Differential Equations
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First Exam Review

1. **Checking if a given function is a solution. Classification.**
   (a) Check if $y = x^2$ and $y = 2 + e^{-x^3}$ are solutions of differential equation $y' + 3x^2y = 6x^2$.
   (b) Classify the equation $y'' - 3y' + 2y = 0$ based on the order and linearity and show that $y = ce^{2x}$ is a solution of this differential equation for every constant c.
   (c) Show that $y = c_1e^x + c_2e^{2x}$ is a solution of differential equation $y'' - 3y' + 2y = 0$ for any value of c_1 and c_2. Then, find the constants c_1 and c_2 such that the initial conditions $y(0) = 2$ and $y'(0) = 5$ are satisfied.
   (d) Determine all values of r for which $6y'' - 7y' - 3y = 0$ has a solution of the form $y = e^{rt}$.
   (e) Find value of constants A, B and C for which the function $y = Ax^2 + Bx + C$ is the solution of the equation $y'' - y' + 4y = 8x^2$.
   (f) Find value of constant A for which the function $y = Ae^{3x}$ is the solution of the equation $y'' - 3y' + 2y = 6e^{3x}$.

2. **Solving the first order equations.** Solve the following differential equations.
   (a) $y' - 2y = x$
   (b) $y' = y^2xe^{2x}$
   (c) $x^3y^4 + (x^4y^3 + 2y)y' = 0$
   (d) $y' = \sqrt{4x + 8}, \quad y(-2) = 3$
   (e) $x^2y' + 2xy = y^3$
   (f) $y' = \frac{x^2+xy+y^2}{x^2}$ (Hint: rewrite right side as $1 + \frac{y}{x} + (\frac{y}{x})^2$)
   (g) $2x + y + (x - 2y)y' = 0$
   (h) $xy' + 2y = x^3$
   (i) $y' = \frac{xy}{x^2+1}, \quad y(0) = 2$
   (j) $y' - y + 2y^3 = 0$
   (k) $y' = \frac{4y-3x}{2x-y}$ (Hint: rewrite right side as $\frac{4(y/x)-3}{2-(y/x)}$)

3. **Autonomous equations.** Find equilibrium solutions and determine their stability. Sketch the graph of solutions of the following equations.
   (a) $y' = y(y + 1)(y - 2)$
   (b) $y' = y(2 - y)^2(5 - y)^3$
   (c) $y' = (y - a)(y - b)$
   (d) $y' = y(ay^2 - b)$, $a > 0$.

4. **Modeling with differential equations.**

(a) A population of field mice inhabits a certain rural area. In the absence of predators, the mice population increases so that each month, the population increases by 50%. However, several owls live in the same area and they kill 15 mice per day. Find an equation describing the population size and use it to predict the long term behavior of the population. Find the general solution.

(b) A population of bacteria grows at a rate proportional to the size of population. The proportionality constant is 0.7. Initially, the population consist of two members. Find the population size after six days.

(c) A glucose solution is administered intravenously into the bloodstream at a constant rate r. As the glucose is added, it is converted into other substances and removed from the bloodstream at a rate proportional to the amount present at that time.

(a) Set up a differential equation that models this situation.

(b) If $r = 4$ and the proportionality constant is 2, find the equilibrium solution, examine the stability and determine the amount of the glucose present after a long period of time. Sketch the graphs of general solutions.

(c) If 1 mg is present initially, find the formula describing the amount present after t hours.

(d) If the current temperature of an object is T_o and the room temperature is T_r, the Newton's Law of Cooling states that the rate of cooling of the object is proportional to the temperature difference between the room temperature the temperature of the object.

(a) Write a differential equation and initial condition that model the temperature of the object as a function of time. Find the equilibrium temperature and check the stability.

(b) Let us consider a 95° C coffee cup that is in a 20° C room. Assume that the proportionality constant is 0.1. Solve the differential equation in part (a) Use the solution to estimate the temperature of the coffee after 20 minutes.

(e) Suppose that a 10-kg object is *dropped* from the initial height. Assume that the drag is proportional to the velocity with the drag coefficient of 2 kg/sec.

(a) Formulate a differential equation describing the velocity of the object. Find the limiting velocity by analyzing the equilibrium solutions.

(b) Find the formula describing the velocity of the object.

(f) A tank initially contains 15 thousands gallons of pure water. A mixture of water and dye enters the tank at the rate of 3 thousands gallons per day and the mixture flows out at the same rate. The concentration of dye in the incoming water is increasing in time t according to the expression $0.5t$ grams per gallon. Determine the differential equation and an appropriate initial condition that model this situation. Find the corresponding particular solution and use it to determine the amount of dye in the tank after 3 days.

5. **Exact equations with parameters.** Find the value of parameters for which the following equations are exact and solve them using those parameter values.

(a) $(ae^{x^2} + 2y)y' - 2x^{-3} + 2xe^{x^2}y = 0$

(b) $2x \sin ay + (x^2 \cos y - by^2)y' = 0$

(c) $ay^2e^{3x} + 2x^2y + (4ye^{3x} + \frac{2}{3}x^3 + 12e^{4y})y' = 0$

Solutions

1. (a) $y = x^2 \Rightarrow y' = 2x$. Plug the function and its derivative into the equation $y' + 3x^2 y = 6x^2 \Rightarrow 2x + 3x^2(x^2) = 6x^2 \Rightarrow 2x + 3x^4 = 6x^2$. This equation does not hold for every value of x (for example if $x = 1$ the equation false identity $2+3 = 6$) so $y = x^2$ is not a solution of the given equation.

   $y = 2 + e^{-x^3} \Rightarrow y' = -3x^2 e^{-x^3}$. Plug the function and its derivative into the equation $y' + 3x^2 y = 6x^2 \Rightarrow -3x^2 e^{-x^3} + 3x^2(2 + e^{-x^3}) = 6x^2 \Rightarrow -3x^2 e^{-x^3} + 6x^2 + 3x^2 e^{-x^3} = 6x^2 \Rightarrow 6x^2 = 6x^2$. This identity holds for every x so the given function is a solution of the equation.

   (b) Linear, second order ordinary differential equation. $y = ce^{2x} \Rightarrow y' = 2ce^{2x} \Rightarrow y'' = 4ce^{2x}$. Plug into the equation $y'' - 3y' + 2y = 0 \Rightarrow 4ce^{2x} - 6ce^{2x} + 2ce^{2x} = 0 \Rightarrow (4-6+2)ce^{2x} = 0 \Rightarrow 0 = 0$. The given function is a solution.

   (c) First part is similar to the previous problem. Use the initial conditions to get $2 = c_1 e^0 + c_2 e^0$ and $5 = c_1 e^0 + 2c_2 e^0 \Rightarrow c_1 + c_2 = 2$ and $c_1 + 2c_2 = 5$. Solve for c_1 and c_2 and get $c_1 = -1, c_2 = 3$.

   (d) If $y = e^{rt}$, then $y' = re^{rt}$, and $y'' = r^2 e^{rt}$. Plugging that into the equation $6y'' - 7y' - 3y = 0$ gives you $6r^2 e^{rt} - 7re^{rt} - 3e^{rt} = 0$. Factor e^{rt}. Get $e^{rt}(6r^2 - 7r - 3) = 0$. Since e^{rt} is larger than zero for any value of t, $6r^2 - 7r - 3$ has to be zero. This happens just when $r = -1/3$ and $r = 3/2$. Thus, $y = e^{rt}$ is a solution for $r = -1/3$ and $r = 3/2$.

   (e) Find the derivatives of $y = Ax^2 + Bx + C$ to be $y' = 2Ax + B$ and $y'' = 2A$ and plug them into the equation $y'' - y' + 4y = 8x^2$ to get $2A - 2Ax - B + 4Ax^2 + 4Bx + 4C = 8x^2$. Note that both sides are polynomial functions which need to be equal for *all* values of x. This is possible just if the coefficient of polynomials with each term are equal. Thus,

   - equating the terms with x^2 obtain that $4A = 8 \Rightarrow A = 2$.
   - Equating the terms with x obtain that $-2A + 4B = 0$. Since $A = 2$, $-4 + 4B = 0 \Rightarrow B = 1$.
   - Equating the terms with no x obtain that $2A - B + 4C = 0 \Rightarrow 4 - 1 + 4C = 0 \Rightarrow C = \frac{-3}{4}$.

   Thus, $y = 2x^2 + x - \frac{3}{4}$ is a solution of differential equation.

   (f) Find the derivatives of $y = Ae^{3x}$ to be $y' = 3Ae^{3x}$ and $y'' = 9Ae^{3x}$ and substitute them into the equation $y'' - 3y' + 2y = 6e^{3x}$ to get

$$9Ae^{3x} - 9Ae^{3x} + 2Ae^{3x} = 6e^{3x} \Rightarrow 2Ae^{3x} = 6e^{3x} \Rightarrow 2A = 6 \Rightarrow A = 3$$

   Thus, $y = 3e^{3x}$ is a solution of differential equation.

2. (a) Linear equation. $P = -2$. $I = e^{\int -2dx} = e^{-2x}$. $y'e^{-2x} - 2e^{-2x}y = xe^{-2x} \Rightarrow (ye^{-2x})' = xe^{-2x} \Rightarrow ye^{-2x} = \int xe^{-2x}dx$. Use the integration by parts with $u = x$ and $dv = e^{-2x}dx$. Obtain that $ye^{-2x} = \frac{-x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + c \Rightarrow y = \frac{\frac{-x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + c}{e^{-2x}} = \frac{-x}{2} - \frac{1}{4} + ce^{2x}$.

   (b) Separable. $y' = y^2 xe^{2x} \Rightarrow \frac{dy}{dx} = y^2 xe^{2x} \Rightarrow \frac{dy}{y^2} = xe^{2x}dx \Rightarrow \frac{-1}{y} = \int xe^{2x}dx$. Use the integration by parts with $u = x$ and $dv = e^{2x}dx$ for this integral to get $\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$. Thus $y = \frac{-1}{\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c}$ or $y = \frac{1}{\frac{-1}{2}xe^{2x} + \frac{1}{4}e^{2x} + c}$.

(c) Exact equation since $M_y = 4x^3y^3 = N_x$. $F = \int x^3y^4dx = \frac{1}{4}x^4y^4 + g(y)$. $F_y = x^4y^3 + g' = N = x^4y^3 + 2y \Rightarrow g' = 2y \Rightarrow g = y^2 + c$. So, $F = \frac{1}{4}x^4y^4 + y^2 + c$ and the solution is $\frac{1}{4}x^4y^4 + y^2 + c = 0$.

(d) Separable. $y' = \sqrt{4x+8} \Rightarrow dy = \sqrt{4x+8}dx \Rightarrow y = \int \sqrt{4x+8}dx$. Use $u = 4x+8$ get $y = \frac{1}{6}(4x+8)^{3/2} + c$. Using the initial condition $x = -2, y = 3$, obtain that $3 = 0 + c \Rightarrow c = 3$. So the particular solution is $y = \frac{1}{6}(4x+8)^{3/2} + 3$.

(e) Bernoulli's equation with $n = 3$. Use $u = y^{1-3} = y^{-2} \Rightarrow y = u^{-1/2} \Rightarrow y' = \frac{-1}{2}u^{-3/2}u'$. Substitute into the equation. $\frac{-1}{2}x^2u^{-3/2}u' + 2xu^{-1/2} = u^{-3/2} \Rightarrow \frac{-1}{2}x^2u' + 2xu = 1 \Rightarrow u' - \frac{4}{x}u = \frac{-2}{x^2}$. This is a linear equation. $I = e^{\int -4/x dx} = e^{-4\ln x} = x^{-4}$. After multiplying by I, get $ux^{-4} = \int \frac{-2}{x^6}dx = \frac{-2}{-5x^5} + c \Rightarrow u = \frac{2}{5x} + cx^4 \Rightarrow y = (\frac{2}{5x} + cx^4)^{-1/2} = \frac{1}{\sqrt{\frac{2}{5x}+cx^4}}$.

(f) Homogeneous. Using the hint obtain $y' = 1 + \frac{y}{x} + (\frac{y}{x})^2$. Use the substitution $u = \frac{y}{x} \Rightarrow y = ux \Rightarrow y' = u'x + u$ to get $u'x + u = 1 + u + u^2 \Rightarrow u'x = 1 + u^2$. Separate the variables. $\frac{du}{1+u^2} = \frac{dx}{x} \Rightarrow \tan^{-1} u = \ln|x| + c \Rightarrow u = \tan(\ln|x| + c) \Rightarrow y = x\tan(\ln|x| + c)$.

(g) Exact equation since $M_y = 1 = N_x$ $F = \int(2x+y)dx = x^2 + xy + g(y)$. $F_y = x + g' = N = x - 2y \Rightarrow g' = -2y \Rightarrow g = -y^2 + c$. $F = x^2 + xy - y^2 + c$ and the solution is $x^2 + xy - y^2 + c = 0$.

(h) Linear and you need to divide by x first. $y' + \frac{2}{x}y = x^2$. $P = \frac{2}{x} \Rightarrow I = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2 \Rightarrow y'x^2 + 2xy = x^4 \Rightarrow (yx^2)' = x^4 \Rightarrow yx^2 = \int x^4dx = \frac{x^5}{5} + c \Rightarrow y = \frac{\frac{x^5}{5}+c}{x^2} = \frac{x^3}{5} + \frac{c}{x^2}$.

(i) Separable. $y' = \frac{xy}{x^2+1} \Rightarrow \frac{dy}{y} = \frac{xdx}{x^2+1} \Rightarrow \ln y = \int \frac{xdx}{x^2+1}$. Use the substitution $u = x^2+1$. Obtain that $\ln y = \frac{1}{2}\ln(x^2+1) + c \Rightarrow y = e^{\frac{1}{2}\ln(x^2+1)+c} = e^{\ln(x^2+1)^{1/2}}e^c = C(x^2+1)^{1/2} = C\sqrt{x^2+1}$. Using that $y = 2$ when $x = 0$, $2 = C\sqrt{1} \Rightarrow C = 2$. So the particular solution is $y = 2\sqrt{x^2+1}$.

(j) Bernoulli's equation with $n = 3$. Use $u = y^{1-3} = y^{-2} \Rightarrow y = u^{-1/2}$ and $y' = \frac{-1}{2}u^{-3/2}u'$. Substitute into the equation. Get $\frac{-1}{2}u^{-3/2}u' - u^{-1/2} + 2u^{-3/2} = 0 \Rightarrow u' + 2u = 4$. This is a linear equation with $I = e^{2x}$ and solution $u = 2 + ce^{-2x}$. Thus $y = \frac{1}{\sqrt{2+ce^{-2x}}}$.

(k) Homogeneous. Using the hint get $y' = \frac{4(y/x)-3}{2-(y/x)} \Rightarrow u'x + u = \frac{4u-3}{2-u} \Rightarrow u'x = \frac{4u-3}{2-u} - u = \frac{4u-3-2u+u^2}{2-u} = \frac{u^2+2u-3}{2-u} \Rightarrow \frac{(2-u)du}{u^2+2u-3} = \frac{dx}{x} \Rightarrow \frac{(2-u)du}{(u+3)(u-1)} = \frac{dx}{x}$. Use the partial fractions for the integral of the left side. Obtain $\frac{-5/4}{u+3} + \frac{1/4}{u-1} = \frac{dx}{x} \Rightarrow \frac{-5}{u+3} + \frac{1}{u-1} = \frac{4dx}{x} \Rightarrow -5\ln|u+3| + \ln|u-1| = 4\ln|x| + c \Rightarrow |u+3|^{-5}|u-1| = C|x|^4 \Rightarrow |u-1| = C|x|^4|u+3|^5$. Substitute back $|\frac{y}{x} - 1| = C|x|^4|\frac{y}{x} + 3|^5$. Multiply by x to get $|y-x| = C|y+3x|^5$.
Note that at the step when we multiplied both sides by $|u+3|^5$ we assumed this term is nonzero. In case when this is zero, the line $u = -3 \Rightarrow y = -3x$ is also a solution. So, the general solutions have the form $|y-x| = c|y+3x|^5$ or $y = -3x$.

3. (a) Equilibrium solutions: $y = -1$, $y = 0$ and $y = 2$. Checking the sign of y', conclude that $y = 0$ is stable and $y = -1$ and $y = 2$ are unstable.

(b) $y' = y(2-y)^2(5-y)^3 = 0 \Rightarrow y = 0$, $(2-y)^2 = 0$ or $(5-y)^3 = 0 \Rightarrow y = 0$, $y = 2$ or $y = 5$. Checking the sign of y' conclude that $y = 0$ is unstable, $y = 2$ is semistable and $y = 5$ is stable.

(c) You can consider the following cases: $a > b$, $a < b$ and $a = b$. In the first case, $y = b$ is stable and $y = a$ is unstable. In the second case, $y = a$ is stable and $y = b$ is unstable. If $a = b$, there is just one equilibrium solution and it is semistable.

(d) Note that $y' = y(ay^2 - b) = 0 \Rightarrow y = 0$ or $y^2 = \frac{b}{a}$. This brings you to the following cases: $\frac{b}{a} \leq 0$ and $\frac{b}{a} > 0$. In the first case, $y = 0$ is the only solution and it is unstable. In the second case, there are three solutions $y = 0$ and $y = \pm\sqrt{\frac{b}{a}}$ and $y = 0$ is stable and $y = \pm\sqrt{\frac{b}{a}}$ are unstable.

4. (a) If y stands for the size of the mice population and t for the time in months, the model can be obtained as $\frac{dy}{dt}$=rate in - rate out$= 0.5y - 15 \cdot 30 = 0.5y - 450$. The equilibrium solution is $y = 900$ and it is unstable. So, 900 mice is the threshold level for the population to survive: the number of mice drop to zero if initially there is less of 900. Just if the initial number is above the threshold, the population will survive.

   The equation $y' = 0.5y - 450$ is a separable equation with the solution $y = ce^{t/2} + 900$.

(b) The equation is $\frac{dy}{dt} = 0.7y$. Separating the variables and solving gives you $y = Ce^{0.7t}$. Using the initial condition $y(0) = 2$, the solution becomes $y = 2e^{0.7t}$. Plugging 6 for t produces $y(6) = 133$ protozoa.

(c) (a) Let y denotes the amount present at time t. The rate in is r and the rate out is ky where k is a proportionality constant. So, the differential equation $y' = r - ky$ models this situation.

   (b) $y' = 4 - 2y$. The equilibrium solution is $4 - 2y = 0 \Rightarrow y = 2$ and it is stable. Thus, 2 mg will be present after sufficiently long time period regardless of the initial amount present.

   (c) Separable (also linear). If separating the variables, you have $\frac{dy}{dt} = 4 - 2y \Rightarrow \frac{dy}{4-2y} = dt \Rightarrow \frac{1}{-2}\ln(4-2y) = t + c \Rightarrow \ln(4-2y) = -2t - 2c \Rightarrow 4 - 2y = e^{-2t-2c} \Rightarrow 2y = 4 - e^{-2t-2c} \Rightarrow y = 2 - \frac{1}{2}e^{-2t-2c} = 2 - \frac{e^{-2c}}{2}e^{-2t} = 2 - Ce^{-2t}$.

   If $y(0) = 1$, then $1 = 2 - Ce^0 \Rightarrow 1 = 2 - C \Rightarrow C = 1$. Thus $y = 2 - e^{-2t}$.

(d) (a) $dT/dt = k(T_r - T)$ and $T(0) = T_o$. The equilibrium solution T$= T_r$ is stable. (b) This is a separable differential equation with general solution $T = T_r - ce^{-kt}$. Substituting the initial condition, we obtain $T = T_r - (T_r - T_o)e^{-kt}$. Thus, if $k = .1$, $T_r = 20$ and $T_o = 95$, and $t = 20$, then $T = 30.15$ degrees Centigrade.

(e) (a) Total force = gravitation - drag. Thus, the equation is $m\frac{dv}{dt} = mg - 2v \Rightarrow \frac{dv}{dt} = g - \frac{2}{m}v = 9.8 - \frac{v}{5}$. The equilibrium solution is $v = 49$ and it is stable. Thus, the limiting velocity is 49 meters per second regardless of the initial velocity.

   (b) Separate the variables $\Rightarrow \frac{5dv}{49-v} = dt \Rightarrow -5\ln(49 - v) = t + c \Rightarrow \ln(49 - v) = \frac{-t}{5} + c \Rightarrow 49 - v = e^{-t/5}C$. General solution $v = 49 - Ce^{-t/5}$. As the object is dropped, the initial velocity is 0. Find the particular solution with $v(0) = 0$ from $v = 49 - Ce^{-t/5}$. Get $0 = 49 - C$. So, $C = 49$ and thus $v = 49 - 49e^{-t/5}$.

(f) If Q denotes the amount of dye (in grams) and t the time (in days), the total rate of change of Q is rate in - rate out. Since the rate in is $3 \cdot 10^3 \frac{\text{gal}}{\text{day}} 0.5t \frac{\text{g}}{\text{gal}}$ and the rate out is

$3 \cdot 10^3 \frac{\text{gal}}{\text{day}} \frac{Q}{15 \cdot 10^3} \frac{\text{g}}{\text{gal}}$, the equation $Q' = 1500t - 0.2Q$ models this situation. Since the tank initially contains no dye, the initial condition is $Q(0) = 0$.

The equation $Q' = 1500t - 0.2Q$ is linear. Write it as $Q' + 0.2Q = 1500t$. $I = e^{0.2t}$. So, $Qe^{.2t} = 1500 \int te^{.2t} dt = 1500(\frac{1}{0.2} te^{0.2t} - \frac{1}{0.2^2} e^{0.2t}) + c \Rightarrow Q = 7500(t-5) + ce^{-0.2t}$.

$Q(0) = 0 \Rightarrow 0 = 0 - 37500 + c \Rightarrow c = 37500$. The particular solution is $Q = 7500(t-5) + 37500e^{-0.2t} = 7500(t - 5 + 5e^{-0.2t})$. When $t = 3$, $Q = 5580.44$ grams or 5.58 kg.

5. (a) $M = -2x^{-3} + 2xe^{x^2}y$ and $N = ae^{x^2} + 2y$. $M_y = N_x \Rightarrow M_y = 2xe^{x^2} = 2axe^{x^2} = N_x$. Thus $2 = 2a \Rightarrow a = 1$.

   In this case, $F = \int(-2x^{-3} + 2xe^{x^2}y)dx = x^{-2} + e^{x^2}y + g(y)$. $F_y = e^{x^2} + g'(y) = N = e^{x^2} + 2y$ So $g'(y) = 2y$, giving you that $g(y) = \int 2ydy = y^2 + c$. Thus, the solution is $F = x^{-2} + e^{x^2}y + y^2 + c = 0$. or $e^{x^2}y + x^{-2} + y^2 = C$.

   (b) Let $M = 2x \sin ay$ and $N = x^2 \cos y - by^2$. Then $M_y = 2ax \cos ay$ and $N_x = 2x \cos y$. If $M_y = N_x$, then a has to be 1 and b can take any real value.

   $F = \int 2x \sin y dx = x^2 \sin y + g(y)$. $F_y = x^2 \cos y + g'(y) = N = x^2 \cos y - by^2 \Rightarrow g' = -by^2 \Rightarrow g = -\frac{b}{3}y^3 + c$. Thus, the solution is $x^2 \sin y - \frac{b}{3}y^3 + c = 0$ or $x^2 \sin y - \frac{b}{3}y^3 = C$.

   (c) Let $M = ay^2e^{3x} + 2x^2y$ and $N = 4ye^{3x} + \frac{2}{3}x^3 + 12e^{4y}$. For the equation to be exact, M_y should be equal to N_x. $M_y = 2aye^{3x} + 2x^2$ and $N_x = 12ye^{3x} + 2x^2$. Thus $2a = 12$ and so $a = 6$.

   In this case, $F = \int(6y^2e^{3x} + 2x^2y)dx = 2y^2e^{3x} + \frac{2}{3}x^3y + g(y)$. $F_y = 4ye^{3x} + \frac{2}{3}x^3 + g'(y) = N = 4ye^{3x} + \frac{2}{3}x^3 + 12e^{4y}$ So $g'(y) = 12e^{4y}$, giving you that $g(y) = 3e^{4y} + c$. Thus, the solution is $F = 2y^2e^{3x} + \frac{2}{3}x^3y + 3e^{4y} + c = 0$ or $2y^2e^{3x} + \frac{2}{3}x^3y + 3e^{4y} = C$.