Differential Equations
Lia Vas

Second and higher order differential equations. Homogeneous equations with constant coefficients

A second order differential equation is **linear** if it can be written in the form

$$a(x)y'' + b(x)y' + c(x)y = g(x).$$

The general solution of such equation will depend on two constants. An **initial-value problem** for the second order equation consists of finding the solution of the second order differential equation that satisfies the conditions

$$y(x_0) = y_0 \text{ and } y'(x_0) = y_1.$$

A **boundary-value problem** for the second order equation consists of finding the solution of the second order differential equation that satisfies the conditions

$$y(x_0) = y_0 \text{ and } y(x_1) = y_1.$$

Homogeneous equations with constant coefficients. A linear differential equation is called **homogeneous** if $g(x) = 0$. To find the general solution of such differential equation, it is sufficient to find two solution $y_1(x)$ and $y_2(x)$ which are not constant multiple of one another (linearly independent solutions). Such two solutions are said to be **fundamental solutions**. Then, the general solution has the form

$$y(x) = c_1y_1(x) + c_2y_2(x).$$

If a homogeneous equation has **constant coefficients** (that is if a, b and c are constants,) then the function of the form $y = e^{rx}$ is a solution iff r is a solution of the **characteristic equation**

$$ar^2 + br + c = 0.$$

This is indeed so since plugging the function $y = e^{rx}$ and its derivatives $y' = re^{rx}$ and $y'' = r^2e^{rx}$ into the equation $ay'' + by' + cy = 0$ produces $ar^2e^{rx} + bre^{rx} + ce^{rx} = 0 \Rightarrow (ar^2 + br + c)e^{rx} = 0$. Note that this last relation holds if and only if r is a solution of the characteristic equation $ar^2 + br + c = 0$.

The equation $ar^2 + br + c = 0$ has either two real solutions r_1 and r_2, single real solution $r_1 = r_2$, or a pair of complex solutions $p \pm iq$. These cases correspond exactly to the the discriminant $b^2 - 4ac$ being either positive, zero, or negative. Let us consider the impact of the three cases to the solution of the original, differential equation $ay'' + by' + cy = 0$.

Case 1 If the characteristic equation has two real and distinct roots r_1 and r_2, then e^{r_1x} and e^{r_2x} are two fundamental solutions and the general solution is $y = c_1e^{r_1x} + c_2e^{r_2x}$.

Case 2 If the characteristic equation has one real root $r_1 = r_2$, then e^{r_1x} and xe^{r_1x} are two fundamental solutions and the general solution is $y = c_1e^{r_1x} + c_2xe^{r_1x}$.

Case 3 If the characteristic equation has two complex roots $p \pm iq$, then $e^{px}\cos qx$ and $e^{px}\sin qx$ are two fundamental solutions and the general solution is $y = c_1e^{px}\cos qx + c_2e^{px}\sin qx$.

Practice Problems.

a) Solve the following differential equations.

1. $y'' - 6y' + 8y = 0$
2. $y'' - y' - 6y = 0$
3. $y'' - 2y' + y = 0$
4. $y'' - 4y' + 4y = 0$
5. $y'' - 2y' + 2y = 0$
6. $y'' + 4y = 0$
7. $y'' - 2y' + 5y = 0$

b) Solve the following initial-value problems.

1. $y'' - 6y' + 8y = 0, \;\; y(0) = 0, \;\; y'(0) = 2$
2. $y'' - 2y' + y = 0, \;\; y(0) = 2, \;\; y'(0) = 3$
3. $y'' + 4y = 0, \quad y(0) = 2, \;\; y'(0) = 2$

c) Solve the following boundary-value problems.

1. $y'' - 6y' + 8y = 0 \;\; y(0) = 0, \;\; y(1) = e^2$
2. $y'' - 2y' + y = 0, \;\; y(0) = 2, \;\; y(1) = 0$
3. $y'' + 4y = 0, \quad y(0) = 3, \;\; y(\pi/4) = -2$

Solutions.

a)
1. The characteristic equation is $r^2 - 6r + 8 = 0 \Rightarrow (r-4)(r-2) = 0 \Rightarrow r = 4, r = 2$. Thus $y_1 = e^{4x}$ and $y_2 = e^{2x}$ so the general solution is $y = c_1e^{4x} + c_2e^{2x}$.
2. The characteristic equation is $r^2 - r - 6 = 0 \Rightarrow r = 3, r = -2$. Thus $y = c_1e^{3x} + c_2e^{-2x}$.
3. The characteristic equation is $r^2 - 2r + 1 = 0 \Rightarrow r = 1$ is a double zero. Hence $y_1 = e^x$ and $y_2 = xe^x$ and the general solution is $y = c_1e^x + c_2xe^x$.
4. $r = 2$ is a double zero of the characteristic equation and so $y = c_1e^{2x} + c_2xe^{2x}$.
5. The characteristic equation is $r^2 - 2r + 2 = 0 \Rightarrow r = \frac{2\pm\sqrt{4-8}}{2} = \frac{2\pm 2i}{2} = 1 \pm i$. Hence $y_1 = e^x \cos x$ and $y_2 = e^x \sin x$ and the general solution is $y = c_1e^x \cos x + c_2e^x \sin x$.
6. The characteristic equation is $r^2 + 4 = 0 \Rightarrow r^2 = -4 \Rightarrow r = \pm 2i$. Hence $y = c_1 \cos 2x + c_2 \sin 2x$.
7. The characteristic equation is $y'' - 2y' + 5y = 0 \Rightarrow r^2 - 2r + 5 = 0 \Rightarrow r = \frac{2\pm\sqrt{-16}}{2} = 1 \pm 2i$. Hence $y_1 = e^x \cos 2x$ and $y_2 = e^x \sin 2x$ so the general solution is $y = c_1e^x \cos 2x + c_2e^x \sin 2x$.

b)
1. Note that the general solution is $y = c_1e^{4x} + c_2e^{2x}$ so that $y' = 4c_1e^{4x} + 2c_2e^{2x}$. From the first condition $c_1 + c_2 = 0$. From the second $4c_1 + 2c_2 = 2$. Solving this system of equations produces $c_1 =$ and $c_2 =$ Hence $y = e^{4x} - e^{2x}$.

2. The general solution is $y = c_1e^x + c_2xe^x$ so that $y' = c_1e^x + c_2e^x + c_2xe^x$. Using the initial conditions produces $y = 2e^x + xe^x$.

3. The general solution is $y = c_1 \cos 2x + c_2 \sin 2x$ so that $y' = -2c_1 \sin 2x + 2c_2 \cos 2x$. Using the initial conditions produces $y = 2\cos(2x) + \sin(2x)$.

c) 1. The general solution is $y = c_1e^{4x} + c_2e^{2x}$. From the first boundary condition $c_1 + c_2 = 0$. From the second $c_1e^4 + c_2e^2 = e^2$. Thus $c_2 = -c_1$ and $c_1e^4 - c_1e^2 = e^2 \Rightarrow c_1e^2 - c_1 = 1 \Rightarrow c_1 = \frac{1}{e^2-1} \Rightarrow c_2 = \frac{-1}{e^2-1}$ $y = \frac{1}{e^2-1}e^{4x} - \frac{1}{e^2-1}e^{2x}$.

2. The general solution is $y = c_1e^x + c_2xe^x$. Using the boundary conditions produces $y = 2e^x - 2xe^x$

3. The general solution is $y = c_1 \cos 2x + c_2 \sin 2x$. Using the boundary conditions produces $y = 3\cos(2x) - 2\sin(2x)$

Higher Order Linear Differential Equations

The method of solving homogeneous differential equations of second order generalizes for solving homogeneous differential equations of higher order with constant coefficients.

Recall that a linear higher order differential equation is of the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_0(x)y = g(x).$$

A homogeneous linear differential equation with constant coefficients has the form

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0y = 0.$$

Its characteristic equation

$$a_nr^n + a_{n-1}r^{n-1} + \ldots + a_0 = 0$$

has n solutions which produce n fundamental solutions $y_1, y_2, \ldots, y_n$. The general solution is

$$y = c_1y_1 + c_2y_2 + \ldots + c_ny_n.$$

The following cases match the three cases of the case $n = 2$.

1. If $r_1, r_2, \ldots, r_m$ are different real solutions for some integer $m \leq n$, then e^{r_1x}, $e^{r_2x}, \ldots,$ e^{r_mx} are fundamental solutions.

2. If $r_1 = r_2 = \ldots = r_m$ for some integer $m \leq n$, then e^{r_1x}, $xe^{r_1x}, \ldots,$ $x^me^{r_mx}$ are fundamental solutions.

3. If $p_1 \pm q_1i, p_2 \pm q_2i, \ldots, p_m \pm q_mi$ are solutions of characteristic equation for some integer m such that $2m \leq n$, then $e^{p_1x}\cos q_1x$, $e^{p_1x}\sin q_1x$, $e^{p_2x}\cos q_2x$, $e^{p_2x}\sin q_2x, \ldots,$ $e^{p_mx}\cos q_mx$, $e^{p_mx}\sin q_mx$ are fundamental solutions.

Examples. Find general solutions of the following equations.

1. $y''' - 2y'' - y' + 2y = 0$ 2. $y''' - 2y'' + y' = 0$ 3. $y^{(4)} + 8y'' - 9y = 0$

Solutions.

1. The characteristic equation is $r^3 - 2r^2 - r + 2 = 0$. The left side factors as $r^3 - 2r^2 - r + 2 = r^2(r-2) - (r-2) = (r-2)(r^2-1) = (r-2)(r-1)(r+1)$ Hence $r = 1, -1, 2$. Thus, e^x, e^{-x} and e^{2x} are three fundamental solutions and the general solution is $y = c_1e^x + c_2e^{-x} + c_3e^{2x}$.

2. The characteristic equation is $r^3 - 2r^2 + r = 0$. Factoring this equation, we get $r(r-1)^2 = 0$, so $r = 0$ is a solution and $r = 1$ is a double solution. Thus, $e^{0x} = 1$, e^x and xe^x are three fundamental solutions and the general solution is $y = c_1 + c_2e^x + c_3xe^x$.

3. The characteristic equation is $r^4 + 8r^2 - 9 = 0$. The equation of the type $ar^4 + br^2 + c = 0$ is called a **biquadratic equation**. You can solve it by using the formula for quadratic equation except that the formula produces solutions for r^2, not r. In this case, you can also factor and obtain $(r^2-1)(r^2+9) = 0$ which produces the solutions $r^2 = 1$ and $r^2 = -9$. The first equation produces $r = \pm 1$ and the second and $r = \pm 3i$. Thus, e^x, e^{-x}, $\cos 3x$ and $\sin 3x$ are four fundamental solutions and the general solution is $y = c_1e^x + c_2e^{-x} + c_3\cos 3x + c_4\sin 3x$.

In order to better understand the complex roots case as well as the methods of finding solutions in case when the characteristic equation is of the form $r^n - a = 0$, we review a few facts about the complex numbers.

Complex Numbers

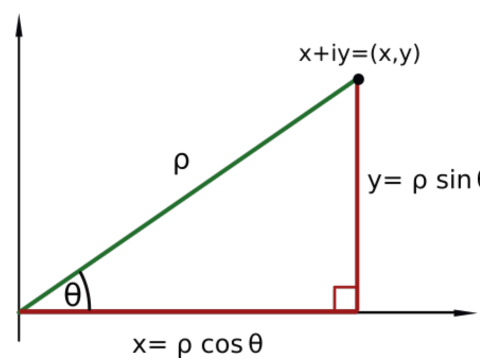
Complex numbers were introduced during the course of the study of algebraic equations and, in particular, the solutions of equations that involve square roots of negative real numbers.

The expression $\sqrt{-1}$ is denoted by i. A **complex number** is any expression of the form $x + iy$ where x and y are real numbers. In this case, x is called the **real** part and y is called the **imaginary** part of the complex number $x + iy$. This complex number can be represented in the real plane as a point with coordinate (x, y). The complex number $x - iy$ is said to be the **complex conjugate** of the number $x + iy$. Note that it is represented by the point $(x, -y)$.

Trigonometric Representations. Let us recall the polar coordinates $x = \rho\cos\theta$ and $y = \rho\sin\theta$. Using this representation, we have that

$$z = x + iy = \rho\cos\theta + i\rho\sin\theta.$$

Recall that ρ [1] is the distance from the point (x, y) to the origin and the angle θ is the angle between the radius vector of (x, y) and the positive part of x-axis.



If $z = x + iy = \rho(\cos\theta + i\sin\theta)$, then ρ is called the **modulus** or the absolute value of z and the angle θ is called the **argument** or the **phase** of z.

[1] We use the notation ρ instead of r used in Calculus 2 not to mix it up with the variable in the characteristic equation.

Euler's formula.

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This formula is especially useful in the solution of differential equations. Euler's formula was proved (in an obscured form) for the first time by Roger Cotes in 1714, then rediscovered and popularized by Euler in 1748. Euler's proof uses the power series for $e^x, \sin x$ and $\cos x$.

Using Euler's formula, we have that

$$z = x + iy = \rho(\cos\theta + i\sin\theta) = \rho e^{i\theta}$$

so a complex number can be represented without a use of addition. This can be especially useful when finding a power or a root of a complex number. For example, the trigonometric representation yields an easy formula for the n-th power of a complex number $z = \rho e^{i\theta}$.

$$z^n = \rho^n e^{in\theta} = \rho^n(\cos(n\theta) + i\sin(n\theta))$$

When solving algebraic equations of the form $z^n = a$ where a is a given complex number $a = \rho(\cos(\theta) + i\sin(\theta))$, we can obtain n solutions of the equation by the formula

$$\sqrt[n]{\rho}e^{\frac{(\theta+2k\pi)i}{n}} = \sqrt[n]{\rho}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right) \text{ for } k = 0, 1, \ldots n-1.$$

These solutions have a nice representation in the complex plane: they form the vertices of a regular polygon with n-sides inscribed in the circle of radius $\sqrt[n]{\rho}$ centered at the origin. We illustrate this in the following examples.

Examples. Find general solutions of the following equations.

1. $y''' + 8y = 0$
2. $y^{(5)} - 32y = 0$

Solutions.

1. The characteristic equation is $r^3 + 8 = 0$. Thus, we need to find all three solutions of the equation $r^3 = -8$. Note that -8 corresponds to the complex number $(-8, 0)$ which is on the negative side of the x-axis so $\theta = \pi$. The distance from $(-8, 0)$ to the origin is 8 so $\rho = 8$. Hence, the three solutions of the characteristic equation can be found by the formula

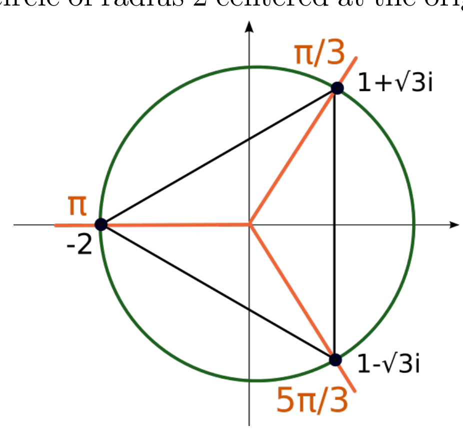
$$\sqrt[3]{8}\, e^{\frac{\pi+2k\pi}{3}i} = 2\, e^{\frac{\pi+2k\pi}{3}i} \text{ for } k = 0, 1, 2.$$

These three solutions form an equilateral triangle on the circle of radius 2 centered at the origin.

$k = 0 \Rightarrow r_0 = 2e^{\frac{\pi}{3}i} = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = 1 + \sqrt{3}i$

$k = 1 \Rightarrow r_1 = 2e^{\frac{3\pi}{3}i} = 2e^{\pi i} = 2(\cos\pi + i\sin\pi) = -2$

$k = 2 \Rightarrow r_2 = 2e^{\frac{5\pi}{3}i} = 2(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}) = 1 - \sqrt{3}i.$



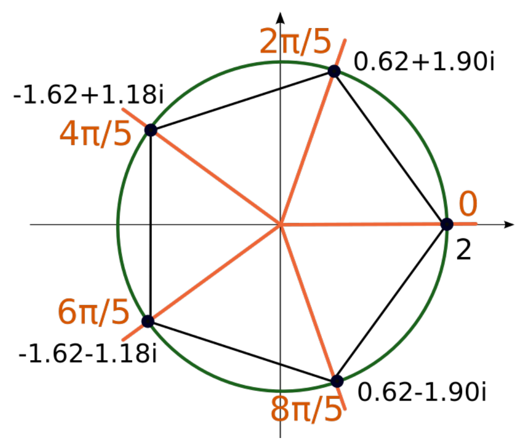
Note that r_1 is real and r_0 and r_2 represent a complex pair. They produce the fundamental solutions $y_1 = e^{-2x}$, $y_2 = e^x \cos\sqrt{3}x$ and $y_3 = e^x \sin\sqrt{3}x$. Thus, the general solution is $y = c_1 e^{-2x} + c_2 e^x \cos\sqrt{3}x + c_3 e^x \sin\sqrt{3}x$.

2. The characteristic equation is $r^5 - 32 = 0$. Thus, we need to find all five solutions of the equation $r^5 = 32$. Note that 32 corresponds to the complex number $(32, 0)$ which is on the positive side of the x-axis so $\theta = 0$. The distance from $(32, 0)$ to the origin is 32 so $\rho = 32$. Hence, the five solutions of the characteristic equation can be found by the formula

$$\sqrt[5]{32} e^{\frac{0+2k\pi}{5} i} = 2e^{\frac{2k\pi}{5} i} \text{ for } k = 0, 1, \ldots, 4.$$

These five solutions form a regular polygon with five sides on the circle of radius 2 centered at the origin.

$$\begin{aligned}
k = 0 \Rightarrow \quad & r_0 = 2e^{0i} \quad = 2,\\
k = 1 \Rightarrow \quad & r_1 = 2e^{\frac{2\pi}{5} i} = 2(\cos\tfrac{2\pi}{5} + i \sin\tfrac{2\pi}{5}) \approx\\
& \qquad 0.62 + 1.90i,\\
k = 2 \Rightarrow \quad & r_2 = 2e^{\frac{4\pi}{5} i} = 2(\cos\tfrac{4\pi}{5} + i \sin\tfrac{4\pi}{5}) \approx\\
& \qquad -1.62 + 1.18i,\\
k = 3 \Rightarrow \quad & r_3 = 2e^{\frac{6\pi}{5} i} = 2(\cos\tfrac{6\pi}{5} + i \sin\tfrac{6\pi}{5}) \approx\\
& \qquad -1.62 - 1.18i,\\
k = 4 \Rightarrow \quad & r_4 = 2e^{\frac{8\pi}{5} i} = 2(\cos\tfrac{8\pi}{5} + i \sin\tfrac{8\pi}{5}) \approx\\
& \qquad 0.62 - 1.90i.
\end{aligned}$$



Solution 2 corresponds to $y_1 = e^{2x}$. Roots r_1 and r_4 are conjugated producing two fundamental solutions $y_2 = e^{0.62x} \cos 1.90x$ and $y_3 = e^{0.62x} \sin 1.90x$. Roots r_2 and r_3 are conjugated, producing another pair of fundamental solutions $y_4 = e^{-1.62x} \cos 1.18x$ and $y_5 = e^{-1.62x} \sin 1.18x$. Thus, the general solution is
$y = c_1 e^{2x} + c_2 e^{0.62x} \cos 1.90x + c_3 e^{0.62x} \sin 1.90x + c_4 e^{-1.62x} \cos 1.18x + c_5 e^{-1.62x} \sin 1.18x$.

Fundamental Theorem of Algebra. A quadratic equation $ax^2 + bx + c = 0$ can have two (possibly equal) real solutions or *no* real solutions. As opposed to this situation, in the complex plane, every quadratic equation has *exactly* two solutions (possibly equal). Similar claim holds for every polynomial: Every polynomial (with complex coefficients) of degree n has exactly n solutions (some possibly equal) in the complex plane. This is statement is known as the Fundamental Theorem of Algebra.

Moreover, if an n-th degree polynomial with **real coefficient** has a complex root $a + ib$, then its complex conjugate $a - ib$ is also the root of a polynomial. Thus, the complex roots appear in *conjugated pairs.* Thus, if $r_1 = a + ib$ and $r_2 = a - ib$ constitute a conjugated complex pair, then two fundamental solutions that correspond to this conjugated pair originate from

$$e^{(a+ib)x} = e^{ax} e^{ibx} = e^{ax}(\cos bx + i \sin bx).$$

Since the solutions are real-valued functions, two solutions can be taken to be $e^{ax} \cos bx$ and $e^{ax} \sin bx$.

Finding zeros of polynomials in Matlab. Unlike the situation for quadratic equation, there is no general formula for polynomials of degrees higher than 4 (more about this in the Abstract Algebra course). Even for cubic or quartic polynomials when such formula exists, it is rather complex to use. Thus, unless a polynomial is easy to factor or to use the n-th root formula, it is convenient to find approximate solutions using Matlab or some other technology.

In Matlab, you can find zeros of polynomial $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$ using the command **roots**. Represent the polynomial as a vector of length $n+1$ with coefficients of the polynomial as the entries

$$\mathbf{p=}[a_n \;\; a_{n-1} \;\; \ldots \;\; a_1 \;\; a_0]$$

and then use the command

$$\mathbf{roots(p)}$$

Example. Find general solution of the equation $-90y^{(4)} + 100y''' - 54y' + 16y = 0$ by using Matlab to find solutions of the characteristic equation.

Solution. The characteristic equation is $-90r^4 + 100r^3 - 54r + 16 = 0$. Represent the polynomial on the left side in Matlab as **p=[-90 100 0 -54 16]** and use the command **roots(p)** to get the solutions $r =$ -0.6900, 0.3511 and $0.7250 \pm 0.4562i$. This gives you four fundamental solutions $y_1 = e^{-0.69x}$, $y_2 = e^{0.3511x}$ $y_3 = e^{0.7250x}\cos 0.4562x$ and $y_4 = e^{0.7250x}\sin 0.4562x$. So, the general solution is $y = c_1 e^{-0.69x} + c_2 e^{0.3511x} + c_3 e^{0.7250x}\cos 0.4562x + c_4 e^{0.7250x}\sin 0.4562x$.

Practice Problems. Find general solutions of the following differential equations.

1. $y^{(4)} - y = 0$.
2. $y^{(4)} - 5y'' - 36y = 0$.
3. $y''' - 8y = 0$
4. $y^{(5)} + 32y = 0$.
5. $-18y^{(5)} + 25y^{(4)} - 27y'' + 16y' + 20y = 0$. Use Matlab to find the solutions of the characteristic equation.

Solutions.

1. The characteristic equation is $r^4 - 1 = 0 \Rightarrow (r^2-1)(r^2+1) = 0 \Rightarrow r = \pm 1$ and $r = \pm i$. The first pair $r = \pm 1$ produces the fundamental solutions $y_1 = e^x$ and $y_2 = e^{-x}$ and the second $r = \pm i$ produces $y_3 = \cos x$ and $y_4 = \sin x$. So, the general solution is $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$.

   Alternatively, you can find the four solutions by considering four solutions of the equation $r^4 = 1 = 1e^{0i}$ by the formula $\sqrt[4]{1} e^{\frac{2k\pi}{4}i} = 1\, e^{\frac{k\pi}{2}i}$ for $k = 0, 1, 2, 3$. Obtain that $r_0 = 1, r_1 = i, r_2 = -1$ and $r_3 = -i$ yield the same general solution.

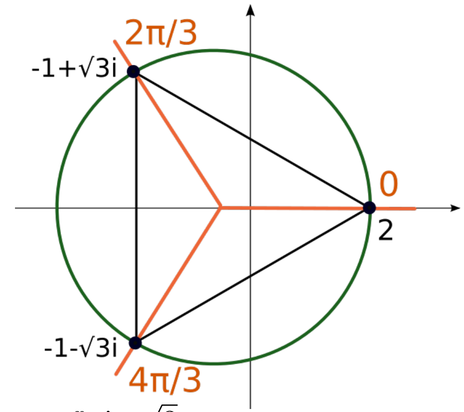
2. The characteristic equation is $r^4 - 5r^2 - 36 = 0 \Rightarrow (r^2 - 9)(r^2 + 4) = 0 \Rightarrow r = \pm 3, r = \pm 2i$. The first pair $r = \pm 3$ produces the fundamental solutions $y_1 = e^{3x}$ and $y_2 = e^{-3x}$ and the second $r = \pm 2i$ produces $y_3 = \cos 2x$ and $y_4 = \sin 2x$. So, the general solution is $y = c_1 e^{3x} + c_2 e^{-3x} + c_3 \cos 2x + c_4 \sin 2x$.

3. The characteristic equation is $r^3 - 8 = 0$. You can approach this equation on two ways.

   One way is to find the solutions of $r^3 = 8 = 8e^{0i}$ by the formula $\sqrt[3]{8}\, e^{\frac{2k\pi}{3}i} = 2e^{\frac{2k\pi}{3}i}$ for $k = 0, 1, 2$.

$k = 0 \ \Rightarrow r_0 = 2e^{0i} = 2$
$k = 1 \Rightarrow r_1 = 2e^{\frac{2\pi}{3}i} = 2(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}) = -1+\sqrt{3}i$
$k = 2 \Rightarrow r_2 = 2e^{\frac{4\pi}{3}i} = 2(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}) = -1-\sqrt{3}i.$

These solutions correspond to the fundamental solutions $y_1 = e^{2x}$, $y_2 = e^{-x}\cos\sqrt{3}x$ and $y_3 = e^{-x}\sin\sqrt{3}x$.

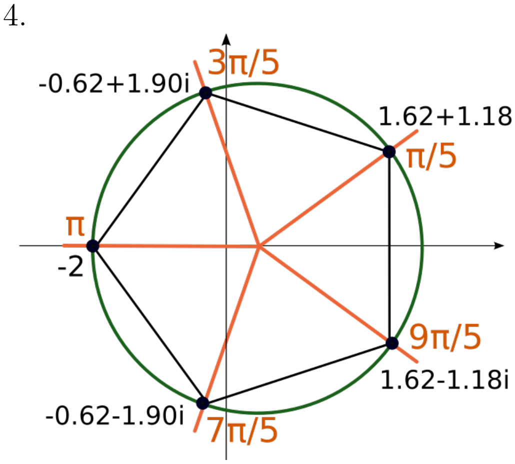


Thus, the general solution is $y = c_1e^{2x} + c_2e^{-x}\cos\sqrt{3}x + c_3e^{-x}\sin\sqrt{3}x$.

Alternatively, you can factor $r^3 - 8$ as $(r-2)(r^2+2r+4)$ and use the quadratic formula to find zeros of the second term. Obtain $r = 2, r = -1 \pm i\sqrt{3}$ which produce the same general solution as above.

4. The characteristic equation is $r^5 + 32 = 0 \Rightarrow r^5 = -32 = 32e^{\pi i}$. Hence, using the formula produces $r_k = \sqrt[5]{32}e^{\frac{\pi+2k\pi}{5}i} = 2e^{\frac{(2k+1)\pi}{5}i}$ for $k = 0, 1, \ldots, 4$.

$k = 0 \Rightarrow \quad r_0 = 2e^{\frac{\pi}{5}i} = 2(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}) \approx 1.62 + 1.18i,$
$k = 1 \Rightarrow \quad r_1 = 2e^{\frac{3\pi}{5}i} = 2(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}) \approx -0.62 + 1.90i,$
$k = 2 \Rightarrow \quad r_2 = 2e^{\frac{5\pi}{5}i} = 2e^{\pi i} = 2(\cos\pi + i\sin\pi) = -2,$
$k = 3 \Rightarrow \quad r_3 = 2e^{\frac{7\pi}{5}i} = 2(\cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}) \approx -0.62 - 1.90i,$
$k = 4 \Rightarrow \quad r_4 = 2e^{\frac{7\pi}{5}i} = 2(\cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}) \approx 1.62 - 1.18i.$



Roots r_0 and r_4 are conjugated and r_1 and r_3 are conjugated. The general solution is $y = c_1e^{-2x} + c_2e^{1.62x}\cos 1.18x + c_3e^{1.62x}\sin 1.18x + c_4e^{-0.62x}\cos 1.90x + c_5e^{-0.62x}\sin 1.90x$.

5. The characteristic equations corresponds to a polynomial p that can be represented in Matlab as **p=[-18 25 0 -27 16 20]**. The command **roots(p)** gives you the following values: 1.2971, $0.7664 \pm 0.9707i$ and $-0.7205 \pm 0.2023i$. Thus, the general solutions is $y = c_1e^{1.2971x} + c_2e^{0.7664x}\cos 0.9707x + c_3e^{0.7664x}\sin 0.9707x + c_4e^{-0.7205x}\cos 0.2023x + c_5e^{-0.7205x}\sin 0.2023x$.