Differential Equations
Lia Vas

Formulas for Exams 1 and 2

1. **Homogeneous equation** $ay'' + by' + cy = 0$.

   Three cases depending on the solutions r_1 and r_2 of the characteristic equation:

	solutions of the characteristic equation $ar^2 + br + c = 0$	fundamental solution of the equation $ay'' + by' + cy = 0$
1.	$r_1 \neq r_2$ real	$y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$
2.	$r_1 = r_2$ real	$y_1 = e^{r_1 x}$ and $y_2 = xe^{r_1 x}$
3.	r_1, r_2 complex $p \pm qi$	$y_1 = e^{px} \cos qx$ and $y_2 = e^{px} \sin qx$

   Polar representation of complex numbers and Euler's formula:

$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = r\ e^{\theta i}.$$

   Finding n-values of the n-th root of a complex number $a = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}$,

$$\sqrt[n]{a} = \sqrt[n]{r}\ e^{\frac{(\theta+2k\pi)i}{n}} \qquad \text{for } k = 0, 1, \ldots n-1.$$

2. **Non-homogeneous equation** $ay'' + by' + cy = g(x)$.

   The general solution is $y = y_h + y_p$ where y_h is the homogeneous and y_p the particular solution.

   (a) **Variation of Parameters.** If $y_h = c_1 y_1 + c_2 y_2$ is the solution of the homogeneous part of find the particular solution in the form $y_p = v_1 y_1 + v_2 y_2$. Find the unknown functions v_1 and v_2 from the equations

$$\begin{aligned} v_1' y_1 + v_2' y_2 &= 0 \\ a v_1' y_1' + a v_2' y_2' &= g(x) \end{aligned}$$

   (b) **Undetermined Coefficients.** Below $p_k(x)$ is a polynomial $a_k x^k + a_{k-1}x^{k-1} + \ldots + a_0$ of degree k and p and q are real numbers.

   Case 1 If $g(x) = p_k(x)e^{px}$, then $y_p = x^s(A_k x^k + A_{k-1}x^{k-1} + \ldots + A_0)e^{px}$ where s is the number of times p appears on the list of zeros of the characteristic equation and $A_0, \ldots, A_k$ are undetermined coefficients.

   Case 2 $g(x) = p_k(x)e^{px}\cos qx$ or $g(x) = p_k(x)e^{px}\sin qx$, then

$$y_p = x^s(A_k x^k + A_{k-1}x^{k-1} + \ldots + A_0)e^{px}\cos qx + x^s(B_k x^k + B_{k-1}x^{k-1} + \ldots + B_0)e^{px}\sin qx$$

   where s is the number of times $p + iq$ appears on the list of zeros of the characteristic equation and $A_0, \ldots, A_k$ and $B_0, \ldots, B_k$ are undetermined coefficients.

   If $g(x)$ is a sum of functions $g(x) = g_1(x) + g_2(x) + \ldots + g_m(x)$ and each function $g_1(x)$, $g_2(x), \ldots, g_m(x)$ is a function described under two cases above, then the particular solution y_p is the sum of particular solutions

$$y_p = y_{p1} + y_{p2} + \ldots + y_{pm}$$

   where each solution y_{pi}, $i = 1, \ldots, m$ is obtained as in Case 1 or 2 described above.

3. **Applications.**

   (a) Equation of the motion of a harmonic oscillator of mass m, the damping constant γ, spring constant k ($k = \frac{mg}{L}$ where L is the total elongation) and an external force $F(t)$ at time t.

   $$mu'' + \gamma u' + ku = F(t).$$

   If $\gamma = 0$, the oscillations are undamped. If $\gamma \neq 0$ and characteristic equation has real zeros, the motion is overdamped. If $\gamma \neq 0$ and characteristic equation has complex zeros, the oscillations are underdamped.

   (b) Equation of the charge of an electric circuit with the resistance R, the capacitance C and the inductance L containing a battery producing the voltage $E(t)$ at time t.

   $$LQ'' + RQ' + \frac{1}{C}Q = E(t).$$

4. **Derivatives and Integrals.**

y	x^n	e^x	b^x	$\ln x$	$\log_b x$	$\sin x$	$\cos x$	$\sin^{-1} x$	$\tan^{-1} x$	$\sec^{-1} x$
y'	nx^{n-1}	e^x	$b^x \ln b$	$\frac{1}{x}$	$\frac{1}{x} \cdot \frac{1}{\ln b}$	$\cos x$	$-\sin x$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{1+x^2}$	$\frac{1}{x\sqrt{x^2-1}}$

y	x^n	e^x	b^x	$\frac{1}{x}$	$\sin x$	$\cos x$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{1+x^2}$
$\int y\, dx$	$\frac{1}{n+1}x^{n+1}$	e^x	$\frac{1}{\ln b}\, b^x$	$\ln \lvert x \rvert$	$-\cos x$	$\sin x$	$\sin^{-1} x$	$\tan^{-1} x$

   (a) Product rule: $y = f \cdot g \Rightarrow y' = f' \cdot g + g' \cdot f$

   (b) Quotient rule: $y = \frac{f}{g} \Rightarrow y' = \frac{f' \cdot g - g' \cdot f}{g^2}$ (c) Chain rule: $y = f(g(x)) \Rightarrow y' = f'(g(x)) \cdot g'(x)$

   Integration by parts formula: $\int u \ dv = u\, v - \int v\, du$

5. **Separable Differential Equation:** $P(x)dx = Q(x)dy$. Integrate both sides.

6. **Linear Differential Equation:** $y' + P(x)y = Q(x)$. Integrating factor: $I(x) = e^{\int P(x)dx}$. After multiplying with $I(x)$, left side of the equation is equal to derivative of $I(x) \cdot y$.

7. **Homogeneous Differential Equation:** $y' = f(\frac{y}{x})$. Use substitution $u = \frac{y}{x}$ to reduce to a separable equation.

8. **Bernoulli's Differential Equation:** $y' + P(x)y = Q(x)y^n$. Use substitution $u = y^{1-n}$ to reduce to a linear equation. In this case $y = u^{1/(1-n)}$.

9. **Exact Differential Equation:** $Mdx + Ndy = 0$ if $M_y = N_x$. Find F as $\int Mdx$ and equate F_y with N. Solution is of the form $F = 0$.

10. **Autonomous Differential Equation:** $\frac{dy}{dt} = f(y)$. Can sketch the solutions without solving it. Find equilibrium solutions and check the sign of $f(y)$.

11. **Basic Differential Equation Models:**

    (a) Rate proportional to the size: $y' = ky$. Here k is a proportionality constant.

    (b) Total rate equal to the difference of rate in and rate out.

    (c) Total force equal to the sum of all acting forces.