Differential Equations
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The First Exam Review

1. **Checking if a given function is a solution. Classification.**
   (a) Check if $y = x^2$ and $y = 2 + e^{-x^3}$ are solutions of differential equation $y' + 3x^2y = 6x^2$.
   (b) Classify the equation $y'' - 3y' + 2y = 0$ based on the order and linearity and show that $y = ce^{2x}$ is a solution of this differential equation for every constant c.
   (c) Show that $y = c_1e^x + c_2e^{2x}$ is a solution of differential equation $y'' - 3y' + 2y = 0$ for any value of c_1 and c_2. Then, find the constants c_1 and c_2 such that the initial conditions $y(0) = 2$ and $y'(0) = 5$ are satisfied.
   (d) Determine all values of r for which $6y'' - 7y' - 3y = 0$ has a solution of the form $y = e^{rt}$.
   (e) Find value of constants A, B and C for which the function $y = Ax^2 + Bx + C$ is the solution of the equation $y'' - y' + 4y = 8x^2$.
   (f) Find value of constant A for which the function $y = Ae^{3x}$ is the solution of the equation $y'' - 3y' + 2y = 6e^{3x}$.

2. **Solving the first order equations.** Solve the following differential equations.
   (a) $y' - 2y = x$
   (b) $y' = y^2xe^{2x}$
   (c) $x^3y^4 + (x^4y^3 + 2y)y' = 0$
   (d) $y' = 3y\sqrt{5 - 2x}, \quad y(\frac{5}{2}) = 3$
   (e) $x^2y' + 2xy = y^3$
   (f) $y' = \frac{x^2+xy+y^2}{x^2}$ (Hint: rewrite right side as $1 + \frac{y}{x} + (\frac{y}{x})^2$)
   (g) $2x + y + (x - 2y)y' = 0$
   (h) $xy' + 2y = x^3$
   (i) $y' = \frac{xy}{x^2+1}, \quad y(0) = 2$
   (j) $y' - y + 2y^3 = 0$
   (k) $y' = \frac{4y-3x}{2x-y}$ (Hint: rewrite right side as $\frac{4(y/x)-3}{2-(y/x)}$)

3. **Exact equations with parameters.** Find the value of parameters for which the following equations are exact and solve them using those parameter values.
   (a) $(ae^{x^2} + 2y)y' - 2x^{-3} + 2xe^{x^2}y = 0$
   (b) $2x \sin ay + (x^2 \cos y - by^2)y' = 0$
   (c) $ay^2e^{3x} + 2x^2y + (4ye^{3x} + \frac{2}{3}x^3 + 12e^{4y})y' = 0$

4. **Autonomous equations.** Find equilibrium solutions and determine their stability. Sketch the graph of solutions of the following equations.

(a) $y' = y(y+1)(y-2)$

(b) $y' = y(2-y)^2(5-y)^3$

(c) $y' = (y-a)(y-b)$

(d) $y' = y(ay^2 - b)$, $a > 0$.

5. **Modeling with differential equations.**

   (a) A population of field mice inhabits a certain rural area. In the absence of predators, the mice population increases so that each month, the population increases by 50%. However, several owls live in the same area and they kill 15 mice per day. Find an equation describing the population size and use it to predict the long term behavior of the population. Find the general solution.

   (b) A glucose solution is administered intravenously into the bloodstream at a constant rate of r mg per minute. As the glucose is added, it is converted into other substances and removed from the bloodstream at a rate proportional to the amount of glucose at that time.

   (a) Set up a differential equation that models this situation.

   (b) If $r = 4$ and the proportionality constant is 2, sketch the graphs of the general solution and examine the stability. Determine the amount of glucose present after a long period of time.

   (c) If 1 mg is present initially, find the formula describing the amount present after t hours.

   (c) The mass of a bacteria colony has been monitored. Let $M(t)$ denote the mass (in mg) of the colony at time t (in days). The mass M is increasing by 20% per day. On the other hand, extreme dryness to which the colony is exposed makes the mass M decrease at a rate of 3 mg per day.

   (a) Write a differential equation whose solution describes the mass M at any given time t using the assumptions from the paragraph above.

   (b) Sketch the graphs of the solutions (note that the equation is autonomous), find the equilibrium solution(s) and examine the stability. Explain what happens with the mass of the bacteria colony in the long run for any initial value of the mass (use complete sentences in your explanation).

   (c) Find the general solution of the differential equation from part (a).

   (d) Determine the bacteria mass after 5 days if the initial mass is 10 mg.

   (d) Let $T(t)$ represents the temperature of an object at time t. If the current temperature of the object is T_0 and the room temperature is T_r, the Newton's Law of Cooling states that the rate of cooling $\frac{dT}{dt}$ is proportional to the temperature difference between the room temperature and the temperature of the object at time t.

   (a) Write a differential equation and the initial condition whose solution would provide a formula for $T(t)$. Make a sketch of the general solutions. Use the graph to conclude what happens to the temperature as time goes by for all possible values of T_0.

   (b) Let us consider a 95° C coffee cup that is in a 20° C room. Assume that the proportionality constant is 0.1. Solve the differential equation from part (a) to find the function describing the temperature of the coffee as a function of time (in minutes). Use the solution to estimate the temperature of the coffee after 20 minutes.

   (e) A tank initially contains 15 thousands gallons of pure water. A mixture of water and dye enters the tank at the rate of 3 thousands gallons per day and the mixture flows out at the same rate. The concentration of dye in the incoming water is increasing in time t according to the expression $0.5t$ grams per gallon. Thus, if y denotes the amount of dye in grams, the rate in is $3000 \cdot 0.5t = 1500t$ and the rate out is $3000 \cdot \frac{y}{15000} = \frac{1}{5}y$. Determine the differential equation and an appropriate initial condition that model this situation. Find the corresponding particular solution and use it to determine the amount of dye in the tank after 3 days.

Solutions

1. (a) $y = x^2 \Rightarrow y' = 2x$. Plug the function and its derivative into the equation $y' + 3x^2y = 6x^2 \Rightarrow 2x + 3x^2(x^2) = 6x^2 \Rightarrow 2x + 3x^4 = 6x^2$. This equation does not hold for every value of x (for example if $x = 1$ the equation false identity $2+3 = 6$) so $y = x^2$ is not a solution of the given equation.

   $y = 2 + e^{-x^3} \Rightarrow y' = -3x^2e^{-x^3}$. Plug the function and its derivative into the equation $y' + 3x^2y = 6x^2 \Rightarrow -3x^2e^{-x^3} + 3x^2(2 + e^{-x^3}) = 6x^2 \Rightarrow -3x^2e^{-x^3} + 6x^2 + 3x^2e^{-x^3} = 6x^2 \Rightarrow 6x^2 = 6x^2$. This identity holds for every x so the given function is a solution of the equation.

   (b) Linear, second order ordinary differential equation. $y = ce^{2x} \Rightarrow y' = 2ce^{2x} \Rightarrow y'' = 4ce^{2x}$. Plug into the equation $y'' - 3y' + 2y = 0 \Rightarrow 4ce^{2x} - 6ce^{2x} + 2ce^{2x} = 0 \Rightarrow (4-6+2)ce^{2x} = 0 \Rightarrow 0 = 0$. The given function is a solution.

   (c) First part is similar to the previous problem. Use the initial conditions to get $2 = c_1e^0 + c_2e^0$ and $5 = c_1e^0 + 2c_2e^0 \Rightarrow c_1 + c_2 = 2$ and $c_1 + 2c_2 = 5$. Solve for c_1 and c_2 and get $c_1 = -1, c_2 = 3$.

   (d) If $y = e^{rt}$, then $y' = re^{rt}$, and $y'' = r^2e^{rt}$. Plugging that into the equation $6y'' - 7y' - 3y = 0$ gives you $6r^2e^{rt} - 7re^{rt} - 3e^{rt} = 0$. Factor e^{rt}. Get $e^{rt}(6r^2 - 7r - 3) = 0$. Since e^{rt} is larger than zero for any value of t, $6r^2 - 7r - 3$ has to be zero. This happens just when $r = -1/3$ and $r = 3/2$. Thus, $y = e^{rt}$ is a solution for $r = -1/3$ and $r = 3/2$.

   (e) Find the derivatives of $y = Ax^2 + Bx + C$ to be $y' = 2Ax + B$ and $y'' = 2A$ and plug them into the equation $y'' - y' + 4y = 8x^2$ to get $2A - 2Ax - B + 4Ax^2 + 4Bx + 4C = 8x^2$. Note that both sides are polynomial functions which need to be equal for *all* values of x. This is possible just if the coefficient of polynomials with each term are equal. Thus,

   - equating the terms with x^2 obtain that $4A = 8 \Rightarrow A = 2$.
   - Equating the terms with x obtain that $-2A + 4B = 0$. Since $A = 2$, $-4 + 4B = 0 \Rightarrow B = 1$.
   - Equating the terms with no x obtain that $2A - B + 4C = 0 \Rightarrow 4 - 1 + 4C = 0 \Rightarrow C = \frac{-3}{4}$.

   Thus, $y = 2x^2 + x - \frac{3}{4}$ is a solution of differential equation.

   (f) Find the derivatives of $y = Ae^{3x}$ to be $y' = 3Ae^{3x}$ and $y'' = 9Ae^{3x}$ and substitute them into the equation $y'' - 3y' + 2y = 6e^{3x}$ to get

   $$9Ae^{3x} - 9Ae^{3x} + 2Ae^{3x} = 6e^{3x} \Rightarrow 2Ae^{3x} = 6e^{3x} \Rightarrow 2A = 6 \Rightarrow A = 3$$

   Thus, $y = 3e^{3x}$ is a solution of differential equation.

2. (a) Linear equation. $P = -2$. $I = e^{\int -2dx} = e^{-2x}$. $y'e^{-2x} - 2e^{-2x}y = xe^{-2x} \Rightarrow (ye^{-2x})' = xe^{-2x} \Rightarrow ye^{-2x} = \int xe^{-2x}dx$. Use the integration by parts with $u = x$ and $dv = e^{-2x}dx$. Obtain that $ye^{-2x} = \frac{-x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + c \Rightarrow y = \frac{\frac{-x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + c}{e^{-2x}} = \frac{-x}{2} - \frac{1}{4} + ce^{2x}$.

(b) Separable. $y' = y^2xe^{2x} \Rightarrow \frac{dy}{dx} = y^2xe^{2x} \Rightarrow \frac{dy}{y^2} = xe^{2x}dx \Rightarrow \frac{-1}{y} = \int xe^{2x}dx$. Use the integration by parts with $u = x$ and $dv = e^{2x}dx$ for this integral to get $\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$. Thus $y = \frac{-1}{\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+c}$ or $y = \frac{1}{\frac{-1}{2}xe^{2x}+\frac{1}{4}e^{2x}+c}$.

(c) Exact equation since $M_y = 4x^3y^3 = N_x$. $F = \int x^3y^4dx = \frac{1}{4}x^4y^4 + g(y)$. $F_y = x^4y^3 + g' = N = x^4y^3 + 2y \Rightarrow g' = 2y \Rightarrow g = y^2 + c$. So, $F = \frac{1}{4}x^4y^4 + y^2 + c$ and the solution is $\frac{1}{4}x^4y^4 + y^2 + c = 0$.

(d) Separable. $y' = 3y\sqrt{5-2x} \Rightarrow \frac{dy}{y} = 3\sqrt{5-2x}dx$. Use substitution with $u = 5-2x$ for the right side. Obtain that $\ln|y| = \frac{-3}{2}\frac{2}{3}(5-2x)^{3/2}+c = -(5-2x)^{3/2}+c \Rightarrow y = \pm e^{-(5-2x)^{3/2}+c} = \pm e^{-(5-2x)^{3/2}}e^c = Ce^{-(5-2x)^{3/2}}$. Using that $y(\frac{5}{2}) = 3$, we have that $3 = Ce^0 \Rightarrow C = 3$. So, $y = 3e^{-(5-2x)^{3/2}}$ or $y = 3e^{-\sqrt{(5-2x)^3}}$.

(e) Bernoulli's equation with $n = 3$. Use $u = y^{1-3} = y^{-2} \Rightarrow y = u^{-1/2} \Rightarrow y' = \frac{-1}{2}u^{-3/2}u'$. Substitute into the equation. $\frac{-1}{2}x^2u^{-3/2}u' + 2xu^{-1/2} = u^{-3/2} \Rightarrow \frac{-1}{2}x^2u' + 2xu = 1 \Rightarrow u' - \frac{4}{x}u = \frac{-2}{x^2}$. This is a linear equation. $I = e^{\int -4/x dx} = e^{-4\ln x} = x^{-4}$. After multiplying by I, get $ux^{-4} = \int \frac{-2}{x^6}dx = \frac{-2}{-5x^5} + c \Rightarrow u = \frac{2}{5x} + cx^4 \Rightarrow y = (\frac{2}{5x} + cx^4)^{-1/2} = \frac{1}{\sqrt{\frac{2}{5x}+cx^4}}$.

(f) Homogeneous. Using the hint obtain $y' = 1 + \frac{y}{x} + (\frac{y}{x})^2$. Use the substitution $u = \frac{y}{x} \Rightarrow y = ux \Rightarrow y' = u'x + u$ to get $u'x + u = 1 + u + u^2 \Rightarrow u'x = 1 + u^2$. Separate the variables. $\frac{du}{1+u^2} = \frac{dx}{x} \Rightarrow \tan^{-1}u = \ln|x| + c \Rightarrow u = \tan(\ln|x| + c) \Rightarrow y = x\tan(\ln|x| + c)$.

(g) Exact equation since $M_y = 1 = N_x$ $F = \int(2x + y)dx = x^2 + xy + g(y)$. $F_y = x + g' = N = x - 2y \Rightarrow g' = -2y \Rightarrow g = -y^2 + c$. $F = x^2 + xy - y^2 + c$ and the solution is $x^2 + xy - y^2 + c = 0$.

(h) Linear and you need to divide by x first. $y' + \frac{2}{x}y = x^2$. $P = \frac{2}{x} \Rightarrow I = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2 \Rightarrow y'x^2 + 2xy = x^4 \Rightarrow (yx^2)' = x^4 \Rightarrow yx^2 = \int x^4dx = \frac{x^5}{5} + c \Rightarrow y = \frac{\frac{x^5}{5}+c}{x^2} = \frac{x^3}{5} + \frac{c}{x^2}$.

(i) Separable. $y' = \frac{xy}{x^2+1} \Rightarrow \frac{dy}{y} = \frac{xdx}{x^2+1} \Rightarrow \ln|y| = \int \frac{xdx}{x^2+1}$. Use the substitution $u = x^2 + 1$ for this last integral. Obtain that $\ln|y| = \frac{1}{2}\ln(x^2 + 1) + c$. Note that $x^2 + 1$ is positive, so no absolute value is needed on the right side. Thus, $|y| = e^{\frac{1}{2}\ln(x^2+1)+c} \Rightarrow y = \pm e^{\ln(x^2+1)^{1/2}}e^c = Ce^{\ln(x^2+1)^{1/2}} = C(x^2 + 1)^{1/2} = C\sqrt{x^2 + 1}$. Using that $y = 2$ when $x = 0$, obtain that $2 = C\sqrt{1} \Rightarrow C = 2$. So, the particular solution is $y = 2\sqrt{x^2 + 1}$.

(j) Bernoulli's equation with $n = 3$. Use $u = y^{1-3} = y^{-2} \Rightarrow y = u^{-1/2}$ and $y' = \frac{-1}{2}u^{-3/2}u'$. Substitute into the equation. Get $\frac{-1}{2}u^{-3/2}u' - u^{-1/2} + 2u^{-3/2} = 0 \Rightarrow u' + 2u = 4$. This is a linear equation with $I = e^{2x}$ and solution $u = 2 + ce^{-2x}$. Thus $y = \frac{1}{\sqrt{2+ce^{-2x}}}$.

(k) Homogeneous. Using the hint get $y' = \frac{4(y/x)-3}{2-(y/x)} \Rightarrow u'x + u = \frac{4u-3}{2-u} \Rightarrow u'x = \frac{4u-3}{2-u} - u = \frac{4u-3-2u+u^2}{2-u} = \frac{u^2+2u-3}{2-u} \Rightarrow \frac{(2-u)du}{u^2+2u-3} = \frac{dx}{x} \Rightarrow \frac{(2-u)du}{(u+3)(u-1)} = \frac{dx}{x}$. Use the partial fractions for the integral of the left side. Obtain $\frac{-5/4}{u+3} + \frac{1/4}{u-1} = \frac{dx}{x} \Rightarrow \frac{-5}{u+3} + \frac{1}{u-1} = \frac{4dx}{x} \Rightarrow -5\ln|u + 3| + \ln|u - 1| = 4\ln|x| + c \Rightarrow |u + 3|^{-5}|u - 1| = C|x|^4 \Rightarrow |u - 1| = C|x|^4|u + 3|^5$. Substitute back $|\frac{y}{x} - 1| = C|x|^4|\frac{y}{x} + 3|^5$. Multiply by x to get $|y - x| = C|y + 3x|^5$.

Note that at the step when we multiplied both sides by $|u + 3|^5$ we assumed this term is nonzero. In case when this is zero, the line $u = -3 \Rightarrow y = -3x$ is also a solution. So, the general solutions have the form $|y - x| = c|y + 3x|^5$ or $y = -3x$.

3. (a) $M = -2x^{-3} + 2xe^{x^2}y$ and $N = ae^{x^2} + 2y$. $M_y = N_x \Rightarrow M_y = 2xe^{x^2} = 2axe^{x^2} = N_x$. Thus $2 = 2a \Rightarrow a = 1$.
   In this case, $F = \int(-2x^{-3} + 2xe^{x^2}y)dx = x^{-2} + e^{x^2}y + g(y)$. $F_y = e^{x^2} + g'(y) = N = e^{x^2} + 2y$ So $g'(y) = 2y$, giving you that $g(y) = \int 2ydy = y^2 + c$. Thus, the solution is $F = x^{-2} + e^{x^2}y + y^2 + c = 0$. or $e^{x^2}y + x^{-2} + y^2 = C$.

   (b) Let $M = 2x\sin ay$ and $N = x^2\cos y - by^2$. Then $M_y = 2ax\cos ay$ and $N_x = 2x\cos y$. If $M_y = N_x$, then a has to be 1 and b can take any real value.
   $F = \int 2x\sin y dx = x^2\sin y + g(y)$. $F_y = x^2\cos y + g'(y) = N = x^2\cos y - by^2 \Rightarrow g' = -by^2 \Rightarrow g = -\frac{b}{3}y^3 + c$. Thus, the solution is $x^2\sin y - \frac{b}{3}y^3 + c = 0$ or $x^2\sin y - \frac{b}{3}y^3 = C$.

   (c) Let $M = ay^2e^{3x} + 2x^2y$ and $N = 4ye^{3x} + \frac{2}{3}x^3 + 12e^{4y}$. For the equation to be exact, M_y should be equal to N_x. $M_y = 2aye^{3x} + 2x^2$ and $N_x = 12ye^{3x} + 2x^2$. Thus $2a = 12$ and so $a = 6$.
   In this case, $F = \int(6y^2e^{3x} + 2x^2y)dx = 2y^2e^{3x} + \frac{2}{3}x^3y + g(y)$. $F_y = 4ye^{3x} + \frac{2}{3}x^3 + g'(y) = N = 4ye^{3x} + \frac{2}{3}x^3 + 12e^{4y}$ So $g'(y) = 12e^{4y}$, giving you that $g(y) = 3e^{4y} + c$. Thus, the solution is $F = 2y^2e^{3x} + \frac{2}{3}x^3y + 3e^{4y} + c = 0$ or $2y^2e^{3x} + \frac{2}{3}x^3y + 3e^{4y} = C$.

4. (a) Equilibrium solutions: $y = -1$, $y = 0$ and $y = 2$. Checking the sign of y', conclude that $y = 0$ is stable and $y = -1$ and $y = 2$ are unstable.

   (b) $y' = y(2-y)^2(5-y)^3 = 0 \Rightarrow y = 0$, $(2-y)^2 = 0$ or $(5-y)^3 = 0 \Rightarrow y = 0$, $y = 2$ or $y = 5$. Checking the sign of y' conclude that $y = 0$ is unstable, $y = 2$ is semistable and $y = 5$ is stable.

   (c) You can consider the following cases: $a > b$, $a < b$ and $a = b$. In the first case, $y = b$ is stable and $y = a$ is unstable. In the second case, $y = a$ is stable and $y = b$ is unstable. If $a = b$, there is just one equilibrium solution and it is semistable.

   (d) Note that $y' = y(ay^2 - b) = 0 \Rightarrow y = 0$ or $y^2 = \frac{b}{a}$. This brings you to the following cases: $\frac{b}{a} \leq 0$ and $\frac{b}{a} > 0$. In the first case, $y = 0$ is the only solution and it is unstable. In the second case, there are three solutions $y = 0$ and $y = \pm\sqrt{\frac{b}{a}}$ and $y = 0$ is stable and $y = \pm\sqrt{\frac{b}{a}}$ are unstable.

5. (a) If y stands for the size of the mice population and t for the time in months, the model can be obtained as $\frac{dy}{dt}$=rate in - rate out$= 0.5y - 15 \cdot 30 = 0.5y - 450$. The equilibrium solution is $y = 900$ and it is unstable. So, 900 mice is the threshold level for the population to survive: the number of mice drop to zero if initially there is less of 900. Just if the initial number is above the threshold, the population will survive.
   The equation is both separable and linear. If separating the variables, get $\frac{dy}{.5y-450} = dt \Rightarrow 2\ln|.5y - 450| = t + c \Rightarrow \ln|.5y - 450| = \frac{t}{2} + \frac{c}{2} \Rightarrow |.5y - 450| = e^{t/2+c/2} \Rightarrow .5y - 450 = \pm e^{t/2+c/2} = \pm e^{c/2}e^{t/2} = Ce^{t/2} \Rightarrow .5y = Ce^{t/2} + 450 \Rightarrow y = Ce^{t/2} + 900$ (in this last step, we denoted $2C$ by C again).

   (b) (a) Let y denotes the amount present at time t. The rate in is r and the rate out is ky where k is a proportionality constant. So, $y' = r - ky$ models this situation.
   (b) $y' = 4 - 2y$. The equilibrium solution is $4 - 2y = 0 \Rightarrow y = 2$ and it is stable. Thus, 2 mg will be present after sufficiently long time period regardless of the initial amount.
   (c) Separable (also linear) equation. If separating the variables, you have $\frac{dy}{4-2y} = dt$. Integrate both sides. Use $u = 4 - 2y$ for the left side. Obtain $\frac{1}{-2}\ln|4 - 2y| = t + c \Rightarrow$

$\ln|4-2y| = -2t-2c \Rightarrow |4-2y| = e^{-2t-2c} \Rightarrow 4-2y = \pm e^{-2t-2c} = \pm e^{-2c}e^{-2t} = Ce^{-2t} \Rightarrow -2y = -4+Ce^{-2t} \Rightarrow y = 2-\frac{1}{2}Ce^{-2t}$. Replacing $\frac{-1}{2}C$ by c, we obtain the general solution $y = 2+ce^{-2t}$. Note that the term ce^{-2t} converges to 0 and so $y \to 2$ for $t \to \infty$ regardless of the value of the constant c. This agrees with the conclusion from part b).

If $y(0) = 1$, then $1 = 2+ce^0 \Rightarrow 1 = 2+c \Rightarrow c = -1$. Thus $y = 2-e^{-2t}$. This function passes $(0,1)$ and it is increasing to 2 when $t \to \infty$.

(c) (a) Since the rate in is $0.2M$ and the rate out is 3 the total rate $\frac{dM}{dt}$ = rate in - rate out $= 0.2M-3$.

(b) The equilibrium solution is $0.2M-3 = 0 \Rightarrow M = 15$ mg. Examining the sign of the derivative $M' = 0.2M-3$ obtain $\overset{-\qquad +}{\underset{15}{\overline{\qquad\qquad}}}$ and conclude that $M = 15$ is an unstable solution. Thus, if the initial mass is below 15 mg, the population will eventually die out. If the initial size is above 15 mg, the population will increase without a bound.

(c) The equation is separable. Separate the variables. Get $\frac{dM}{0.2M-3} = dt$. Integrate both sides. Get $\frac{1}{0.2}\ln|0.2M-3| = t+c \Rightarrow \ln|0.2M-3| = .2t+.2c \Rightarrow |0.2M-3| = e^{.2t+.2c} \Rightarrow 0.2M-3 = \pm e^{.2t+.2c} = \pm e^{.2c}e^{.2t} = Ce^{.2t} \Rightarrow .2M = Ce^{.2t}+3 \Rightarrow M = \frac{1}{0.2}(Ce^{0.2t}+3) \Rightarrow M = Ce^{0.2t}+15$. Note that in this last step, we use C for $\frac{C}{.2}$ from the previous step.

The equation $M'-0.2M = -3$ is also linear and you can solve by using $P = -0.2 \Rightarrow I = e^{-0.2t} \Rightarrow Me^{-0.2t} = \int -3e^{-0.2t}dt \Rightarrow Me^{-0.2t} = 15e^{-0.2t}+c \Rightarrow M = 15+ce^{0.2t}$.

(d) The initial condition is $M(0) = 10$. Thus, with solution $M = 15+ce^{0.2t}$, we have that $10 = 15+c \Rightarrow c = -5$. Hence, the particular solution is $M = 15-5e^{0.2t}$. When $t = 5$ days, $M = 15-5e \approx 1.41$ mg.

(d) (a) The equation is $\frac{dT}{dt} = k(T_r-T)$ with the initial condition $T(0) = T_0$. Note that the equation is *not* $\frac{dT}{dt} = k(T_r-T_0)$ because the right side of the equation is constant and the object is not cooling at a constant rate. There is one equilibrium solution $T = T_r$.

The equilibrium solution $T = T_r$ is stable: if the object has initial temperature higher than that of the room, it will cool down to T_r, and if the object has initial temperature lower than that of the room, it will warm up to T_r. If the initial temperature is the room temperature, the temperature stays constant at T_r.

(b) The equation is separable. Separating the variables get $\frac{dT}{T_r-T} = kdt$. Integrating both sides get $-\ln|T_r-T| = kt+c$. Solving for T get $\ln|T_r-T| = -kt-c \Rightarrow T_r-T = \pm e^{-kt-c} = \pm e^{-c}e^{-kt} = Ce^{-kt} \Rightarrow T = T_r-Ce^{-kt}$. Alternatively, you can have $T = T_r+Ce^{-kt}$ if you use C for $-C$ of the previous version.

With the initial condition $T(0) = T_0$ and $T = T_r+Ce^{-kt}$, you have that $T_0 = T_r+C$, thus $C = T_0-T_r$ and so $T = T_r+(T_0-T_r)e^{-kt}$. With $T_0 = 95$, $T_r = 20$, and $k = 0.1$, the solution is $T = 20+75e^{-0.1t}$. To estimate the temperature of the coffee after 20 minutes, plug $t = 20$ into the equation for T. Get $T = 20+75e^{-2} = 30.15$ degrees Centigrade.

(e) The equation is $y' = 1500t-\frac{1}{5}y$. Since the tank initially contains no dye, the initial condition is $y(0) = 0$.

The equation is linear. Write it as $y'+\frac{1}{5}y = 1500t$ so that $I = e^{t/5}$. Thus, $ye^{t/5} = 1500\int te^{t/5}dt$. Use the integration by parts for the integral on the right and obtain that $ye^{t/5} = 1500(5te^{t/5}-25e^{t/5})+c \Rightarrow y = 7500(t-5)+ce^{-t/5}$.

$y(0) = 0 \Rightarrow 0 = 0-37500+c \Rightarrow c = 37500$. Thus, $y = 7500(t-5)+37500e^{-t/5} = 7500(t-5+5e^{-t/5})$. After three days, $t = 3$ and so $y = 5580.44$ grams or 5.58 kg.