Differential Equations
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Nonhomogeneous equations with constant coefficients. Methods. Applications

Consider a nonhomogeneous linear equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y = g(x).$$

The general solution of such equation is of the form

$$y = y_h + y_p$$

where y_h is the general solution of homogeneous equation and y_p is called the **particular solution** and depends on the nonhomogeneous part. There are two main methods for finding a particular solutions of nonhomogeneous equations.

1. **Variation of parameters.** This method is completely general, but sometimes tends to lead to difficult integrals.

2. **Undetermined coefficients**. This method is easier, but it works just when the function $g(x)$ is of a specific form and, thus, it is not general.

We present the methods for the case $n = 2$. Both methods can be generalized to higher orders.

Variation of parameters

Consider the equation $ay'' + by' + cy = g(x)$ and assume that y_1 and y_2 are solutions of the homogeneous part so that $y_h = c_1 y_1 + c_2 y_2$ is the general solution of the homogeneous part. The particular solution y_p is obtained by assuming that c_1 and c_2 are not constants but functions that depend on x. Since there is just one equation and we are introducing two new functions, we can impose one condition on them with no risk of loosing generality. Let us denote the two new functions by v_1 and v_2 so that

$$y_p = v_1 y_1 + v_2 y_2.$$

To find the unknown functions v_1 and v_2, find the derivatives of y_p.

$$y_p' = v_1' y_1 + v_1 y_1' + v_2' y_2 + v_2 y_2'$$

and impose the condition that $v_1' y_1 + v_2' y_2 = 0$. Thus $y_p' = v_1 y_1' + v_2 y_2'$ and so

$$y_p'' = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''$$

Substituting derivatives in the equation and keeping in mind that y_1 and y_2 are solutions of homogeneous part, we obtain

$$av_1' y_1' + av_1 y_1'' + av_2' y_2' + av_2 y_2'' + bv_1 y_1' + bv_2 y_2' + cv_1 y_1 + cv_2 y_2 =$$

$$= v_1(ay_1'' + by_1' + cy_1) + v_2(ay_2'' + by_2' + cy_2) + av_1'y_1' + av_2'y_2' =$$
$$= av_1'y_1' + av_2'y_2' = g.$$

Thus, to determine the functions v_1 and v_2, we need to solve two equations

$$v_1'y_1 + v_2'y_2 = 0 \quad \text{and} \quad av_1'y_1' + av_2'y_2' = g$$

First, solve the equations algebraically for v_1' and v_2' and then obtain v_1 and v_2 by integrating.

This illustrates that the steps of this method are the following.

1. Find the solution y_h of the homogeneous part in the form $c_1y_1 + c_2y_2$.
2. To find a particular solution y_p, replace c_1 and c_2 in y_h with two unknown functions v_1 and v_2 and write down the two equations in v_1' and v_2'.
3. Solve the equations for v_1' and v_2' and then obtain v_1 and v_2 by integrating.
4. Finally, put v_1 and v_2 back into $y_p = v_1y_1 + v_2y_2$.

We illustrate this method in the following example.

Example. Solve the equation $y'' - y' - 2y = e^{3x}$.

Solution. The characteristic equation is $r^2 - r - 2 = 0$. The roots are 2 and -1, so that $y_1 = e^{2x}$, $y_2 = e^{-x}$ and the homogeneous solution is $y_h = c_1y^{2x} + c_2y^{-x}$ and we can find a particular solution y_p in the form $y_p = v_1y^{2x} + v_2y^{-x}$. The two equations for the unknown functions are

$$v_1'e^{2x} + v_2'e^{-x} = 0 \quad \text{and} \quad 2v_1'e^{2x} - v_2'e^{-x} = e^{3x}$$

Solving the first equation for v_2' produces $v_2'e^{-x} = -v_1'e^{2x} \Rightarrow v_2' = -v_1'e^{2x}e^{x} = -v_1e^{3x}$. Substitute that in the second equation to get $2v_1'e^{2x} + v_1'e^{2x} = e^{3x} \Rightarrow 3v_1'e^{2x} = e^{3x} \Rightarrow v_1' = \frac{1}{3}e^{x}$. Hence, $v_2' = -v_1e^{3x} = \frac{-1}{3}e^{4x}$.

Integrate v_1' and v_2' to obtain v_1 and v_2, respectively. We have that $v_1 = \int \frac{1}{3}e^{x}\,dx = \frac{1}{3}e^{x}$ and $v_2 = \int -\frac{1}{3}e^{4x}dx = -\frac{1}{12}e^{4x}$. This gives a particular solution $y_p = \frac{1}{3}e^{x}e^{2x} - \frac{1}{12}e^{4x}e^{-x} = \frac{1}{3}e^{3x} - \frac{1}{12}e^{3x} = \frac{1}{4}e^{3x}$. Hence, the general solution of the differential equation is

$$y = c_1e^{2x} + c_2e^{-x} + \frac{1}{4}e^{3x}.$$

Practice Problems. Solve the differential equations.

1. $y'' - 6y' + 9y = x^{-3}e^{3x}$.
2. $y'' - 5y' + 6y = 2e^{x}$
3. $y'' + 4y' + 4y = x^{-2}e^{-2x}$

Solutions.

1. The characteristic equation is $r^2 - 6r + 9 = 0$. It factors as $(r-3)(r-3) = 0$ and so 3 is a double zero. Thus, $y_1 = e^{3x}$ and $y_2 = xe^{3x}$ and the homogeneous solution is $y_h = c_1e^{3x} + c_2xe^{3x}$. We can find y_p in the form $y_p = v_1e^{3x} + v_2xe^{3x}$. Two equations in derivatives v_1' and v_2' are

$$v_1'e^{3x} + v_2'xe^{3x} = 0 \quad \text{and} \quad 3v_1'e^{3x} + v_2'e^{3x} + 3v_2'xe^{3x} = x^{-3}e^{3x}.$$

Cancelling e^{3x} we have that

$$v_1' + v_2'x = 0 \quad \text{and} \quad 3v_1' + v_2' + 3v_2'x = x^{-3}.$$

From the first equation, $v_1' = -xv_2'$. Plugging that in the second produces $-3xv_2' + v_2' + 3xv_2' = x^{-3} \Rightarrow v_2' = x^{-3}$. Substituting back in $v_1' = -xv_2'$ gives us that $v_1' = -x^{-2}$. Hence, $v_2 = \int x^{-3}dx = -\frac{1}{2x^2}$ and $v_1 = \int -x^{-2}dx = \frac{1}{x}$. So, the general solution is $y = c_1e^{3x} + c_2xe^{3x} + \frac{1}{x}e^{3x} - \frac{1}{2x^2}xe^{3x} = c_1e^{3x} + c_2xe^{3x} + \frac{1}{2x}e^{3x}$.

2. The characteristic equation is $r^2 - 5r + 6 = 0$. The roots are 2 and 3, so that $y_1 = e^{2x}$, $y_2 = e^{3x}$ and the homogeneous solution is $y_h = c_1y^{2x} + c_2y^{3x}$ and we can find y_p in the form $y_p = v_1y^{2x} + v_2y^{3x}$. The two equations for the unknown functions are

$$v_1'e^{2x} + v_2'e^{3x} = 0 \quad \text{and} \quad 2v_1'e^{2x} + 3v_2'e^{3x} = 2e^{x}$$

Solving the first equation for v_2' produces $v_2'e^{3x} = -v_1'e^{2x} \Rightarrow v_2' = -v_1'e^{2x}e^{-3x} = -v_1e^{-x}$. Substitute that in the second equation to get $2v_1'e^{2x} - 3v_1'e^{2x} = 2e^{x} \Rightarrow -v_1'e^{2x} = 2e^{x} \Rightarrow v_1' = -2e^{-x}$. Hence, $v_2' = -v_1e^{-x} = 2e^{-2x}$.

Integrate v_1' and v_2' to obtain v_1 and v_2, respectively. We have that $v_1 = \int -2e^{-x}\,dx = 2e^{-x}$ and $v_2 = \int 2e^{-2x}dx = -e^{-2x}$. This gives a particular solution $y_p = 2e^{-x}e^{2x} - e^{-2x}e^{3x} = 2e^{x} - e^{x} = e^{x}$. Hence, the general solution of the differential equation is $y = c_1e^{2x} + c_2e^{3x} + e^{x}$.

3. The characteristic equation is $r^2 + 4r + 4 = 0$. It factors as $(r+2)(r+2) = 0$ so -2 is a double zero. Thus, $y_1 = e^{-2x}$ and $y_2 = xe^{-2x}$ and the homogeneous solution is $y_h = c_1e^{-2x} + c_2xe^{-2x}$. We can find y_p in the form $y_p = v_1e^{-2x} + v_2xe^{-2x}$. Two equations in derivatives v_1' and v_2' are

$$v_1'e^{-2x} + v_2'xe^{-2x} = 0 \quad \text{and} \quad -2v_1'e^{-2x} + v_2'e^{-2x} - 2xv_2'e^{-2x} = x^{-2}e^{-2x}$$

Cancelling e^{-2x} we have that

$$v_1' + v_2'x = 0 \quad \text{and} \quad -2v_1' + v_2' - 2xv_2' = x^{-2}.$$

From the first equation, $v_1' = -xv_2'$. Plugging that in the second produces $2xv_2' + v_2' - 2xv_2' = x^{-2} \Rightarrow v_2' = x^{-2}$. Substituting back in $v_1' = -xv_2'$ gives us that $v_1' = -x^{-1} = -\frac{1}{x}$. Hence, $v_2 = \int x^{-2}dx = -x^{-1}$ and $v_1 = \int -\frac{1}{x}dx = -\ln x$. This gives a particular solution $y_p = -\ln xe^{-2x} - x^{-1}xe^{-2x} = -\ln xe^{-2x} - e^{-2x}$. So, the general solution is $y = c_1e^{-2x} + c_2xe^{-2x} - \ln xe^{-2x} - e^{-2x}$. This is completely acceptable as your final answer. Note, though, that you can combine the terms c_1e^{-2x} and $-e^{-2x}$ as $(c_1 - 1)e^{-2x}$ and still use c_1e^{-2x} for this term. Thus, the solution can also be written as $y = c_1e^{-2x} + c_2xe^{-2x} - \ln xe^{-2x}$.

Undetermined Coefficients

The method of Undetermined Coefficients determines the particular solution y_p of a nonhomogeneous linear equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y = g(x)$$

in case when one of the two cases below hold.

Case 1 $g(x)$ is a product of a polynomial and exponential function.

Case 2 $g(x)$ is a product of a polynomial, exponential function and a trigonometric function.

In particular, let $p_k(x)$ be a polynomial $a_k x^k + a_{k-1} x^{k-1} + \ldots + a_0$ of degree k and let p and q be real numbers.

Case 1 If $g(x) = p_k(x) e^{px}$, then

$$y_p = x^s (A_k x^k + A_{k-1} x^{k-1} + \ldots + A_0) e^{px}$$

where s is the number of times p appears on the list of zeros of the characteristic equation and $A_0, \ldots, A_k$ are undetermined coefficients of a general polynomial of the same degree k as $p_k(x)$.

Case 2 $g(x) = p_k(x) e^{px} \cos qx$ or $g(x) = p_k(x) e^{px} \sin qx$, then

$$y_p = x^s (A_k x^k + A_{k-1} x^{k-1} + \ldots + A_0) e^{px} \cos qx + x^s (B_k x^k + B_{k-1} x^{k-1} + \ldots + B_0) e^{px} \sin qx$$

where s is the number of times $p + iq$ appears on the list of zeros of the characteristic equation and $A_0, \ldots, A_k$ and $B_0, \ldots, B_k$ are undetermined coefficients of two different general polynomials of the same degree k as $p_k(x)$.

Note that while $g(x)$ can contain only sine or only cosine term, the solution y_p should contain *both* sine and cosine terms.

To find the undetermined coefficients, plug the particular solution and its derivatives into the original equation and determine the coefficients from there by equating the polynomials (on the same way as when solving partial fractions in a Calculus 2 course). The *number* of unknown coefficients should always be *the same* as the number of equations you get by equating the coefficients of the same powers of x or of the same trigonometric function. This is an indicator that the *form* of the particular solution you are trying to compute is correct.

If $g(x)$ is a sum of functions $g(x) = g_1(x) + g_2(x) + \ldots + g_m(x)$ and each function $g_1(x), g_2(x), \ldots, g_m(x)$ is a function described under two cases above, then the particular solution y_p is the sum of particular solutions

$$y_p = y_{p1} + y_{p2} + \ldots + y_{pm}$$

where each solution y_{pi}, $i = 1, \ldots, m$ is obtained as in Case 1 or 2 described above.

In the first example, we illustrate the method itself. By using an example from the previous section, we also illustrate that this method, if applicable, is much shorter and simpler than the Variations of Parameters.

Case 1 Example. Solve the differential equation $y'' - y' - 2y = e^{3x}$.

Solution. The characteristic equation is $0 = r^2 - r - 2 = (r-2)(r+1)$ so $y_h = c_1e^{2x} + c_2e^{-x}$. The function $g(x)$ is e^{3x} and so $p = 3$ and the polynomial $p_k(x)$ is 1 so it is a constant polynomial (i.e. of degree zero). Hence, $y_p = x^sAe^{3x}$ for some constant A. To determine s, note that $p = 3$ is not a zero of the characteristic equation. Thus, $s = 0$ and a particular solution is of the form

$$y_p = Ae^{3x}$$

Finding the derivatives $y_p' = 3Ae^{3x}$ and $y_p'' = 9Ae^{3x}$ and substituting them into the equation yields

$$9Ae^{3x} - 3Ae^{3x} - 2Ae^{3x} = e^{3x} \quad\Rightarrow\quad 9A - 3A - 2A = 1 \quad\Rightarrow\quad 4A = 1 \Rightarrow A = \frac{1}{4}.$$

Thus, $y_p = \frac{1}{4}e^{3x}$ and the general solution is $y = c_1e^{2x} + c_2e^{-x} + \frac{1}{4}e^{3x}$.

Comparing the length and the amount of algebra involved in the solution of this example and in the example of the previous section, you can see that the Undetermined Coefficients is much shorter and simpler. We emphasize that it is not applicable to use this method in practice problems 1 and 3 of the previous section since the parts x^{-3} and x^{-2} are not polynomials. This illustrates that the Variation of Parameters is also necessary.

The next example illustrates the second case.

Case 2 Example. Solve the differential equation $y'' - 3y' - 4y = 25\sin 3x$.

Solution. The characteristic equation is $0 = r^2 - 3r - 4 = (r-4)(r+1)$ so 4 and -1 are zeros and $y_h = c_1e^{4x} + c_2e^{-x}$. The function $g(x)$ is $25\sin 3x = 25e^{0x}\sin 3x$ and so $p + qi = 0 + 3i = 3i$ and the polynomial $p_k(x)$ is 25 so it is a constant polynomial (i.e. of degree zero). Hence,

$$y_p = x^sAe^{0x}\cos 3x + x^sBe^{0x}\sin 3x = x^sA\cos 3x + x^sB\sin 3x$$

for some constants A and B. To determine s, note that $p + iq = i$ is not a zero of the characteristic equation. Thus, $s = 0$ and a particular solution is of the form

$$y_p = A\cos 3x + B\sin 3x.$$

Finding the derivatives $y_p' = -3A\sin 3x + 3B\cos 3x$ and $y_p'' = -9A\cos 3x - 9B\sin 3x$ and substituting them into the equation produces

$$-9A\cos 3x - 9B\sin 3x + 9A\sin 3x - 9B\cos 3x - 4A\cos 3x - 4B\sin 3x = 25\sin 3x$$

Equate the terms with $\cos 3x$ and the terms with $\sin 3x$. This produces two equations in two unknowns.

$$-9A - 9B - 4A = 0 \quad\text{and}\quad -9B + 9A - 4B = 25$$

From the first equation $B = -\frac{13}{9}A$ and from the second $9A + \frac{169}{9}A = 25$ so $\frac{250}{9}A = 25 \Rightarrow A = \frac{9}{10}$. Thus $B = -\frac{13}{10}$ and $y_p = \frac{9}{10}\cos 3x - \frac{13}{10}\sin 3x$ and the general solution is $y = c_1e^{4x} + c_2e^{-x} + \frac{9}{10}\cos 3x - \frac{13}{10}\sin 3x$.

Case s = 1 Example. Solve the differential equation $y'' - 3y' - 4y = 5xe^{4x}$.

Solution. From previous examples, 4 and -1 are zeros of characteristic equation and $y_h = c_1e^{4x} + c_2e^{-x}$. The function $g(x)$ is $5xe^{4x}$ and so $p = 4$ and the polynomial $p_k(x)$ is a linear polynomial.

Hence, $y_p = x^s(Ax+B)e^{4x}$ for some constants A and B. To determine s, note that 4 is a (single) zero of the characteristic equation. Thus, $s=1$ and so a particular solution is of the form

$$y_p = x^1(Ax+B)e^{4x} = (Ax^2+Bx)e^{4x}.$$

Finding the derivatives $y_p' = (2Ax+B)e^{4x} + 4(Ax^2+Bx)e^{4x} = (4Ax^2+4Bx+2Ax+B)e^{4x}$ and $y_p'' = (8Ax+4B+2A)e^{4x} + 4(4Ax^2+4Bx+2Ax+B)e^{4x} = (16Ax^2+16Ax+16Bx+8B+2A)e^{4x}$ and substituting them into the equation yields

$$(16Ax^2+16Ax+16Bx+8B+2A-12Ax^2-12Bx-6Ax-3B-4Ax^2-4Bx)e^{4x} = 5xe^{4x}$$

Thus $16Ax^2+16Ax+16Bx+8B+2A-12Ax^2-12Bx-6Ax-3B-4Ax^2-4Bx = 5x$. Equating the similar terms of the polynomials on the left and right side yields two equations in two unknowns

$$+16A+16B-12B-6A-4B = 5 \text{ and } 8B+2A-3B = 0$$

Thus $10A=5$ and $5B=2A$ giving us that $A=\frac{1}{2}$ and $B=\frac{1}{5}$. So, $y_p = (\frac{1}{2}x^2+\frac{1}{5}x)e^{4x}$ and the general solution is $y = c_1e^{4x}+c_2e^{-x}+(\frac{1}{2}x^2+\frac{1}{5}x)e^{4x}$.

Two non-homogeneous functions Example. Solve the differential equation $y''-3y'-4y = 5xe^{4x}+25\sin 3x$.

Solution. Consider the function $g(x)$ as the sum of two separate parts: $5xe^{4x}$ and $25\sin 3x$. Then look for the particular solution in the form $y_{p1}+y_{p2}$ where the particular solution y_{p1} is determined by the function $5xe^{4x}$ and the particular solution y_{p2} is determined by the function $25\sin x$ as in the previous two examples. By these examples, $y_h = c_1e^{4x}+c_2e^{-x}$, $y_{p1} = (\frac{1}{2}x^2+\frac{1}{5}x)e^{4x}$ and $y_{p2} = \frac{9}{10}\cos 3x - \frac{13}{10}\sin 3x$ Thus, the general solution is

$$y = c_1e^{4x}+c_2e^{-x}+(\frac{1}{2}x^2+\frac{1}{5}x)e^{4x}+\frac{9}{10}\cos 3x-\frac{13}{10}\sin 3x.$$

Case s = 2 Example. Solve the differential equation $y''-4y'+4y = 6e^{2x}$.

Solution. The characteristic equation is $0 = r^2-4r+4 = (r-2)(r-2)$ so $y_h = c_1e^{2x}+c_2xe^{2x}$. The function $g(x)$ is $6e^{2x}$ and so $p=2$ and the polynomial $p_k(x)$ is 6 so it is a constant polynomial. Hence, $y_p = x^sAe^{2x}$ for some constant A. To determine s, note that $p=2$ is a *double* zero of the characteristic equation. Thus, $s=2$ and a particular solution is of the form

$$y_p = Ax^2e^{2x}$$

Finding the derivatives $y_p' = 2Axe^{2x}+2Ax^2e^{2x}$ and $y_p'' = 2Ae^{2x}+4Axe^{2x}+4Axe^{2x}+4Ax^2e^{2x}$ and substituting them into the equation yields

$$2Ae^{2x}+4Axe^{2x}+4Axe^{2x}+4Ax^2e^{2x}-8Axe^{2x}-8Ax^2e^{2x}+4Ax^2e^{2x} = 6e^{2x} \Rightarrow$$

$$2A+4Ax+4Ax+4Ax^2-8Ax-8Ax^2+4Ax^2 = 6 \Rightarrow 2A = 6 \Rightarrow A = 3.$$

Thus, $y_p = 3x^2e^{2x}$ and the general solution is $y = c_1e^{2x}+c_2xe^{2x}+3x^2e^{2x}$.

Practice Problems. Find the general solutions of problems 1 – 6. In problems 7 – 9, find the *form* of particular solutions and the general solutions. You **do not** have to solve for unknown coefficients in particular solutions.

1. $y'' + y = 3e^{-x}$
2. $y'' - 5y' - 6y = 4e^{2x}$
3. $y'' - 5y' + 6y = 4e^{2x}$
4. $y'' + 4y = 5x^2e^x$
5. $y'' - 2y' + y = 7xe^x$
6. $y'' + 2y' - 3y = 5\sin 3x$
7. $y'' - 3y' - 10y = 3xe^{2x} + 5e^{-2x}$
8. $y'' - 8y' + 16y = 3x^2 - 5e^{4x}$
9. $y'' + 4y' + 13y = -2\sin 3x + e^{-2x}\cos 3x$

Solutions.

1. The zeros of the characteristic equation are $r = \pm i$ and so $y_h = c_1\cos x + c_2\sin x$. We have that $p = -1$ and $p_k(x) = 3$ so $y_p = x^sAe^{-x}$. Since -1 is not a zero of the characteristic equation, $s = 0$ and so $y_p = Ae^{-x}$. Find the derivatives $y_p' = -Ae^{-x}$ and $y_p'' = Ae^{-x}$ and plug the derivatives and y_p the into the equation. Obtain that $Ae^{-x} + Ae^{-x} = 3e^{-x} \Rightarrow 2A = 3 \Rightarrow A = \frac{3}{2}$. Thus, $y_p = \frac{3}{2}e^{-x}$ and the general solution is $y = c_1\cos x + c_2\sin x + \frac{3}{2}e^{-x}$.

2. The zeros of the characteristic equation are $r = 6$ and $r = -1$ and so $y_h = c_1e^{6x} + c_2e^{-x}$. We have that $p = 2$ and $p_k(x) = 4$ so $y_p = x^sAe^{2x}$. Since 2 is not a zero of the characteristic equation, $s = 0$ and so $y_p = Ae^{2x}$. Find the derivatives $y_p' = 2Ae^{2x}$ and $y_p'' = 4Ae^{2x}$ and plug them and y_p the into the equation. Obtain that $4Ae^{2x} - 10Ae^{2x} - 6Ae^{2x} = 4e^{2x} \Rightarrow 4A - 10A - 6A = 4 \Rightarrow -12A = 4 \Rightarrow A = -\frac{1}{3}$. Thus, $y_p = -\frac{1}{3}e^{2x}$ and the general solution is $y = c_1e^{6x} + c_2e^{-x} - \frac{1}{3}e^{2x}$.

3. The zeros of the characteristic equation are $r = 2$ and $r = 3$ and $y_h = c_1e^{2x} + c_2e^{3x}$. We have that $p = 2$ and $p_k(x) = 4$ so $y_p = x^sAe^{2x}$. Since 2 is a (single) zero of the characteristic equation, $s = 1$ and so $y_p = Axe^{2x}$. Find the derivatives $y_p' = Ae^{2x} + 2Axe^{2x}$ and $y_p'' = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x} = 4Ae^{2x} + 4Axe^{2x}$ and plug them and y_p the into the equation. Obtain that $4Ae^{2x} + 4Axe^{2x} - 5Ae^{2x} - 10Axe^{2x} + 6Axe^{2x} = 4e^{2x} \Rightarrow 4A + 4Ax - 5A - 10Ax + 6Ax = 4 \Rightarrow 4A - 5A = 4 \Rightarrow A = -4$. Thus, $y_p = -4xe^{2x}$ and the general solution is $y = c_1e^{2x} + c_2e^{3x} - 4xe^{2x}$.

4. The zeros of the characteristic equation are $r = \pm 2i$ and so $y_h = c_1\cos 2x + c_2\sin 2x$. We have that $p = 1$ and $p_k(x) = 5x^2$ so $y_p = x^s(Ax^2 + Bx + C)e^x$. Since 1 is not a zero of the characteristic equation, $s = 0$ and so $y_p = (Ax^2 + Bx + C)e^x$. Find the derivatives $y_p' = (2Ax + B)e^x + (Ax^2 + Bx + C)e^x = (2Ax + B + Ax^2 + Bx + C)e^x$ and $y_p'' = (2A + 2Ax + B)e^x + (2Ax + B + Ax^2 + Bx + C)e^x$ and plug them and y_p the into the equation. Obtain that $(2A + 2Ax + B)e^x + (2Ax + B + Ax^2 + Bx + C)e^x + (4Ax^2 + 4Bx + 4C)e^x = 5x^2e^x \Rightarrow$

$$2A + 2Ax + B + 2Ax + B + Ax^2 + Bx + C + 4Ax^2 + 4Bx + 4C = 5x^2$$

   Equating the terms with x^2, we obtain that $5A = 5$ so $A = 1$. Equating the terms with x, we obtain that $4A + 5B = 0 \Rightarrow 5B = -4A = -4$ since $A = 1$. Hence, $B = -\frac{4}{5}$. Equating the terms with no x, we obtain that $= 2A + 2B + 5C = 0 \Rightarrow 5C = -2A - 2B = -2 + \frac{8}{5} = \frac{-2}{5}$ so $C = -\frac{2}{25}$. Thus, $y_p = (x^2 - \frac{4}{5}x - \frac{2}{25})e^x$ and the general solution is $y = c_1\cos 2x + c_2\sin 2x + (x^2 - \frac{4}{5}x - \frac{2}{25})e^x$.

5. The characteristic equation is $r^2 - 2r + 1 = 0 \Rightarrow (r-1)(r-1) = 0$ so $r = 1$ is a double zero and $y_h = c_1e^x + c_2xe^x$. We have that $p = 1$ and $p_k(x) = 7x$ so $y_p = x^s(Ax + B)e^x$. Since 1 is a double zero of the characteristic equation $s = 2$ and so $y_p = x^2(Ax + B)e^x = (Ax^3 + Bx^2)e^x$. Similarly as in previous problems, find the derivatives y_p' and y_p'' and plug them and y_p into the equation. Obtain that $A = \frac{7}{6}$, and $B = 0$ so $y_p = \frac{7}{6}x^3e^x$ and the general solution is $y = c_1e^x + c_2xe^x + \frac{7}{6}x^3e^x$.

6. The zeros of the characteristic equation are $r = -3$ and $r = 1$ and so $y_h = c_1 e^{-3x} + c_2 e^x$. We have that $p + iq = 0 + 3i$ and $p_k(x) = 5$ so $y_p = x^s A e^{0x} \cos 3x + x^s B e^{0x} \sin 3x$. Since $3i$ is not a zero of the characteristic equation, $s = 0$ and so $y_p = A\cos 3x + B\sin 3x$. From here $y_p' = -3A\sin 3x + 3B\cos 3x$ and $y_p'' = -9A\cos 3x - 9B\sin 3x$ and the equation becomes $-9A\cos 3x - 9B\sin 3x - 6A\sin 3x + 6B\cos 3x - 3A\cos 3x - 3B\sin 3x = 5\sin 3x$. Equate the terms with $\cos 3x$ and the terms with $\sin 3x$ to obtain two equations in two unknowns.

$$-9A + 6B - 3A = 0 \quad \text{and} \quad -9B - 6A - 3B = 5$$

From the first equation $B = 2A$ and from the second $-6A - 24A = 5 \Rightarrow -30A = 5 \Rightarrow A = -\frac{1}{6}$. Thus $B = -\frac{1}{3}$ and $y_p = -\frac{1}{6}\cos 3x - \frac{1}{3}\sin 3x$. The general solution is $y = c_1 e^{-3x} + c_2 e^x - \frac{1}{6}\cos 3x - \frac{1}{3}\sin 3x$.

7. The roots of the characteristic equation $r^2 - 3r - 10 = (r-5)(r+2) = 0$ are 5 and -2 so the homogeneous solution is $y_h = c_1 e^{5x} + c_2 e^{-2x}$.

   You have to consider functions $g_1(x) = 3xe^{2x}$ and $g_2(x) = 5e^{-2x}$ separately and obtain two separate particular solutions y_{p1} and y_{p2}.

   For $g_1(x) = 3xe^{2x}$, $p = 2$ and $p_k(x) = 3x$ so $y_{p1} = x^s(Ax + B)e^{2x}$. Since 2 is not a solution of the characteristic equation, $s = 0$ and so $y_{p1} = (Ax + B)e^{2x}$.

   For $g_2(x) = 5e^{-2x}$, $p = -2$ and $p_k(x) = 5$ so $y_{p2} = x^s C e^{-2x}$. Since -2 is a (single) solution of the characteristic equation, $s = 1$ and so $y_{p2} = x^1 C e^{-2x} = Cxe^{-2x}$.

   The general solution has the form $y = c_1 e^{5x} + c_2 e^{-2x} + (Ax + B)e^{2x} + Cxe^{-2x}$.

8. The characteristic equation is $r^2 - 8r + 16 = (r-4)(r-4) = 0$, so $r = 4$ is a double zero. The homogeneous solution is $y_h = c_1 e^{4x} + c_2 x e^{4x}$. Consider the functions $g_1(x) = 3x^2$ and $g_2(x) = -5e^{4x}$ separately and obtain two separate particular solutions y_{p1} and y_{p2}.

   For $g_1(x) = 3x^2 = 3x^2 e^{0x}$, $p = 0$ and $p_k(x) = 3x^2$ so $y_{p1} = x^s(Ax^2 + Bx + C)e^{0x}$. Since 0 is not a solution of the characteristic equation, $s = 0$ and so $y_{p1} = Ax^2 + Bx + C$.

   For $g_2(x) = -5e^{4x}$, $p = 4$ and $p_k(x) = -5$ so $y_{p2} = x^s D e^{4x}$. Since 4 is a double zero of the characteristic equation, $s = 2$, and so $y_{p2} = x^2 D e^{4x} = Dx^2 e^{4x}$.

   The general solution has the form $y = c_1 e^{4x} + c_2 x e^{4x} + Ax^2 + Bx + C + Dx^2 e^{4x}$.

9. The characteristic equation $r^2 + 4r + 13 = 0$ has solutions $r = \frac{-4 \pm \sqrt{16-52}}{2} = \frac{-4 \pm 6i}{2} = -2 \pm 3i$. So, the homogeneous solution is $y_h = c_1 e^{-2x}\cos 3x + c_2 e^{-2x}\sin 3x$.

   For $g_1(x) = -2\sin 3x = -2e^{0i}\sin 3x$, $p + iq = 0 + 3i$ and $p_k(x) = -2$ so $y_{p1} = x^s A e^{0x}\cos 3x + x^s B e^{0x}\sin 3x$. Since $0 + 3i$ is not a solution of the characteristic equation, $s = 0$ and so $y_{p1} = A\cos 3x + B\sin 3x$.

   For $g_2(x) = e^{-2x}\cos 3x$, $p + iq = 0 + 3i$ and $p_k(x) = 1$ so $y_{p1} = x^s C e^{-2x}\cos 3x + x^s D e^{-2x}\sin 3x$. Since $-2 + 3i$ is a solution of the characteristic equation, $s = 1$ and so $y_{p2} = x(Ce^{-2x}\cos 3x + De^{-2x}\sin 3x) = Cxe^{-2x}\cos 3x + Dxe^{-2x}\sin 3x$.

   The general solution has the form $y = c_1 e^{-2x}\cos 3x + c_2 e^{-2x}\sin 3x + A\cos 3x + B\sin 3x + Cxe^{-2x}\cos 3x + Dxe^{-2x}\sin 3x$.

Applications

Many physical processes can be modeled by linear differential equations. For example, mechanical oscillations, electric circuits and more.

Mechanical oscillations. Consider a mass m on a spring. Let $u(t)$ denotes the position at time t. The following forces act on the mass.

1. The gravitational force mg.

2. The spring force F_s that is proportional to the natural length L plus any additional elongation $u(t)$, so $F_s = -k(L+u)$ By Hooke's law, $mg = kL$ where k is a spring constant

$$k = \frac{mg}{L}$$

so that this force is $F_s = -mg - ku$.

3. The damping or resistive force F_d that may arise because of resistance from the air, internal energy dissipation, friction between the mass and possible guides etc. It is proportional to the speed of the mass $F_d = \pm\gamma u'(t)$. The constant γ is called the damping constant, and the sign $\pm$ depends on the choice of the coordinate system for the motion (recall the problem with a falling object from the First Order Diff. Eq. handout – the same consideration applies here). We can choose the coordinate system so that this force acts in the opposite direction from mg and so $F_d = -\gamma u'(t)$.

4. A possible external force $F(t)$.

The total force $F = ma = mu''$, the product of the mass and the acceleration, is equal to the sum of all four acting forces

$$mu'' = mg - k(L+u) - \gamma u' + F(t) = mg - mg - ku - \gamma u' + F(t) = -ku - \gamma u' + F(t).$$

Placing all the terms with u, u' or u'' on the same side, produces the following equation.

$$mu'' + \gamma u' + ku = F(t)$$

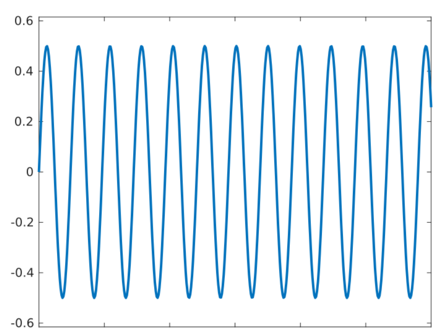
If $\gamma = 0$, the oscillations are said to be **undamped**, otherwise they are **damped**.
If $F(t) = 0$, the oscillations are said to be **free**, otherwise they are **forced.**

Undamped free oscillations. The equation of motion for undamped free oscillations is

$$mu'' + ku = 0$$

Note that the characteristic equation of this differential equation is $mr^2 + k = 0$ and has solutions $r = \pm\sqrt{\frac{k}{m}}\,i$. If we denote $\sqrt{\frac{k}{m}}$ by ω_0, the general solution of this equation is

$$u = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$$



Undamped Free Oscillations

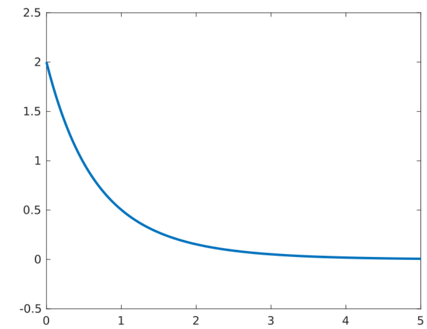
The graph of such function is a periodic function with **constant amplitude**. The values of c_1 and c_2 impact the values of the amplitude and the phase [1] The constant ω_0 is called the **natural frequency** the constant $\frac{2\pi}{\omega_0}$ represents the **period** of the motion.

Damped free oscillations. The equation of motion for damped free oscillations is

$$mu'' + \gamma u' + ku = 0$$

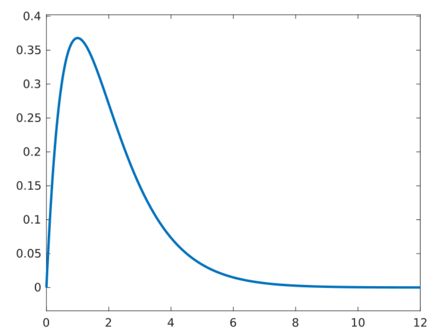
The solutions of the characteristic equations are $r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$. Let us consider the sign of the term under the root.

(i) If $\gamma^2 - 4mk > 0$, the solutions are real, different and negative (because $\gamma^2 - 4mk < \gamma^2 \Rightarrow \sqrt{\gamma^2 - 4mk} < \gamma \Rightarrow -\gamma + \sqrt{\gamma^2 - 4mk} < 0$). So, the solution is $u = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ for some $r_1, r_2 < 0$. The limit of u is zero when $t \to \infty$ since $r_1, r_2 < 0$. Hence, the mass goes back to original position and does not oscillate because no periodic functions are present. This motion is said to be **overdamped**.



Overdamped Case

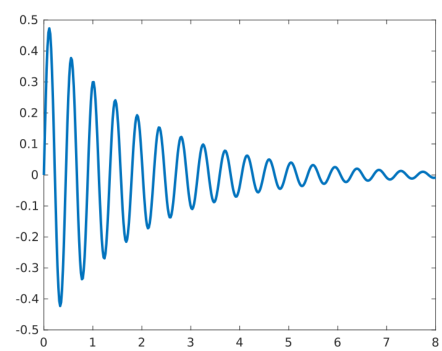
(ii) $\gamma^2 - 4mk = 0$, the solutions are real, equal ($r_1 = r_2$) and negative. The solution is $u = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$ and u also converges to zero when $t \to \infty$. Just as in the previous case, there are no periodic functions present in the solution so the mass also does not oscillate. The value of γ that makes $\gamma^2 - 4mk = 0$ is called the **critical damping.**



Critically damped Case

In these two cases, there are **no oscillations**.

(iii) $\gamma^2 - 4mk < 0$, the solutions are complex $r_1, r_2 = \frac{-\gamma}{2m} \pm i\frac{\sqrt{4km - \gamma^2}}{2m}$. So, the solutions is $u = e^{-\gamma t/(2m)}(c_1 \cos \mu t + c_2 \sin \mu t)$ where $\mu = \frac{\sqrt{4km - \gamma^2}}{2m}$. The presence of periodic functions in the solution indicates the oscillations. The term $e^{-\gamma t/(2m)}$ converges to zero when $t \to \infty$, and so do the solution u as well as its amplitude.



Underdamped Case

[1] In particular, if we put $c_1 = R\cos\delta$ and $c_2 = R\sin\delta$, then R is the amplitude, δ is the phase, and the solution is

$$u = R\cos\delta\cos\omega_0 t + R\sin\delta\sin\omega_0 t = R\cos(\omega_0 t - \delta).$$

So, the mass oscillates about the original position with a **decreasing amplitude** and the oscillations are getting smaller and smaller as time passes by.

This case occurs when the damping is relatively small (i.e. $\gamma < \sqrt{4mk}$) and it is referred to as **underdamping**. The parameter μ is called the **quasi frequency** and $\frac{2\pi}{\mu}$ is called the **quasi period**. The values of c_1 and c_2 impact the amplitude and the phase. [2]

Undamped forced oscillations. The equation of motion for undamped forced oscillations is

$$mu'' + ku = F(t)$$

If the force F is periodic, we can write it as $F = F_0 \cos \omega t$ (or $F_0 \sin \omega t$). Recall that the characteristic equation has solutions $\pm\sqrt{\frac{k}{m}}i = \pm\omega_0 i$ so that the homogeneous solution is $u_h = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$.

The particular solution can be found using the Undetermined Coefficients method. The particular solution has the form

$$u_p = t^s(A \cos \omega t + B \sin \omega t)$$

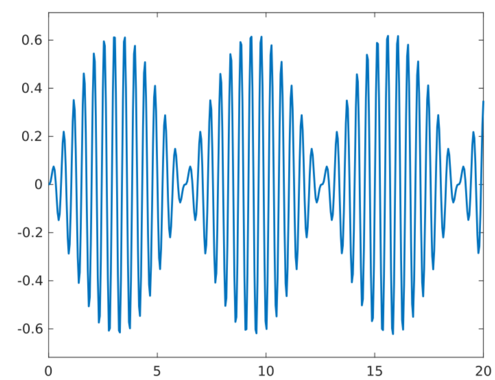
where

- $s = 0$ if ωi is not a solution of the characteristic equation i.e $\omega \neq \omega_0$ and
- $s = 1$ if ωi is a solution of the characteristic equation i.e $\omega = \omega_0$.

Case $\omega_0 \neq \omega$. In this case, the general solution has the form

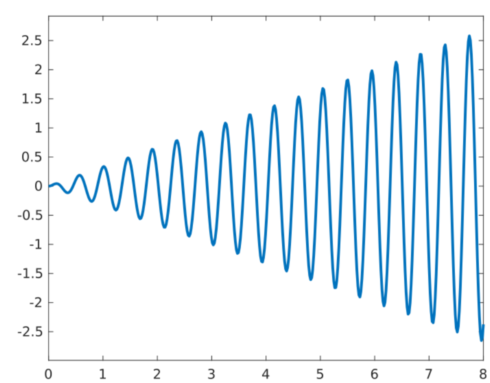
$$u = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + A \cos \omega t + B \sin \omega t$$

A function of this form is a periodic function with **periodic amplitude**. This type of motion is known as oscillations with **beats**.



Beats

Case $\omega_0 = \omega$. In this case, the frequency of the force is the same as the natural frequency and the general solution has the form
$u = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + t(A \cos \omega t + B \sin \omega t)$
Because of the term t which multiplies the trigonometric functions in the particular solution, the amplitude of the solution increases when $t \to \infty$. Thus, a function of this form is a periodic function with an **increasing amplitude**. This type of motion is known as oscillations with a **resonance**. Examples of such motion can be found in mechanics and acoustics. Mechanical resonance may cause swaying motions leading to a catastrophic failure of structures such as bridges, buildings, and vehicles. To prevent this from happening, such objects should



Resonance

[2] In particular, if we put $c_1 = R \cos \delta$ and $c_2 = R \sin \delta$, the amplitude is given by the function $Re^{-\gamma t/(2m)}$.

be designed so that the mechanical resonance frequencies of the component parts do not match the frequencies of any oscillating parts. Like mechanical resonance, acoustic resonance can result in catastrophic failure of the object at resonance, such as breaking a glass with sound. This happens when the sound wave has the same frequency as the natural frequency of the glass, the frequency at which the glass easily vibrates. If the force from the sound wave making the glass vibrate is big enough, the size of the vibration becomes so large that the glass fractures.

Example 1. Consider a motion of an object modeled by the equation $u'' + \frac{1}{4}u' + u = 0$ where the position u (in meters) is a function of time (in seconds). Assume that the object is set in motion from equilibrium with an initial velocity of 1 meter per second.

(a) Determine the position u as a function of time.

(b) Graph the solution and classify the type of motion the graph displays by noting what happens with the amplitude of the solution.

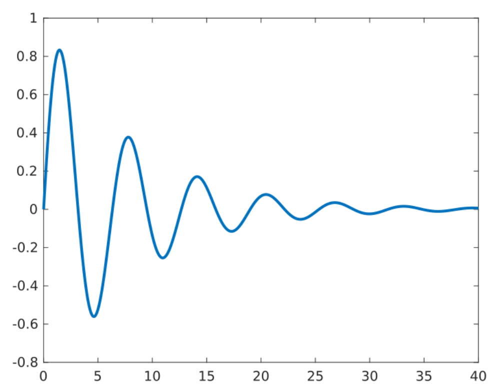
(c) Find the time when the amplitude of the oscillations becomes smaller than .1 meter.

(d) Find the time the mass returns to the equilibrium position for the first time.

Solution. (a) The characteristic equation is $r^2 + \frac{1}{4}r + 1 = 0$ and it has solutions $\frac{-\frac{1}{4} \pm \sqrt{\frac{1}{16} - 4}}{2} = \frac{-1}{8} \pm \frac{\sqrt{\frac{-63}{16}}}{2} = \frac{-1}{8} \pm \frac{\sqrt{63}i}{8} - .125 \pm .992i$. Thus, the general solution is $u = c_1 e^{-.125t}\cos .992t + c_2 e^{-.125t}\sin .992t$.

Since the object is set in motion from the equilibrium position, $u(0) = 0$. Since it is set in motion with an initial velocity of 1 m/s, $u'(0) = 1$. Use the condition $u(0) = 0$ (plug 0 for t and set u equal to 0), we have that $0 = c_1(1) + c_2(0) = c_1$. To use the condition $u'(0) = 1$, find u' first, then set it to 1 and plug 0 for t. As $u' = -.125c_1 e^{-.125t}\cos .992t - .992c_1 e^{-.125t}\sin .992t - .125c_2 e^{-.125t}\sin .992t + .992c_2 e^{-.125t}\cos .992t$ and $c_1 = 0$, we have that $1 = -.125(0)(1) - .992(0)(0) - .125c_2(0) + .992c_2(1) = .992c_2 \Rightarrow c_2 = \frac{1}{.992} = 1.008$. Hence, $u = 1.008e^{-.125t}\sin .992t$.

(b) The presence of sine and cosine in the solution means that the motion is not overdamped so there are oscillations. Since $e^{-.125t} \to 0$ for $t \to \infty$ and $e^{-.125t}$ is present in both terms, u converges to 0 meaning that the oscillations have a *decreasing amplitude*. Thus, this is an underdamped free oscillator.



(c) The expression $1.008e^{-.125t}$ represents the amplitude of the solution. Thus, the oscillations become smaller than .1 meter after the time when $1.008e^{-.125t} = .1 \Rightarrow e^{-.125t} = .0992 \Rightarrow -.125t = -2.31 \Rightarrow t = 18.48$. So, about 18.5 seconds after the mass is set in motion, the oscillations become smaller than .1 meter.

(d) The mass is at the equilibrium position when $u = 0 \Rightarrow 1.008e^{-.125t}\sin .992t = 0$. Since $1.008e^{-.125t}$ is never zero, this is possible only when $\sin .992t = 0$. Solving for t produces $.992t = \sin^{-1}(0) = 0$ and, the second solution $.992t = \pi - 0 = \pi$. The first solution corresponds to the starting position and the second solution $t = \frac{\pi}{.992} \approx 3.17$ seconds is the time when the mass returns to the equilibrium position for the first time after the starting position.

Example 2. Consider a motion of a 1-kg mass which stretches a spring by 9.8 meters. Use the value of 9.8 m/sec^2 for g. Assume that there is no damping and that the mass is acted on by an external force of $\frac{1}{2}\cos 0.8t$ newtons.

(a) Write down an equation which models the motion.

(b) Assume that the mass is set in motion from *resting* at its *equilibrium position.* Determine the position u as a function of time t.

(c) Graph the solution, and classify the type of motion the graph displays by noting what happens with the amplitude of the solution.

Solution. (a) The general equation of motion is $mu'' + \gamma u' + ku = F(t)$. We are given that $m = 1, \gamma = 0, L = 9.8$, and $F(t) = \frac{1}{2}\cos 0.8t$. Compute k using the formula $k = \frac{mg}{L}$. Thus, $k = \frac{1(9.8)}{9.8} = 1$. Hence the equation $u'' + u = \frac{1}{2}\cos 0.8t$ models this motion.

(b) The characteristic equation is $r^2 + 1 = 0$ and has solutions $r = \pm i$. Hence, the homogeneous solution is $u_h = c_1 \cos t + c_2 \sin t$. For $F(t) = \frac{1}{2}\cos 0.8t$, $p + iq = 0 + 0.8i$, $p_k(t) = \frac{1}{2}$ and so $u_p = t^s(A\cos 0.8t + B\sin 0.8t)$. Since $0.8i$ is not a solution of the characteristic equation, $s = 0$ and $u_p = A\cos 0.8t + B\sin 0.8t$. Find $u_p' = -.8A\sin 0.8t + .8B\cos 0.8t$ and $u_p'' = -.64A\cos 0.8t - .64B\sin 0.8t$ and plug them in the equation to have

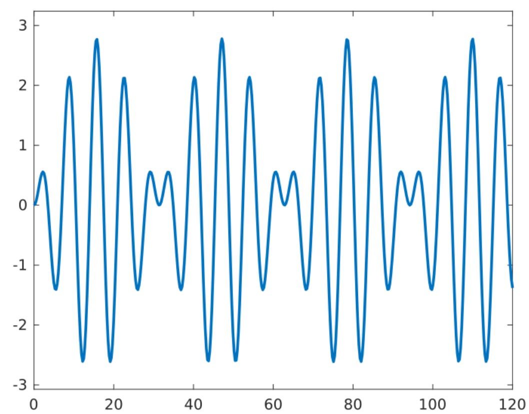
$$-.64A\cos 0.8t - .64B\sin 0.8t + A\cos 0.8t + B\sin 0.8t = \frac{1}{2}\cos 0.8t \Rightarrow .36A\cos 0.8t + .36B\sin 0.8t = \frac{1}{2}\cos 0.8t.$$

Equating the terms with cosine, $.36A = \frac{1}{2} \Rightarrow A = \frac{25}{18} \approx 1.39$. Equating the terms with sine, $B = 0$. Thus $u = c_1\cos t + c_2 \sin t + \frac{25}{18}\cos 0.8t$.

Since the object is set in motion from the equilibrium position, $u(0) = 0$. Since it is set in motion from rest, $u'(0) = 0$. Use the condition $u(0) = 0$ (plug 0 for t and set u equal to 0), we have that $0 = c_1(1) + c_2(0) + \frac{25}{18}(1) = c_1 + \frac{25}{18} \Rightarrow c_1 = -\frac{25}{18}$. To use the condition $u'(0) = 0$, find u' first, then set it to zero and plug 0 for t. As $u' = -c_1 \sin t + c_2 \cos t - 0.8\,\frac{25}{18}\sin 0.8t$ and $c_1 = -\frac{25}{18}$, we have that $0 = -c_1(0) + c_2(1) - 0.8\,\frac{25}{18}(0) = c_2 \Rightarrow c_2 = 0$. Hence,

$$u = -\frac{25}{18}\cos t + \frac{25}{18}\cos 0.8t.$$

(c) The presence of trigonometric functions indicate oscillations. The presence of two different frequencies 1 and 0.8 indicate oscillates with a *periodic amplitude.* So, the oscillations are with *beats.*



Electric circuits. Consider an electric circuit with the resistance R, the capacitance C and the inductance L containing a battery producing the voltage $E(t)$ at time t. The current I and the charge Q are related by $I = \frac{dQ}{dt}$. The second Kirchhoff's law tells us that the applied voltage $E(t)$ is equal to the sum of voltage drops in the rest of the circuit. Since

- The voltage drop across the resistor is IR,

- The voltage drop across the capacitor is $\frac{Q}{C}$, and
- The voltage drop across the inductor is $L\frac{dI}{dt}$,

the following equation models this set up.

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t)$$

Since $I = \frac{dQ}{dt}$, $\frac{dI}{dt} = \frac{d^2Q}{dt^2}$ and so we have a second order linear differential equation

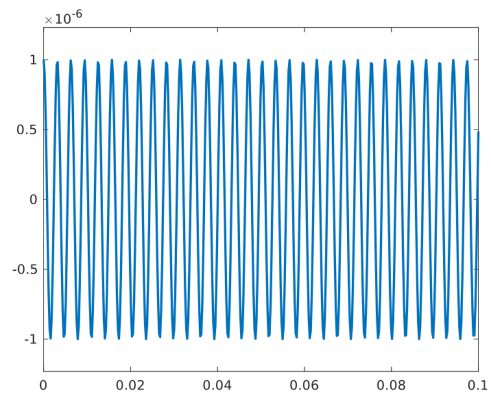
$$LQ'' + RQ' + \frac{1}{C}Q = E(t).$$

The analysis of this equation is completely analogous to the analysis of the equation of mechanical motion $mu'' + \gamma u' + ku = F(t)$.

Example 3. A series circuit has capacitor of $C = 0.25 \cdot 10^{-6}$ farad and inductor of $L = 1$ henry. If the initial charge on the capacitor is 10^{-6} coulomb and there is no initial current, find the charge Q as a function of t. Graph the solution and classify the type of motion the graph displays by noting what happens with the amplitude of the solution.

Solution. Note that $R = 0$, $\frac{1}{C} = 4 \cdot 10^6$, and there is no applied voltage so $E(t) = 0$. Thus, the general circuit equation $LQ''+RQ'+\frac{1}{C}Q = E(t)$ becomes $Q''+4\cdot 10^6 Q = 0$. The characteristic equation $r^2 + 4 \cdot 10^6 = 0$ has solutions $r = \pm 2000i$ and so the general solution is $Q = c_1 \cos 2000t + c_2 \sin 2000t$.

The initial conditions are $Q(0) = 10^{-6}$ and $Q'(0) = 0$. Plugging the first in the equation produces $10^6 = c_1(1) + c_2(0) = c_1$. The derivative is $Q' = -2000c_1 \sin 2000t + 2000c_2 \cos 2000t$ so the second condition produces $0 = -2000c_1(0) + 2000c_2(1) = 2000c_2 \Rightarrow c_2 = 0$. Thus, $Q = 10^{-6} \cos 2000t$. This is an undamped free oscillator and the solution is a periodic function with a *constant amplitude.*



Hyperbolic Sine and Cosine. In many cases, the solutions of differential equations are represented in terms of hyperbolic sine and cosine rather than in terms of exponential functions. The hyperbolic sine and cosine are defined as

$$\sinh t = \frac{e^t - e^{-t}}{2} \quad \text{and} \quad \cosh t = \frac{e^t + e^{-t}}{2}$$

The name "hyperbolic" comes from the fact that $(\cosh t, \sinh t)$ form a hyperbola, analogously to the fact that the points $(\cos t, \sin t)$ form a circle.

Using the definitions of the hyperbolic functions, the following identities can be obtained.

$$\sinh t + \cosh t = e^t \quad \text{and} \quad \cosh t - \sinh t = e^{-t}$$

Thus,

$$\sinh at + \cosh at = e^{at} \quad \text{and} \quad \cosh at - \sinh at = e^{-at}$$

Using the hyperbolic functions, we can see the solutions of the equation $y'' - a^2 y = 0$ as completely analogous to $y'' + a^2 y = 0$, where a is positive. Let us compare these solutions.

Recall that the equation $y'' + a^2 y = 0$ has characteristic roots $\pm ai$ yielding the general solution $y = c_1 \cos at + c_2 \sin at$. The equation, $y'' - a^2 y = 0$ has characteristic roots $\pm a$ yielding the general solution $y = c_1 e^{at} + c_2 e^{-at}$. Represent this solution using hyperbolic functions and the above identities: $y = c_1(\sinh at + \cosh at) + c_2(\cosh at - \sinh at) = (c_1 + c_2)\cosh at + (c_1 - c_2)\sinh at$. Denoting $C_1 = c_1 + c_2$ and $C_2 = c_1 - c_2$, we obtain the solution in the form

$$y = C_1 \cosh at + C_2 \sinh at$$

that parallels the solutions $y = c_1 \cos at + c_2 \sin at$ of $y'' + a^2 y = 0$.

Converting Higher Order Equations into Systems of First Order Equations

Recall that a system of n first order differential equations has the form

$$y_1' = F_1(t, y_1, \ldots, y_n), \;\; y_2' = F_2(t, y_1, \ldots, y_n), \;\; \ldots, \;\; y_n' = F_n(t, y_1, \ldots, y_n),$$

Every **differential equation of order** n can be converted into a **system of** n **first order equations**. Thus, studying systems encompasses the study of higher order differential equations as well. In particular, finding numerical solution of higher order equations using Matlab command **ode45** requires this procedure.

A general n-th order differential equation $F(y^{(n)}, y^{(n-1)}, \ldots, y', y, t) = 0$ can be converted into a system of n differential equations of the first order in unknown functions $y_1, y_2, \ldots, y_n$ by considering

the substitution $\quad y_1 = y, \quad y_2 = y' = y_1', \quad y_3 = y'' = y_2', \ldots, \quad y_n = y^{(n-1)} = y_{n-1}'.$

The $n-1$ equations above starting from the second to the last one represent $n-1$ equations of the new first order system. The n-th equation of the system is obtained from the original equation which, using the substitution becomes

$$F(y_n', y_n, y_{n-1}, \ldots, y_2, y_1, t) = 0.$$

If solving for y_n' produces the equation $y_n' = f(x_n, x_{n-1}, \ldots, x_2, x_1, t)$, this becomes the n-th equation of the new system. So, the **new system of** n **first order equations** is the following.

$$\begin{array}{lcl} y_1' & = & y_2 \\ y_2' & = & y_3 \\ & & \ldots \\ y_{n-1}' & = & y_n \\ y_n' & = & = f(y_n, y_{n-1}, \ldots, y_2, y_1, t) \end{array}$$

Example 4. Convert the following differential equations into a system of first order equations.

1. $y'' - ty' + 7y = \sin t + t^2$
2. $y''' + 3y' - 2y = e^t$

Solution. (1) We need to convert the given second order differential equation into a system of two first order equations. The substitution $y_1 = y$ and $y_2 = y'$ converts the given equation in y into a system in y_1 and y_2. The two new variables are related by $y_1' = y_2$ and this relation is the first equation of the new system. With this substitution the given equation becomes $y_2' - ty_2 + 7y_1 = \sin t + t^2 \Rightarrow y_2' = ty_2 - 7y_1 + \sin t + t^2$ and this last equation is the second equation of the new system. So, the new system is

$$y_1' = y_2, \quad y_2' = ty_2 - 7y_1 + \sin t + t^2.$$

(2) We need to convert the given third order differential equation into a system of three first order equations. The substitution $y_1 = y$, $y_2 = y'$, and $y_3 = y''$ converts the given equation in y into a system in y_1, y_2, and y_3. The three new variables are related by $y_1' = y_2$ and $y_2' = y_3$ these two relations are the first two equations of the new system. With this substitution the given equation becomes $y_3' + 3y_2 - 2y_1 = e^t \Rightarrow y_3' = -3y_2 + 2y_1 + e^t$ and this last equation is the third equation of the new system. So, the new system is

$$y_1' = y_2, \quad y_2' = y_3, \quad \text{and} \quad y_3' = -3y_2 + 2y_1 + e^t.$$

Practice Problems.

1. Consider a motion of an object modeled by the equation $u'' + 2u' + u = 0$ where the position u (in meters) is a function of time (in seconds). Assume that the object is set in motion from resting at 1 meter from the equilibrium position. Determine the position u as a function of time. Graph the solution and classify the type of motion the graph displays by noting what happens with the amplitude of the solution.

2. Consider a motion of a 1-kg mass which stretches a spring by 9.8 meters. Use the value of 9.8 m/sec^2 for g. Assume that there is no damping and that the mass is acted on by an external force of $\frac{1}{2}\cos t$ newtons.
   (a) Write down an equation which models the motion.
   (b) Assume that the mass is set in motion by pulling it 1 meter from the equilibrium position and then releasing it from rest. Determine the position u as a function of time t.
   (c) Graph the solution, and classify the type of motion the graph displays by noting what happens with the amplitude of the solution.

3. Determine the values of γ for which the equation $u'' + \gamma u' + 9u = 0$ has solutions which are not overdamped.

4. A mass of 0.1 kg stretches a spring 0.05 m. If the mass is set in motion from its equilibrium position with a downward velocity of 10 m/sec, and if there is no damping, determine the position u as the function of time t. Graph the solution and classify the type of motion the graph displays by noting what happens with the amplitude of the solution. Note the period and the frequency and find the time when the mass first returns to its equilibrium position.

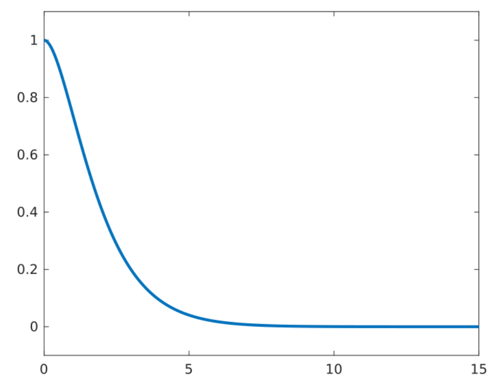
5. A mass of 20 kg is oscillating on a spring with the spring constant of 3920 N/m in a medium with the damping constant of 400 kg/sec. If the mass is pulled down additional 2 m and then released, determine the position u as the function of time t. Graph the solution and classify the type of motion the graph displays by noting what happens with the amplitude of the solution.

6. A mass of 0.5 kg stretches a spring .1 m. The mass is acted on by an external force of $\sin\frac{t}{2}$ newtons and moves in a medium that impacts a viscous force with the damping constant of 5 kg/sec. If the mass is set in motion from its equilibrium position with an initial velocity of 0.03 m/sec, determine the position u as the function of time t. Graph the solution and classify the type of motion the graph displays by noting what happens with the amplitude of the solution.

Solutions.

1. The characteristic equation is $r^2+2r+1=0 \Rightarrow (r+1)(r+1)=0$ so -1 is a double zero and the general solution is $u=c_1e^{-t}+c_2te^{-t}$. Since the mass is set in motion from 1 meter from the equilibrium, $u(0)=1$. Since the mass is set from resting, the initial velocity is zero and so $u'(0)=0$.

   The condition $u(0)=1$ implies $1=c_1(1)+c_2(0) \Rightarrow c_1=1$. Find the derivative $u'=-c_1e^{-t}+c_2e^{-t}-c_2te^{-t}$ and use $u'(0)=0$ and $c_1=1$ to have $0=-1+c_2(1)-c_2(0)=-1+c_2 \Rightarrow c_2=1$. Thus, $u=e^{-t}+te^{-t}$. The absence of trigonometric functions indicates that there are *no oscillations*. Hence, this is the overdamped case. The mass returns to equilibrium position without oscillations.



2. (a) The general equation of motion is $mu''+\gamma u'+ku=F(t)$. We are given that $m=1, \gamma=0, L=9.8$, and $F(t)=\frac{1}{2}\cos t$. Compute k using the formula $k=\frac{mg}{L}$. Thus, $k=\frac{1(9.8)}{9.8}=1$. Hence the equation $u''+u=\frac{1}{2}\cos t$ models this motion.

   (b) The characteristic equation is $r^2+1=0$ and has solutions $r=\pm i$. Hence, the homogeneous solution is $u_h=c_1\cos t+c_2\sin t$. For $F(t)=\frac{1}{2}\cos t$, $p+qi=0+1i$, $p_k(t)=\frac{1}{2}$ and so $u_p=t^s(A\cos t+B\sin t)$. Since i is a solution of the characteristic equation, $s=1$ and $u_p=At\cos t+Bt\sin t$. Find $u_p'=A\cos t-At\sin t+B\sin t+Bt\cos t$ and $u_p''=-A\sin t-A\sin t-At\cos t+B\cos t+B\cos t-Bt\sin t$ and plug them in the equation to have

$$-A\sin t-A\sin t-At\cos t+B\cos t+B\cos t-Bt\sin t+At\cos t+Bt\sin t=\frac{1}{2}\cos t \Rightarrow$$

$$-2A\sin t+2B\cos t=\frac{1}{2}\cos t \Rightarrow -2A=0 \text{ and } 2B=\frac{1}{2} \Rightarrow A=0 \text{ and } B=\frac{1}{4}.$$

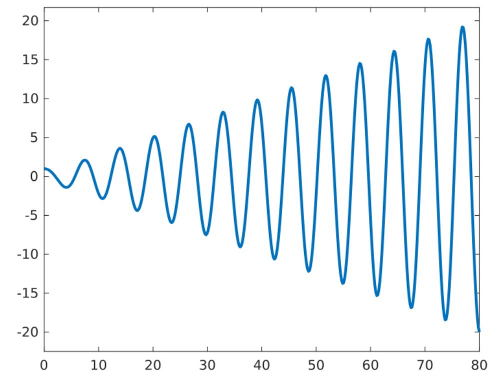
   Thus, $u=c_1\cos t+c_2\sin t+\frac{1}{4}t\sin t$. This example illustrates that the presence of cos function only in the external force does not exclude the presence if sine function only in the particular solution.

   Since the object is set in motion at 1 meter from the equilibrium position, $u(0)=1$. Since it is set in motion from rest, $u'(0)=0$. Using $u(0)=1$, $1=c_1(1)+c_2(0)+\frac{1}{4}(0)=c_1 \Rightarrow c_1=1$.

Find the derivative $u' = -c_1 \sin t + c_2 \cos t + \frac{1}{4}\sin t + \frac{1}{4} t \cos t$ and use the condition $u'(0) = 0$. Thus $0 = -c_1(0) + c_2(1) + 0 - 0 \Rightarrow c_2 = 0$. Hence,

$$u = \cos t + \frac{1}{4} t \sin t.$$

(c) The presence of trigonometric functions indicate oscillations. The presence of t in front of the sine function indicates an *increasing amplitude.* So, the oscillations are with a *resonance.*



3. The solutions are not overdamped if the characteristic equation has complex solutions (since just in this case the solution has periodic functions present). The characteristic equation is $r^2 + \gamma r + 9 = 0$. The solutions are $r = \frac{-\gamma \pm \sqrt{\gamma^2 - 36}}{2}$. Thus, the complex solutions are present just if the expression under the root is negative. So, $\gamma^2 - 36 < 0 \Rightarrow (\gamma - 6)(\gamma + 6) < 0$. This inequality has the solution $-6 < \gamma < 6$. In addition, since γ is nonnegative, this corresponds to the interval $0 \leq \gamma < 6$.

4. Find k first by $k = \frac{mg}{L}$. So, $k = \frac{0.1(9.8)}{0.05} = 19.6$. Since there is no damping and the oscillations are free, the equation of motion is $mu'' + ku = 0$. Thus, $0.1u'' + 19.6u = 0 \Rightarrow u'' + 196u = 0$. The characteristic equation has solutions $r = \pm 14i$ and the general solution is $u = c_1 \cos 14t + c_2 \sin 14t$. Since the mass is set in motion from the equilibrium position $u(0) = 0$. The initial velocity is 10 m/sec so $u'(0) = 10$. From the first initial condition, $0 = c_1(1) + c_2(0) \Rightarrow c_1 = 0$. Since $u' = -14c_1 \sin 14t + 14c_2 \cos 14t$, the second initial condition produces $10 = -14c_1(0) + 14c_2(1) \Rightarrow c_2 = \frac{10}{14} = \frac{5}{7}$.

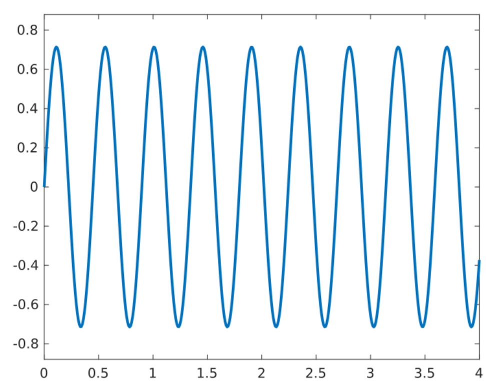
Hence,

$$u = \frac{5}{7} \sin 14t.$$

These are undamped free oscillations: the solution is a periodic function with a *constant amplitude.*

The frequency of oscillations is 14 and the period is $\frac{2\pi}{14} = \frac{\pi}{7}$.

The mass is at the equilibrium when $u = 0$. $\frac{5}{7}\sin 14t = 0 \Rightarrow \sin 14t = 0$.

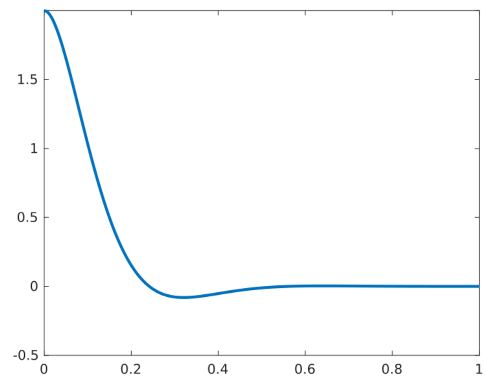


The first solution of this equation is $14t = \sin^{-1}(0) = 0 \Rightarrow t = 0$ which just denotes the first initial condition. The second solution is $14t = \pi - 0 \Rightarrow t = \frac{\pi}{14} = .22$ seconds which is the time when the mass first returns to equilibrium position after it is set in motion.

5. The equation of motion is $20u'' + 400u' + 3920u = 0$. The roots of characteristic equation $20r^2 + 400r + 3920 = 0$ are $-10 \pm 4\sqrt{6}i$. So, the general solution is $u = c_1 e^{-10t} \cos 4\sqrt{6}t + c_2 e^{-10t} \sin 4\sqrt{6}t$. The initial condition are $u(0) = 2$ and $u'(0) = 0$. From the first condition, $2 = c_1(1) + c_2(0) \Rightarrow c_1 = 2$. Find the derivative $u' = -10c_1 e^{-10t} \cos 4\sqrt{6}t - 4\sqrt{6}c_1 e^{-10t} \sin 4\sqrt{6}t -$

$10c_2e^{-10t}\sin 4\sqrt{6}t + 4\sqrt{6}c_2e^{-10t}\cos 4\sqrt{6}t$ and use $u'(0) = 0$ and $c_1 = 2$ to get $0 = -10(2)(1) - (0) - (0) + 4\sqrt{6}c_2(1) \Rightarrow 20 = 4\sqrt{6}c_2(1) \Rightarrow c_2 = \frac{20}{4\sqrt{6}} = \frac{5}{\sqrt{6}}$. Thus, the solution is $u = 2e^{-10t}\cos 4\sqrt{6}t + \frac{5}{\sqrt{6}}e^{-10t}\sin 4\sqrt{6}t$.

(b) The presence of sine and cosine in the solution means that the motion is not overdamped and that there are oscillations. Since $e^{-10t} \to 0$ for $t \to \infty$ and e^{-10t} is present in both terms of the solution, u converges to 0 meaning that the oscillations have a *decreasing amplitude*.

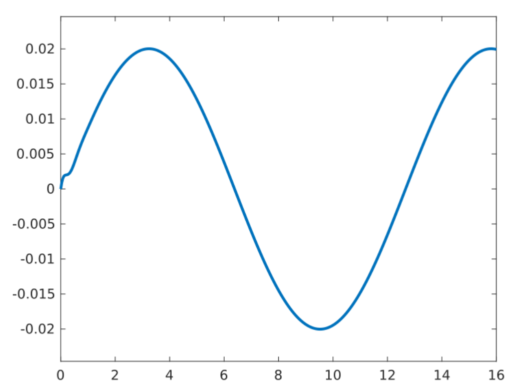


The term e^{-10t} converges to zero rather fast, so the oscillations become negligible in size fast too and the graph resembles that of an overdamped oscillator. Still, the presence of trigonometric functions indicates that there are oscillations so the motion is underdamped, not overdamped.

6. $k = \frac{0.5\cdot 9.8}{.1} = 49$, and $\gamma = 5$ so the equation of motion is $0.5u'' + 5u' + 49u = \sin\frac{t}{2}$ or, to avoid fractions, $u'' + 10u' + 98u = 2\sin\frac{t}{2}$. The characteristic equation is $r^2 + 10r + 98 = 0$ and has zeros $r = -5 \pm \sqrt{73}i$. So, the solution of the homogeneous part is $u_h = c_1e^{-5t}\cos(\sqrt{73}t) + c_2e^{-5t}\sin(\sqrt{73}t)$. Since $\frac{1}{2}i$ is not a zero of the characteristic equation, $s = 0$ and a particular solution is of the form $u_p = A\cos\frac{t}{2} + B\sin\frac{t}{2}$. Find the derivatives, substitute them in the equation and equate the terms with sines and cosines. The cos-terms equation produces $-\frac{A}{4} + 5B + 98A = 0$ and the sine equation produces $-\frac{B}{4} - 5A + 98B = 2$. From the first, $391A + 20B = 0 \Rightarrow B = \frac{-391A}{20}$. Plugging that in the second produces $(-400 - 391^2)A = 160 \Rightarrow A = -0.001$. Hence, $B = 0.0204$ and so $u = c_1e^{-5t}\cos(\sqrt{73}t) + c_2e^{-5t}\sin(\sqrt{73}t) - 0.001\cos\frac{t}{2} + 0.02\sin\frac{t}{2}$.

The initial conditions are $u(0) = 0$, $u'(0) = .03$. Using the first one, $0 = c_1(1) + c_2(0) - 0.001(1) + 0.02(0) \Rightarrow c_1 = 0.001$. Find u' and use the second condition and $c_1 = .001$. Get $0.03 = -5c_1 + \sqrt{73}c_2 + 0.01 \Rightarrow \sqrt{73}c_2 = 0.025 \Rightarrow c_2 = 0.0029$.

Thus, $u = 0.001e^{-5t}\cos(\sqrt{73}t) + 0.0029e^{-5t}\sin(\sqrt{73}t) - 0.001\cos\frac{t}{2} + 0.02\sin\frac{t}{2}$.



Since $e^{-5t} \to 0$ for $t \to \infty$, u_h converges to 0. So, after some time only u_p remains relevant. Note that the graph looks like that of an undamped free oscillator (this is how a graph of u_p also looks like) except of the small part at the beginning (this is the only part where the presence of u_h is visible).