Differential Equations
Lia Vas

The Laplace Transform

The Laplace transform is an integral operator (meaning that it is defined via an integral and that it maps one function to the other). It can be useful when solving differential equations because it transforms a linear differential equation with constant coefficients into **an ordinary equation**. The Laplace transform can also be used for reduction of the order of equation of non-constant coefficients.

Differential Equation in y, constant coefficients	Laplace transform $\longrightarrow$	Ordinary Equation in Y
$\uparrow$		$\downarrow$
Get solution y.	Inverse Laplace transform $\longleftarrow$	Find solution Y.

The Laplace transform of a piecewise continuous and exponentially bounded function $f(t)$, defined for non-negative t-values, is defined to be

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)dt$$

Note that the integral on the right side is not a function of t any more but a function of variable s. We shall denote the resulting function $\mathcal{L}[f(t)]$ by $F(s)$.

Example. Find the Laplace transform of the following functions.

a) $f(t) = 1$ b) $f(t) = e^{at}$ c) $f(t) = t$

d) $f(t) = t^2$ e) $f(t) = \sin at$ f) $f(t) = te^{at}$

Solutions. a) $\mathcal{L}[1] = \int_0^\infty e^{-st}dt = \frac{-1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s}$ for $s > 0$.
b) $\mathcal{L}[e^{at}] = \int_0^\infty e^{at-st}dt = \frac{1}{a-s}e^{at-st}\Big|_0^\infty = \frac{-1}{a-s} = \frac{1}{s-a}$ for $s > a$.
c) $\mathcal{L}[t] = \int_0^\infty te^{-st}dt$. To evaluate this integral, use the integration by parts. We have $\frac{-t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\Big|_0^\infty = \frac{1}{s^2}$ for $s > 0$.
d) Using the integration by parts twice, obtain that $\mathcal{L}[t^2] = \frac{2}{s^3}$. Note that using inductive argument, it can be shown that $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$.
e) $\mathcal{L}(\sin at) = \int_0^\infty e^{-st}\sin(at)dt$. Using the integration by parts twice, we obtain $\frac{a}{s^2+a^2}$ for $s > 0$.
f) $\mathcal{L}[te^{at}] = \int_0^\infty te^{at-st}dt$ similarly as in part c) we obtain $\frac{1}{(s-a)^2}$ for $s > a$.

The Laplace transform is linear (since the integral which defines it is linear) thus

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$$

for any constants a and b.

The following table summarizes the Laplace transform of some frequently used functions.

Function $f(t)$	**Laplace transform** $F(s)$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$

Example. Find the Laplace transform of the following.

a) $t^2e^{6t} - 7t^3 + 8$, b) $e^{-2t}\cos 3t + 2\sin 3t - 5$.

Solution. a) $\frac{2}{(s-6)^3} - \frac{42}{s^4} + \frac{8}{s}$ b) $\frac{s+2}{(s+2)^2+9} + \frac{6}{s^2+9} - \frac{5}{s}$.

The Laplace transform of a differential equation will turn out to be an ordinary equation in all the cases we shall consider. However, in order to obtain a solution from the original equation from the solution of the transformed equation, we need to use the **inverse Laplace transform** $\mathcal{L}^{-1}$. Note that

$$\mathcal{L}^{-1}[F(s)] = f(t) \quad \text{if} \quad \mathcal{L}[f(t)] = F(s)$$

Thus, the inverse Laplace transform of a function in the right column of the table above is the corresponding function in the left column. Note that the expressions in the right column have the form of partial fractions from Calculus 2. So, in most applications, the function $F(s)$ will be a rational function whose partial fraction decomposition will match the functions in the above table. Thus, in order to find $\mathcal{L}^{-1}[F(s)] = f(t)$ one would need to find the partial fractions decomposition of $F(s)$. Before going to examples, we review the partial fractions decomposition from Calculus 2 course.

Partial Fraction Decomposition. If $F(s) = \frac{p(s)}{q(s)}$ where p and q are polynomials such that **the degree of** p **is smaller than the degree of** q, to find the partial fraction decomposition perform the following steps.

1. Factor the denominator into a product of powers of linear terms $as + b$ and quadratic terms $as^2 + bs + c$. The quadratic equation $as^2 + bs + c = 0$ should have no real solutions otherwise you would be able to factor $as^2 + bs + c$ into a product of two linear terms.

2. For each power of a linear term of the form $(as+b)^k$, introduce k partial fractions with unknown coefficients $A_1, A_2, \ldots A_k$.

$$\frac{A_1}{as+b} + \frac{A_2}{(as+b)^2} + \ldots + \frac{A_k}{(as+b)^k}$$

3. For each quadratic term of the form $as^2 + bs + c$, introduce a partial fraction with unknown coefficients A and B.

$$\frac{As+B}{as^2+bs+c}$$

4. Determine the unknown coefficients by combining the partial fractions into a single fraction. Be careful when finding the least common denominator: note that it should be *equal to the initial denominator* $q(s)$. Then equate the coefficients of the numerator you obtain with the coefficients of the initial numerator $p(s)$. This should give you a system in all the unknown coefficients. Note that *the number of equations should match the number of unknowns.*

   When you determine the unknown coefficients, you have found the partial fractions decomposition.

5. Write the given rational function as a sum of partial fractions from the previous step. Each partial fraction matches some function on the right side of the table so you can find the inverse Laplace transform of each of them.

Example. Determine the *form* of the partial fractions of the given function (do not determine the constants).

a) $\dfrac{2s-3}{(s^2-1)(s+2)}$ b) $\dfrac{2s-3}{(s-1)^2(s+1)}$

c) $\dfrac{2s-3}{(s-1)^3(s+1)^2}$ d) $\dfrac{2s-3}{(s-1)(s^2+1)}$

Solution. a) The denominator factors as $(s-1)(s+1)(s+2)$. The function decomposes as $\frac{A}{s-1}+\frac{B}{s+1}+\frac{C}{s+2}$.

b) The denominator factors as $(s-1)^2(s+1)$ and the function decomposes as $\frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{C}{s+1}$.

c) There are two linear terms, the first one is with power 3 and the second with power 2. So, the function decomposes as $\frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{C}{(s-1)^3}+\frac{D}{s+1}+\frac{E}{(s+1)^2}$.

d) There is a linear term and a quadratic term (note that s^2+1 cannot be factored further since $s^2+1=0$ has no real solutions). So, the function decomposes as $\frac{A}{s-1}+\frac{Bs+C}{s^2+1}$.

Example. Find the inverse Laplace transform of the following functions.

a) $\frac{5}{s^2+4}$ b) $\frac{8}{(s-2)^4}$ c) $\frac{10}{s^2+3s-4}$ d) $\frac{s+4}{s^2+2s+5}$ e) $\frac{5s^2+3s-2}{s^3+2s^2}$ f) $\frac{3s^2-4s+5}{(s-1)(s^2+1)}$ g) $\frac{s^2}{(s+1)^3}$

Solution. a) $\mathcal{L}^{-1}[\frac{5}{s^2+4}]=\frac{5}{2}\mathcal{L}^{-1}[\frac{2}{s^2+4}]=\frac{5}{2}\sin 2t$ b) $\mathcal{L}^{-1}[\frac{8}{(s-2)^4}]=\frac{8}{6}\mathcal{L}^{-1}[\frac{3!}{(s-2)^{3+1}}]=\frac{4}{3}t^3e^{2t}$

c) Since s^2+3s-4 factors as $(s+4)(s-1)$, in order to find the Laplace transform, we need to find the partial fractions $\frac{A}{s+4}+\frac{B}{s-1}$. We obtain $A=-2$, $B=2$. So, $\mathcal{L}^{-1}[\frac{-2}{s+4}+\frac{2}{s-1}]=-2e^{-4t}+2e^t$.

d) Since s^2+2s+5 cannot be factored in a product of two linear real terms, we have to complete to a sum of squares and to use the formulas for $e^{at}\cos bt$ and $e^{at}\sin bt$. $s^2+2s+5=(s+2s+1)+4=(s+1)^2+2^2$. $\frac{s+4}{s^2+2s+5}=\frac{s+1+3}{(s+1)^2+2^2}=\frac{s+1}{(s+1)^2+2^2}+\frac{3}{2}\frac{2}{(s+1)^2+2^2}$. Hence the inverse Laplace is $e^{-t}\cos 2t+\frac{3}{2}e^{-t}\sin 2t$.

e) Use the partial fractions. $\frac{5s^2+3s-2}{s^3+2s^2}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+2}=\frac{2}{s}-\frac{1}{s^2}+\frac{3}{s+2}$. $\mathcal{L}^{-1}$ is $2-t+3e^{-2t}$.

f) Use the partial fractions. $\frac{3s^2-4s+5}{(s-1)(s^2+1)}=\frac{A}{s-1}+\frac{Bs+C}{s^2+1}=\frac{2}{s-1}+\frac{s-3}{s^2+1}$ $\mathcal{L}^{-1}$ is $2e^t+\cos t-3\sin t$.

g) $\frac{s^2}{(s+1)^3}=\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{C}{(s+1)^3}=\frac{A(s+1)^2+B(s+1)+C}{(s+1)^3}=\frac{As^2+2As+A+Bs+B+C}{(s+1)^3}\Rightarrow$

From the terms with s^2, $A=1$. From the terms with s, $2A+B=0$. Since $A=1$, $B=-2A=-2$. From the terms with no s, $A+B+C=0\Rightarrow C=-B-A=2-1=1$. Thus, $\mathcal{L}^{-1}\left[\frac{s^2}{(s+1)^3}\right]=$ $\mathcal{L}^{-1}\left[\frac{1}{s+1}+\frac{-2}{(s+1)^2}+\frac{1}{(s+1)^3}\right]=\mathcal{L}^{-1}\left[\frac{1}{s+1}\right]-2\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right]+\frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{(s+1)^3}\right]=e^{-t}-2te^{-t}+\frac{1}{2}t^2e^{-t}$.

In order to transform a differential equation using the Laplace transform, we also need to know the Laplace transform of the derivatives of a function. Let Y denotes $\mathcal{L}[y]$ for some function $y(t)$. Using the definition, the Laplace transform of y' can be computed as $\mathcal{L}[y'] = \int_0^\infty y'e^{-st}dt$. Using the integration by parts with $u = e^{-st}$ and $dv = y'dt$, so that $du = -se^{-st}$ and $v = y$, we have that

$$\mathcal{L}[y'] = \int_0^\infty y'e^{-st}dt = ye^{-st}\Big|_0^\infty + s\int_0^\infty ye^{-st}dt = 0 - y(0)e^{-s(0)} + s\int_0^\infty ye^{-st}dt = -y(0) + s\int_0^\infty ye^{-st}dt.$$

Note that the last integral is the Laplace transform of y, $\mathcal{L}[y]$ which we denoted by Y. Hence

$$\mathcal{L}[y'] = -y(0) + s\mathcal{L}[y] = sY - y(0).$$

In a similar way, one obtains the formulas for the other derivatives.

$$\mathcal{L}[y''] = s\mathcal{L}[y'] - y'(0) = s(s\mathcal{L}[y] - y(0)) - y(0) = s^2Y - sy(0) - y'(0),$$

$$\mathcal{L}[y'''] = s\mathcal{L}[y''] - y''(0) = s^3Y - s^2y(0) - sy'(0) - y''(0).$$

Continuing on this way we obtain that

$$\mathcal{L}[y^{(n)}] = s^nY - s^{n-1}y(0) - \ldots - sy^{(n-2)}(0) - y^{(n-1)}(0).$$

Example. Solve the following initial value problems.

a) $y'' - 2y' + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$ b) $y'' + y = \sin 2t$, $y(0) = 2$, $y'(0) = 1$.

Solution. a) Find the Laplace transform of the entire equation. Denoting $\mathcal{L}[y] = Y$, we get $s^2Y - 1 - 2sY + 2Y = \frac{1}{s+1}$. Solving for Y we get $Y = \frac{s+2}{(s+1)(s^2-2s+2)}$. The partial fraction decomposition is $Y = \frac{1}{5(s+1)} + \frac{1}{5}\frac{-s+8}{(s-1)^2+1} = \frac{1}{5(s+1)} - \frac{1}{5}\frac{s-1}{(s-1)^2+1} + \frac{7}{5}\frac{1}{(s-1)^2+1}$. From here $y = \frac{1}{5}(e^{-t} - e^t\cos t + 7e^t\sin t)$.

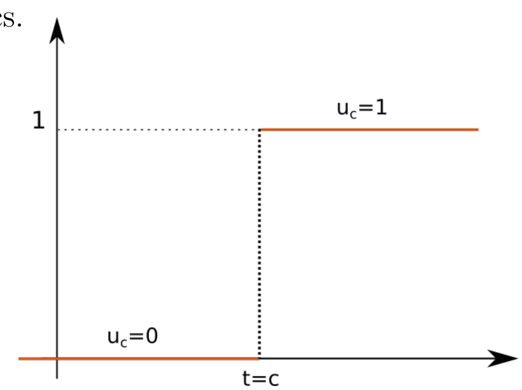
b) Find the Laplace transform of the entire equation. Denoting $\mathcal{L}[y] = Y$, we get $s^2Y - 2s - 1 + Y = \frac{2}{s^2+4}$. Solving for Y we get $Y = \frac{2s^3+s^2+8s+6}{(s^2+4)(s^2+1)}$. The partial fraction decomposition is $Y = \frac{2s+5/3}{s^2+1} + \frac{-2/3}{s^2+4}$. From here $y = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t$.

Step, Boxcar and Delta Functions

In the last two examples we could have used other methods to solve the differential equations (undetermined coefficients or variation of parameters) instead of the Laplace transform. The real importance of the Laplace transform is in its use for finding solutions of the equations for which the other methods fail to be applicable. In particular, Laplace transform can be used to find solutions of differential equations involving **discontinuous functions**. Most frequently used are step, boxcar and impulse functions which frequently appear in physics.

Step Function.

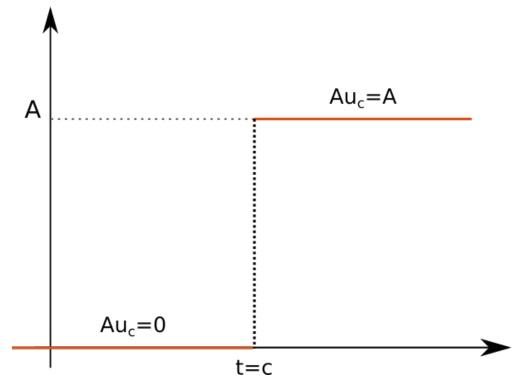
The **unit step or Heaviside function** is the function $u_c(t) = \begin{cases} 0, & t < c, \\ 1, & t \geq c. \end{cases}$



The Laplace transform of u_c is

$$\mathcal{L}[u_c(t)] = \int_c^\infty e^{-st}dt = \frac{-1}{s}e^{-st}\Big|_c^\infty = \frac{e^{-cs}}{s}$$

In some cases, the height of the step may be different than 1. If this height is A, we can represent such function as $Au_c(t)$ because



$$A\,u_c(t) = A\begin{cases} 1, & t \geq c, \\ 0, & t < c. \end{cases} = \begin{cases} A, & t \geq c, \\ 0, & t < c. \end{cases}$$

Using the unit step function, the Laplace transform of a discontinuous function defined by

$$u_c(t)f(t-c) = \begin{cases} f(t-c), & t \geq c, \\ 0, & t < c. \end{cases}$$ can be found to be

$$\mathcal{L}[u_c(t)f(t-c)] = \int_c^\infty e^{-st}f(t-c)dt = \int_0^\infty e^{-s(t+c)}f(t)dt = e^{-sc}\int_0^\infty e^{-st}f(t)dt = e^{-cs}\mathcal{L}[f(t)]$$

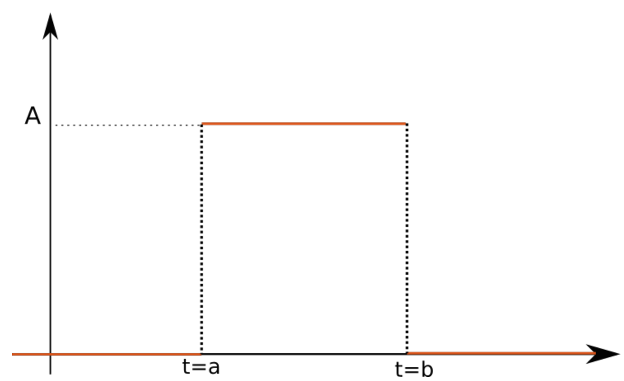
Note that from this follows that

$$\mathcal{L}^{-1}[e^{-cs}F(s)] = u_c(t)f(t-c)$$

Boxcar Function.

For $a < b$, consider a function given by the formula

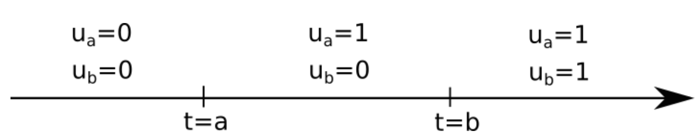
$$\begin{cases} 0, & t < a \\ A, & a \leq t < b, \\ 0, & t \geq b. \end{cases}$$



The graph of such function, on the right, looks like a box, so this function is known as a **boxcar function**.

To find the Laplace transform of a boxcar function, we need to represent it via step functions. Consider the values of u_a and u_b on the three intervals corresponding to the three branches of the boxcar function: $t < a, a \leq t < b$, and $t \geq b$.

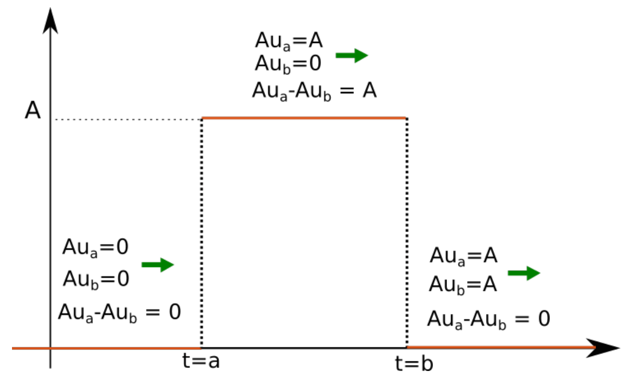
- If $t < a (< b)$, both step functions are 0.
- If $a \leq t < b$, $u_a(t) = 1$ and $u_b(t) = 0$.
- If $t > b (> a)$, both step functions are 1 so $u_a(t) - u_b(t) = 1 - 1 = 0$.



Compare now the values of the function $Au_a(t) - Au_b(t)$ on the three relevant intervals. For $t < a$, $Au_a(t) - Au_b(t) = 0 - 0 = 0$. For $a \leq t < b$, $Au_a(t) - Au_b(t) = A - 0 = A$. For $t \geq b$, $Au_a(t) - Au_b(t) = A - A = 0$. Hence, the original boxcar function is equal to $Au_a(t) - Au_b(t)$.

The Laplace transform can be found as

$$\mathcal{L}[u_a(t) - u_b(t)] = \frac{e^{-as}}{s} - \frac{e^{-bs}}{s}.$$



Example. Find the solution of differential equation

$$y'' + y = \begin{cases} 1, & 5 \leq t < 20, \\ 0, & t < 5 \text{ and } t \geq 20. \end{cases}$$

with the initial conditions $y(0) = 0$ and $y'(0) = 0$. Graph the solution on interval $[0, 30]$.

Solution. The function on the right side is a boxcar function given by $u_5(t) - u_{20}(t)$. Taking the Laplace transform we obtain $s^2Y + Y = \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}$. From here $Y = (e^{-5s} - e^{-20s})\frac{1}{s(s^2+1)}$. In order to find the inverse Laplace transform, it is sufficient to find the inverse Laplace transform of $\frac{1}{s(s^2+1)}$ and to use the property $\mathcal{L}^{-1}[e^{-cs}F(s)] = u_c(t)f(t-c)$. So, let $F(s) = \frac{1}{s(s^2+1)}$ and let us find its partial fractions decomposition $F(s) = \frac{A}{s} + \frac{Bs+C}{s^2+1} = \frac{1}{s} - \frac{s}{s^2+1}$. Then, take the Laplace inverse $\mathcal{L}^{-1}[F(s)] = 1 - \cos t$. Denote this last expression by $f(t)$. Thus, the solution is

$$y = \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}[(e^{-5s}F(s) - e^{-20s}F(s)] = u_5(t)f(t-5) - u_{20}(t)f(t-20) =$$

$$= u_5(t)(1 - \cos(t-5)) - u_{20}(t)(1 - \cos(t-20)).$$

In order to graph the solution, we need to represent it as a piecewise function. Since 5 and 20 divide the number line on three pieces ($t < 5, 5 \leq t20$, and $t \geq 20$), the resulting function consists of three branches.

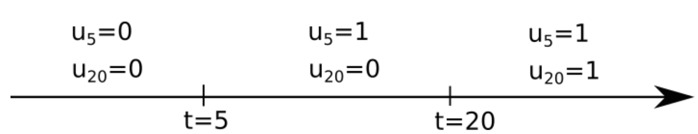
1. Since both $u_5(t)$ and $u_{20}(t)$ are zero for $t < 5$, we obtain the first branch of the function to be zero for $t < 5$.

2. On the interval $5 \leq t < 20$, the function $u_5(t)$ is 1 and $u_{20}(t)$ is 0. Thus, the second branch is $1(1 - \cos(t-5)) - 0(1 - \cos(t-20)) = 1 - \cos(t-5)$ for $5 \leq t < 20$.

3. On the interval $t \geq 20$, both functions $u_5(t)$ and $u_{20}(t)$ are 1. Thus, the third branch is $1(1 - \cos(t-5)) - 1(1 - \cos(t-20)) = -\cos(t-5) + \cos(t-20)$ on $t \geq 20$.

Hence

$$y = \begin{cases} 0, & t < 5 \\ 1 - \cos(t-5), & 5 \leq t < 20, \\ -\cos(t-5) + \cos(t-20), & t \geq 20. \end{cases}$$



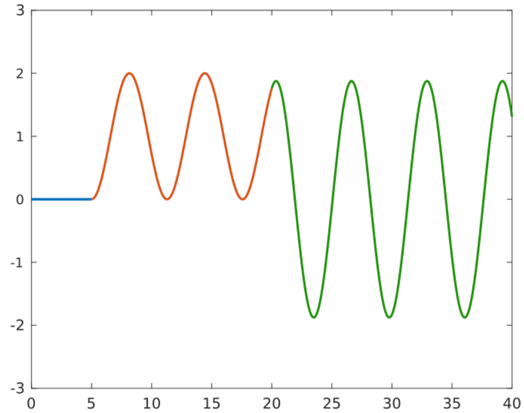
To graph y using Matlab, you can use

syms x

hold on

ezplot(0*x, [0, 5])
ezplot(1-cos(x-5), [5, 20])
ezplot(-cos(x-5)+cos(x-20), [20, 40])
hold off
axis([0 40 -3 3])

To graph y using TI83+, you can enter **0(X<5)+(1-cos(X-5))(5≤ X<20)+(-cos(X-5)+cos(X-20))(X≥ 20)** as a function and graph it. The inequality signs $<$, $\geq$ and other can be found in **2nd Math** menu.



Delta function.

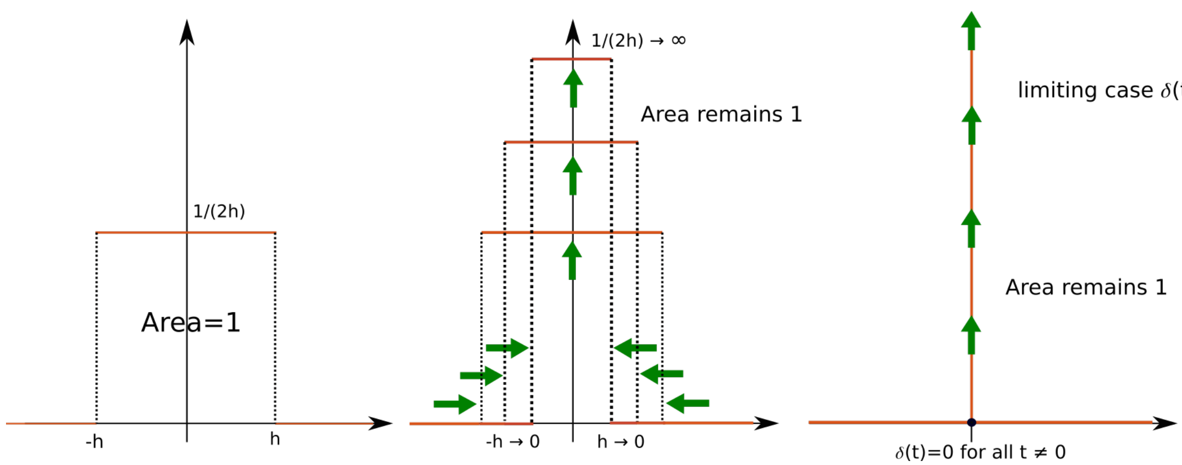
In many applications, it is necessary to represent phenomena of an impulsive nature using mathematical models. For example, voltages of large magnitude that act over a very short period of time. Consider a boxcar function δ_h defined by

$$\delta_h(t) = \begin{cases} \frac{1}{2h}, & -h < t < h, \\ 0, & t \leq -h \text{ and } t \geq h. \end{cases}$$

Note that the area under this function is 1 since the base has length $2h$ and the height is 2h also. Hence, $\int_{-\infty}^{\infty} \delta_h(t)dt = 1$. Now, let $h \to 0$. As h is getting smaller and smaller, the base gets shorter and shorter and the height larger and larger. However, the area under the function still remains 1.

Let $\delta(t)$ denote the limiting case $\delta(t) = \lim_{h\to 0} \delta_h(t)$. Since the base reduces to a single point at $t = 0$, the value of $\delta(t)$ is zero for all values of $t \neq 0$. The value at $t = 0$ is such that the total area under the function $\delta(t)$ is still 1. These two properties can be used to define $\delta(t)$.

1. $\delta(t) = 0$ for all values of $t \neq 0$, and 2. $\int_{-\infty}^{\infty} \delta(t)dt = 1$.



At this point, you probably wonder how having such a “function” is possible because any function equal to zero at all point but 0 has area under it equal to zero, not equal to a positive value such as 1. Many mathematicians wonder about the same thing after Paul Dirac introduced such “function” to model the density of an idealized point mass. Indeed, $\delta(t)$ is not a function in the traditional sense, but a **generalized function.** If a generalized function has area underneath it equal to 1, as $\delta(t)$ does, it is called a **distributions** or a generalized probability function. $\delta(t)$ is also called **unit impulse function** or **Dirac delta function**.

Using the formula for the Laplace transform of a boxcar function from the previous section, $\mathcal{L}[\delta_h(t)] = \frac{e^{hs}}{2hs} - \frac{e^{-hs}}{2hs}$. Using L'Hopital's rule, you can check that $\lim_{h\to 0} \frac{e^{hs}}{2hs} = \frac{s}{2s} = \frac{1}{2}$ and that $\lim_{h\to 0} \frac{e^{-hs}}{2hs} = \frac{-s}{2s} = \frac{-1}{2}$. Hence,

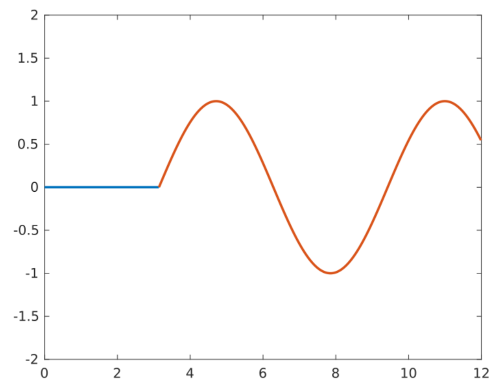
$$\mathcal{L}[\delta(t)] = \mathcal{L}\left[\lim_{h\to 0} \delta_h(t)\right] = \lim_{h\to 0} \frac{e^{hs}}{2hs} - \frac{e^{-hs}}{2hs} = \frac{1}{2} - \frac{-1}{2} = 1.$$

In a similar way, you can check that the Laplace transform of the delta function shifted by c is e^{-cs}.

$$\mathcal{L}[\delta(t-c)] = e^{-cs}$$

Example. Find the solution of differential equation $y'' + y = \delta(t-\pi)$ with the initial conditions $y(0) = 0$ and $y'(0) = 0$. Sketch the graph of the solution.

Solution. Taking the Laplace transform, we obtain $s^2Y + Y = e^{-\pi s}$. Thus $Y = e^{-\pi s}\frac{1}{s^2+1}$ and so $y = u_\pi(t)\sin(t-\pi)$.



To graph the solution, represent it as

$$y = \begin{cases} 0, & t < \pi \\ \sin(t-\pi), & t \geq \pi. \end{cases}$$

Use **ezplot** in combination with **hold on** and **hold off** to graph it in Matlab. On TI83, you can use **0(X**$<\pi$**)+sin(X-**π**)(X**$\geq \pi$**)**.

Integral Equations. The Convolution

The convolution is an operation on two functions $f(t)$ and $g(t)$ defined as follows

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau$$

The significance of this operation for the Laplace transform is because

$$\mathcal{L}[f(t) * g(t)] = \mathcal{L}[f(t)] \cdot \mathcal{L}[g(t)]$$

This property implies that $\mathcal{L}^{-1}[F(s) \cdot G(s)] = \mathcal{L}^{-1}[F(s)] * \mathcal{L}^{-1}[G(s)]$.
Using convolution, the function equations of involving or yielding an integral could also be solved.

Example. Solve the integral equation $y(t) + \int_0^t (t-\tau)y(\tau)d\tau = 1$.

Solution. Note that the equation can be written as $y + t * y = 1$. Let again $Y = \mathcal{L}[y]$. Taking the Laplace transform, we have that $Y + \frac{1}{s^2}Y = \frac{1}{s}$. Solving for Y, we have $Y = \frac{s}{s^2+1}$. Thus, $y = \cos t$.

The equation from the last example belongs to the class of integral equations known as Volterra integral equations. This class of equations was introduced in the early 1900s, by V. Volterra in his study of population growth. The general form of these equations is $y(t) + \int_0^t f(t-\tau)y(\tau)d\tau = g(t)$.

Using convolution, one can also express solution of differential equation involving *any* integrable function $f(t)$ in terms of an integral involving $f(t)$.

Example. Find the solution of the equation $y'' + y = f(t)$, with the initial conditions $y(0) = 0$ and $y'(0) = 0$. Express your answer in terms of an integral involving function $f(t)$.

Solution. Let $Y = \mathcal{L}[y]$ and $F = \mathcal{L}[f]$. The Laplace transform converts the equation to $s^2Y+Y = F$. From here $Y = \frac{F}{s^2+1} = F\frac{1}{s^2+1}$. Applying the inverse Laplace, you obtain

$$y = \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}[F\frac{1}{s^2+1}] = \mathcal{L}^{-1}[F] * \mathcal{L}^{-1}[\frac{1}{s^2+1}] = f * \sin t = \int_0^t f(\tau)\sin(t-\tau)d\tau.$$

Solving Systems Using Laplace Transform

The Laplace transform can be used for solving systems of differential equations. Taking Laplace transform of a system of differential equations produces a system of *ordinary* equations. Solving this new system and taking the inverse Laplace transform of the solution produces the solutions of the original system.

Example. Solve the following system.

$$\begin{array}{ll} x' = -x - 3y & x(0) = 1 \\ y' = -x + y & y(0) = 0 \end{array}$$

Solutions. Let X denote $\mathcal{L}[x]$ and Y denote $\mathcal{L}[y]$. Start by taking the Laplace transform of both equations. Obtain the new system

$$sX - 1 = -X - 3Y \qquad sY = -X + Y.$$

Solving the second equation for X produces $X = Y - sY$. Substitute that in the first equation and solve for Y. $s(Y-sY)-1 = -Y+sY-3Y \Rightarrow sY-s^2Y-1 = sY-4Y \Rightarrow -1 = (s^2-4)Y \Rightarrow Y = \frac{-1}{s^2-4}$. Thus $X = Y - sY = \frac{-1}{s^2-4} + \frac{s}{s^2-4}$. Finding the partial fractions decomposition for X and Y produces $Y = \frac{-1/4}{s-2} + \frac{1/4}{s+2}$ and $X = \frac{1/4}{s-2} + \frac{3/4}{s+2}$. Thus, $x = \mathcal{L}^{-1}[X] = \frac{1}{4}e^{2t} + \frac{3}{4}e^{-2t}$ and $y = \mathcal{L}^{-1}[Y] = \frac{-1}{4}e^{2t} + \frac{1}{4}e^{-2t}$.

Practice Problems.

1. Find the Laplace transform of the following functions.

   (a) $t^4e^{-2t} + \cos 5t - 7$ (b) $\int_0^t \tau^3 e^{t-\tau}d\tau$ (c) $\int_0^t \sin(2\tau)\cos(2t-2\tau)d\tau$

2. Find the inverse Laplace transform of the following functions.

   (a) $\frac{3}{s^2+25}$ (b) $\frac{5}{(s+2)^3}$ (c) $\frac{s+11}{s^2+2s-3}$ (d) $\frac{s-2}{s^2-2s+5}$ (e) $\frac{7s^3-2s^2-3s+6}{s^4-2s^3}$ (f) $\frac{7s^2-41s+84}{(s-1)(s^2-4s+13)}$

3. Use the Laplace transform to solve the following initial value problems.

   (a) $y'' + 3y' + 2y = 0$, $y(0) = 1$, $y'(0) = 0$.

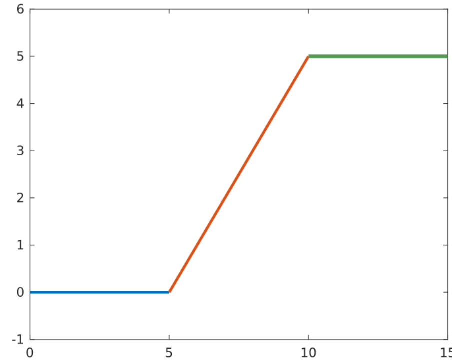
   (b) $y'' - 4y' + 4y = 0$, $y(0) = 1$, $y'(0) = 1$.

   (c) $y'' - 6y' + 5y = 2$, $y(0) = 0$, $y'(0) = -1$.

4. Solve the initial value problem $y'' + 4y = \begin{cases} 0, & t < 5, \\ t-5, & 5 \le t < 10, \\ 5, & t \ge 10, \end{cases}$ with $y(0) = 0$ and $y'(0) = 0$.

The function on the right side of the equation is known as **ramp loading** and can be represented as $u_5(t)(t-5) - u_{10}(t)(t-10)$.

Assume that an undamped harmonic oscillator is described by the given differential equation. Sketch a graph of the solution and comment on the type of motion.



5. Solve the initial value problem $y'' + 4y = \delta(t - 4\pi)$, $y(0) = 1$, $y'(0) = 0$. Sketch a graph of the solution and comment on the type of motion in case the equation describes a harmonic oscillator.

6. Assume that the initial value problem $y'' + 3y' + 4y = \delta(t-3), \quad y(0) = 0, \quad y'(0) = 0$ models the motion y (in cm) of an oscillator as time t (in seconds) passes. Find the solution, write your answer as a piecewise function, sketch its graph and describe the motion of the oscillator.

7. Solve the initial value problem $y'' + 4y = f(t)$, $y(0) = 3$, $y'(0) = -1$. Express your answer in terms of an integral involving function $f(t)$.

8. Solve the integral equation $y(t) + \int_0^t (t-\tau)y(\tau)d\tau = t$.

9. Solve the integro-differential equation $y'(t) + \int_0^t y(t-\tau)e^{-2\tau}d\tau = 1$, $y(0) = 1$.

10. Solve the following systems.

    (a) $\frac{dx}{dt} = -x + y \quad \frac{dy}{dt} = -x - y, \quad x(0) = 1, \quad y(0) = 2$

    (b) $\frac{dx}{dt} = 3x - y \quad \frac{dy}{dt} = 4x - 2y, \quad x(0) = 1, \quad y(0) = 3.$

Solutions.

1. (a) $\frac{24}{(s+2)^5} + \frac{s}{s^2+25} - \frac{7}{s}$ (b) The function is the convolution of t^3 and e^t. Thus the Laplace transform is $\mathcal{L}[t^3]\mathcal{L}[e^t] = \frac{6}{s^4}\frac{1}{s-1} = \frac{6}{s^4(s-1)}$. (c) The function is the convolution of $\sin 2t$ and $\cos 2t$. Thus the Laplace transform is $\mathcal{L}[\sin 2t]\mathcal{L}[\cos 2t] = \frac{2}{s^2+4}\frac{s}{s^2+4} = \frac{2s}{(s^2+4)^2}$.

2. (a) $\frac{3}{5}\sin 5t$ (b) $\frac{5}{2}t^2e^{-2t}$ (c) $3e^t - 2e^{-3t}$ (d) $e^t\cos 2t - \frac{1}{2}e^t\sin 2t$ (e) $1 - \frac{3}{2}t^2 + 6e^{2t}$ (f) $5e^t + 2e^{2t}\cos 3t - 5e^{2t}\sin 3t$

3. (a) The Laplace transform of the equation is $s^2Y - s + 3sY - 3 + 2Y = 0 \Rightarrow Y = \frac{s+3}{s^2+3s+2} = \frac{s+3}{(s+1)(s+2)}$. The partial fraction decomposition is $Y = \frac{2}{s+1} - \frac{1}{s+2}$. Thus $y = 2e^{-t} - e^{-2t}$.

   (b) The Laplace transform of the equation is $s^2Y - s - 1 - 4sY + 4 + 4Y = 0 \Rightarrow Y = \frac{s-3}{s^2-4s+4} = \frac{s-3}{(s-2)^2}$. The partial fraction decomposition is $Y = \frac{1}{s-2} - \frac{1}{(s-2)^2}$. Thus $y = e^{2t} - te^{2t}$.

   (c) The Laplace transform of the equation is $s^2Y + 1 - 6sY + 5Y = \frac{2}{s} \Rightarrow Y(s^2 - 6s + 5) = \frac{2}{s} - 1 \Rightarrow Y(s-1)(s-5) = \frac{2-s}{s} \Rightarrow Y = \frac{2-s}{s(s-5)(s-1)}$. The partial fraction decomposition is $Y = \frac{2}{5s} - \frac{3}{20(s-5)} - \frac{1}{4(s-1)}$. Thus $y = \frac{2}{5} - \frac{3}{20}e^{5t} - \frac{1}{4}e^t$.

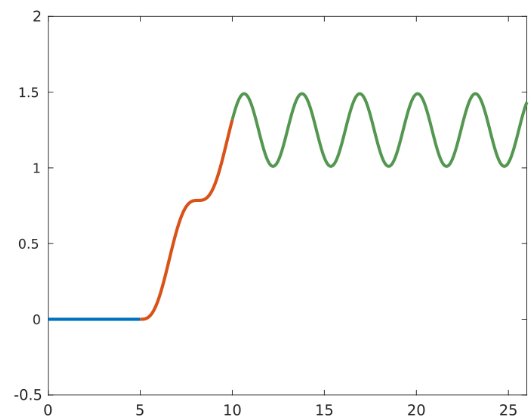
4. Note that the function on the right side of the equation can be represented as $u_5(t)(t-5) - u_{10}(t)(t-10)$. The Laplace transform makes the equation into $s^2Y + 4Y = e^{-5s}\frac{1}{s^2} - e^{-10s}\frac{1}{s^2}$. Thus $Y = (e^{-5s} - e^{-10s})\frac{1}{s^2(s^2+4)}$. Let $F(s) = \frac{1}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}$. Determine that $A = C = 0$, $B = \frac{1}{4}$, and $D = \frac{-1}{4}$. So, $F(s) = \frac{1/4}{s^2} - \frac{1/4}{s^2+4}$ and $f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{4}t - \frac{1}{8}\sin 2t$. Thus,

$$y = \mathcal{L}^{-1}[Y] = u_5(t)f(t-5) - u_{10}(t)f(t-10) =$$

$$u_5(t)(\frac{1}{4}(t-5) - \frac{1}{8}\sin 2(t-5)) - u_{10}(t)(\frac{1}{4}(t-10) - \frac{1}{8}\sin 2(t-10)).$$

Represent this function as a piecewise function. Since 5 and 10 divide the number line on three pieces ($t < 5, 5 \le t10$, and $t \ge 10$), the resulting function consists of three branches.

1. For $t < 5$, both $u_5(t)$ and $u_{10}(t)$ are zero and so the first branch of the function is zero.
2. For $5 \le t < 10$, $u_5(t)$ is 1 and $u_{10}(t)$ is 0. Thus, the second branch is $1(\frac{1}{4}(t-5) - \frac{1}{8}\sin 2(t-5)) = \frac{1}{4}(t-5) - \frac{1}{8}\sin 2(t-5)$.
3. For $t \ge 10$, both $u_5(t)$ and $u_{10}(t)$ are 1. Thus, the third branch is $1(\frac{1}{4}(t-5) - \frac{1}{8}\sin 2(t-5)) = -\cos(t-5) + \cos(t-20)$.



Thus,

$$y = \begin{cases} 0, & 0 \le t < 5, \\ \frac{1}{4}(t-5) - \frac{1}{8}\sin 2(t-5), & 5 \le t < 10, \\ \frac{5}{4} - \frac{1}{8}\sin 2(t-5) + \frac{1}{8}\sin 2(t-10), & t \ge 10. \end{cases}$$

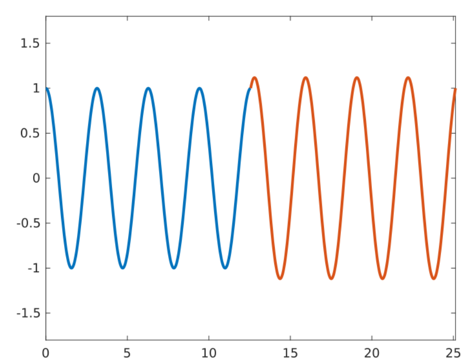
Thus, there are no oscillations before 5. Between 5 and 10 seconds, the mass oscillates about the line $\frac{1}{4}(t-5)$. After 10 seconds, the mass oscillates about $\frac{5}{4}$ with a constant amplitude.

5. The Laplace transform of the equation is

$s^2Y - s + 4Y = e^{-4\pi s}$. Thus $Y = \frac{e^{-4\pi}+s}{s^2+4}$. Then

$$y = u_{4\pi}(t)\frac{1}{2}\sin 2(t-4\pi) + \cos 2t =$$

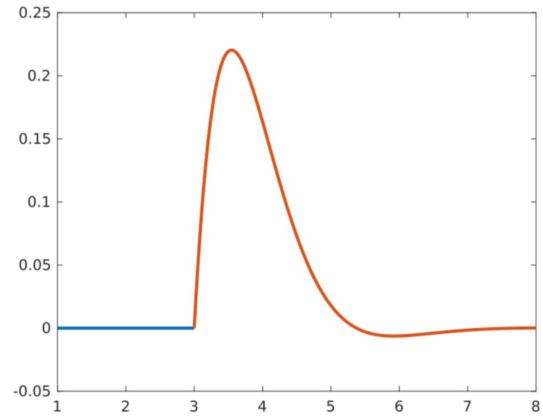
$$\begin{cases} \cos 2t, & t < 4\pi, \\ \cos 2t + \frac{1}{2}\sin 2(t-4\pi) & t \ge 4\pi. \end{cases}$$



6. Let $Y = \mathcal{L}[y]$. Applying the Laplace transform to the equation $y'' + 3y' + 4y = \delta(t-3)$ with $y(0) = y'(0) = 0$ produces $s^2Y + 3sY + 4Y = e^{-3s}$. From here $Y = \frac{e^{-3s}}{s^2+3s+4}$. Complete the denominator of $F(s) = \frac{1}{s^2+3s+4}$ to a sum of squares. $s^2 + 3s + 4 = s^2 + 2s(\frac{3}{2}) + \frac{9}{4} + 4 - \frac{9}{4} = (s+\frac{3}{2})^2 + \frac{7}{4}$. Thus $f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}[\frac{1}{(s+\frac{3}{2})^2+\frac{7}{4}}] = \frac{2}{\sqrt{7}}\mathcal{L}^{-1}[\frac{\frac{\sqrt{7}}{2}}{(s+\frac{3}{2})^2+\frac{7}{4}}] = \frac{2}{\sqrt{7}}e^{-3t/2}\sin\frac{\sqrt{7}}{2}t$.

$y = \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}[\frac{e^{-3s}}{s^2+3s+4}] = \mathcal{L}^{-1}[e^{-3s}F(s)] = u_3(t)\,f(t-3) = u_3(t)\,\frac{2}{\sqrt{7}}e^{-3(t-3)/2}\sin\frac{\sqrt{7}}{2}(t-3)$.

Represent this function as a piecewise function with two branches corresponding to $t<3$ and $t\geq 3$.



1. For $t<3$, $u_3(t)$ is zero and so the first branch of the function is zero.
2. For $t\geq 3$, $u_3(t)$ is 1 and so the second branch is $\frac{2}{\sqrt{7}}e^{-3(t-3)/2}\sin\frac{\sqrt{7}}{2}(t-3)$. Hence,

$$y=\begin{cases} 0, & t<3,\\ \frac{2}{\sqrt{7}}e^{-3(t-3)/2}\sin\frac{\sqrt{7}}{2}(t-3), & t\geq 3.\end{cases}$$

Thus, the object starts oscillating only after 3 seconds. It oscillates with a decreasing amplitude given by $\frac{2}{\sqrt{7}}e^{-3(t-3)/2}$ converging to 0 and the oscillations become negligible in time.

7. Let $Y=\mathcal{L}[y]$ and $F=\mathcal{L}[f]$. The Laplace transform converts the equation to $s^2Y-3s+1+4Y=F\Rightarrow Y(s^2+4)=3s-1+F\Rightarrow Y=\frac{3s-1}{s^2+4}+F\frac{1}{s^2+4}=3\frac{s}{s^2+4}-\frac{1}{2}\frac{2}{s^2+4}+\frac{1}{2}F\frac{2}{s^2+4}$. Applying the inverse Laplace transform, you obtain $y=3\cos 2t-\frac{1}{2}\sin 2t+\frac{1}{2}f(t)*\sin 2t=3\cos 2t-\frac{1}{2}\sin 2t+\frac{1}{2}\int_0^t f(\tau)\sin 2(t-\tau)d\tau$.

8. The equation is $y+t*y=t$. The Laplace transform gives you $Y+\frac{1}{s^2}Y=\frac{1}{s^2}$. Then $Y=\frac{1}{s^2+1}$ and so $y=\sin t$.

9. The equation is $y'+y*e^{-2t}=1$. Thus $sY-1+Y\frac{1}{s+2}=\frac{1}{s}\Rightarrow Y(s+\frac{1}{s+2})=\frac{1}{s}+1\Rightarrow Y\frac{s(s+2)+1}{s+2}=\frac{1+s}{s}\Rightarrow Y=\frac{(1+s)(s+2)}{s(s^2+2s+1)}=\frac{(1+s)(s+2)}{s(s+1)^2}=\frac{s+2}{s(s+1)}$. Find the partial fraction decomposition to be $Y=\frac{2}{s}-\frac{1}{s+1}\Rightarrow y=2-e^{-t}$.

10. (a) Let $X=\mathcal{L}[x]$ and $Y=\mathcal{L}[y]$. Taking $\mathcal{L}$ of both equations gives you $sX-1=-X+Y$ and $sY-2=-X-Y$. From the first equation $Y=sX+X-1$. Plugging that in the second gives you $s(sX+X-1)-2=-X-(sX+X-1)\Rightarrow s^2X+2sX+2X=s+3\Rightarrow X=\frac{s+3}{s^2+2s+2}$. Thus $Y=\frac{s^2+3s+s+3-s^2-2s-2}{s^2+2s+2}=\frac{2s+1}{s^2+2s+2}$.

    Then $x=\mathcal{L}^{-1}[X]=\mathcal{L}^{-1}[\frac{s+1+2}{(s+1)^2+1}]=\mathcal{L}^{-1}[\frac{s+1}{(s+1)^2+1}+2\frac{1}{(s+1)^2+1}]=e^{-t}\cos t+2e^{-t}\sin t$ and $y=\mathcal{L}^{-1}[Y]=\mathcal{L}^{-1}[\frac{2s+1}{s^2+2s+2}]=\mathcal{L}^{-1}[\frac{2s+2-1}{(s+1)^2+1}]=\mathcal{L}^{-1}[2\frac{s+1}{(s+1)^2+1}-\frac{1}{(s+1)^2+1}]=2e^{-t}\cos t-e^{-t}\sin t$.

    (b) If $X=\mathcal{L}[x]$ and $Y=\mathcal{L}[y]$, taking $\mathcal{L}$ of both equations produces $sX-1=3X-Y$ and $sY-3=4X-2Y$. From the first equation, $Y=-sX+3X+1$. Substitute that in the second equation. $s(-sX+3X+1)-3=4X-2(-sX+3X+1)\Rightarrow -s^2X+3sX-4X-2sX+6X=-s+3-2\Rightarrow -s^2X+sX+2X=-s+1\Rightarrow -(s^2-s-2)X=-(s-1)\ X=\frac{s-1}{s^2-s-2}$. Thus $Y=\frac{-s^2+s+3s-3+s^2-s-2}{s^2-s-2}=\frac{3s-5}{s^2-s-2}$. The partial fraction decomposition produces $X=\frac{s-1}{(s-2)(s+1)}=\frac{1/3}{s-2}+\frac{2/3}{s+1}$ and $Y=\frac{3s-5}{(s-2)(s+1)}=\frac{1/3}{s-2}+\frac{8/3}{s+1}$. Taking inverse Laplace transform, $x=\mathcal{L}^{-1}[X]=\frac{1}{3}e^{2t}+\frac{2}{3}e^{-t}$ and $y=\mathcal{L}^{-1}[Y]=\frac{1}{3}e^{2t}+\frac{8}{3}e^{-t}$.