the growth complete.

5. Let a be a positive number. Suppose that y is a function whose derivative is proportional to y^{1+a}. Differential equation that models this situation is

$$y' = ky^{1+a}$$

   This differential equation is called a **doomsday equation** because the exponent of y on the right side of the equation is larger than that for the natural growth ($1+a > 1$). This produces a solution with a vertical asymptote. The x-value corresponding to this asymptote is called **the doomsday** because y-values diverge to infinity when the x-values approach this finite value.

   Assume now that the size of an especially prolific breed of rabbits can be modeled by this differential equation. Suppose that $k = 1$, $a = \frac{1}{100}$ there are 2 rabbits initially, and the time is measured in months. Determine the doomsday i.e. the time when there will be infinitely many rabbits.

Solutions.

1. The equation is $\frac{dy}{dt} = 0.7y$. Separating the variables and solving gives you $y = Ce^{0.7t}$. Using the initial condition $y(0) = 2$, the solution becomes $y = 2e^{0.7t}$. Plugging 6 for t produces $y(6) = 133$ protozoa.

2. a) Let y denotes the concentration at time t. The concentration is increasing at a constant rate r and is decreasing at a rate proportional to y. Let k denote the proportionality constant. Then the rate in is r and the rate out is ky. So, the differential equation $y' = r - ky$ models this situation.

   b) With the given values, the equation becomes $y' = 4 - 2y$. The equilibrium solution is $4 - 2y = 0 \Rightarrow 4 = 2y \Rightarrow y = 2$. $y' = 4 - 2y > 0$ for $y < 2$ and $y' = 4 - 2y < 0$ for $y > 2$. Thus, the solutions are decreasing toward the equilibrium solution if $y_0 > 2$ and increasing towards 2 if $y_0 < 2$. So, $y = 2$ is stable. This means that regardless of the initial concentration, the concentration will become 2 mg/cm^3 after sufficiently long time period.

   c) Separate the variables $\frac{dy}{dt} = 4 - 2y \Rightarrow \frac{dy}{4-2y} = dt$. Integrate both sides. Use $u = 4 - 2y$ for the left side. Obtain $\frac{1}{-2}\ln(4-2y) = t + c \Rightarrow \ln(4-2y) = -2t - 2c \Rightarrow 4 - 2y = e^{-2t-2c} \Rightarrow 2y = 4 - e^{-2t-2c} \Rightarrow y = 2 - \frac{1}{2}e^{-2t-2c} = 2 - \frac{e^{-2c}}{2}e^{-2t}$. Denoting $\frac{e^{-2c}}{2}$ by C, we obtain that $y = 2 - Ce^{-2t}$. Note that the term Ce^{-2t} converges to 0 and so $y \to 2$ for $t \to \infty$ regardless of the value of the constant C. This agrees with the conclusion from part b).

   If $y(0) = 1$, then $1 = 2 - Ce^0 \Rightarrow 1 = 2 - C \Rightarrow C = 1$. Thus $y = 2 - e^{-2t}$.

3. a) The equation is $\frac{dC}{dt} = -.0005C$. Separating the variables and solving gives you $C = C_0e^{-.0005t}$.
   b) Solve $0.9C_0 = C_0e^{-.0005t}$ for t. Get $t = \frac{\ln 0.9}{-.0005} = 210.72$ seconds or about 3.5 minutes.

4. a) The differential equation is $\frac{dA}{dt} = k\sqrt{A}(10 - A)$.

   b) Inline the equation $y' = \frac{1}{4}\sqrt{y}(10 - y)$, and use 0 for x-initial, 1 for y-initial, 10 for x-final, and 0.2 for the step size so that $n = \frac{10-0}{0.2} = 50$. Obtain that the area is 8.3 cm^2 after 3 days, it is 9.67 cm^2 after 5 days and it is 9.995 cm^2 after 10 days. The graph is a logistic curve with the horizontal asymptote $y = 10$ and y-intercept 1.

   c) Display the list of (x, y) values. Since 10-0.01=9.99, you are looking for the smaller x-value in your list for which the y-value is at least 9.99. Note that at $x = 9.2$ the y-value became larger than 9.99. So, after 9.2 days, the growth can be considered complete.

5. $\frac{dy}{dt} = y^{1.01} \Rightarrow y^{-1.01}dy = dt \Rightarrow \frac{y^{-0.01}}{-0.01} = t + c \Rightarrow y^{-0.01} = -.01t - .01c = -0.01t + C \Rightarrow y = (-0.01t + C)^{-100}$. Using the initial condition $y(0) = 2$, obtain that $2 = C^{-100} \Rightarrow C = 2^{-.01} = .993$. So, the particular solution is $y(t) = (-.01t + 0.993)^{-100} = \frac{1}{(-.01t+0.993)^{100}}$. This function has a vertical asymptote at x-value corresponding to a zero of the denominator. $-.01t + 0.993 = 0 \Rightarrow t = \frac{0.993}{0.01} = 99.3$. So, the doomsday is in 99.3 months.