Differential Equations
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The Second Exam Review

1. **Homogeneous equations with constant coefficients.** Solve the following equations.

   (a) $y'' - 2y' + 5y = 0$ (b) $y''' - 2y'' + y' = 0$
   (c) $y^{(4)} - y = 0$ (d) $y^{(4)} - 5y'' - 36y = 0$
   (e) $y^{(5)} - 32y = 0$. (f) $y^{(5)} + 32y = 0$.

2. **Non-homogeneous equations with constant coefficients. Variation of Parameters.** Solve the following differential equations. Note that parts (a) and (c) cannot be solved using Undetermined Coefficients method.

   (a) $y'' - 6y' + 9y = x^{-3}e^{3x}$. (b) $y'' - 5y' + 6y = 2e^{x}$ (c) $y'' + 4y' + 4y = x^{-2}e^{-2x}$

3. **Non-homogeneous equations with constant coefficients. Undetermined Coefficients.** Find general solution of problems (a)–(d). In problems (e)–(g), find the *form* of particular solutions and the general solutions. For (e)–(g), you **do not** have to solve for unknown coefficients in particular solutions.

   (a) $y'' - 5y' + 6y = 4e^{2x}$ (b) $y'' + 4y = 5x^2e^{x}$
   (c) $y'' - 2y' + y = 7xe^{x}$ (d) $y'' + 2y' - 3y = 5\sin 3x$
   (e) $y'' - 3y' - 10y = 3xe^{2x} + 5e^{-2x}$ (f) $y'' - 8y' + 16y = 3x^2 - 5e^{4x}$
   (g) $y'' + 4y' + 13y = -2\sin 3x + e^{-2x}\cos 3x$

4. **Applications of higher order differential equations.**

   (a) Consider a motion of an object modeled by the equation $u'' + \frac{1}{4}u' + u = 0$ where the position u (in meters) is a function of time (in seconds). Assume that the object is set in motion from equilibrium with an initial velocity of 1 meter per second. (i) Determine the position u as a function of time. (ii) Graph the solution and classify the type of motion the graph displays by noting what happens with the amplitude of the solution.

   (b) Consider a motion of a 1-kg mass which stretches a spring by 9.8 meters. Use the value of 9.8 m/sec^2 for g. Assume that there is no damping and that the mass is acted on by an external force of $\frac{1}{2}\cos 0.8t$ newtons. (i) Write down an equation which models the motion. (ii) Assume that the mass is set in motion from resting at its equilibrium position. Determine the position u as a function of time t. (iii) Graph the solution, and classify the type of motion the graph displays by noting what happens with the amplitude of the solution.

   (c) Assume that the force in the previous problem changes to $\frac{1}{2}\cos t$ newtons and that the initial conditions are determined by assuming that the mass is set in motion by pulling it 1 meter from the equilibrium position and then releasing it from rest. Do the parts (i) to (iii) of the previous problem in this case.

(d) Consider a motion of an object modeled by the equation $u'' + 2u' + u = 0$ where the position u (in meters) is a function of time (in seconds). Assume that the object is set in motion from resting at 1 meter from the equilibrium position. Determine the position u as a function of time. Graph the solution and classify the type of motion the graph displays by noting what happens with the amplitude of the solution.

(e) A series circuit has capacitor of $C = 0.25 \cdot 10^{-6}$ farad and inductor of $L = 1$ henry. If the initial charge on the capacitor is 10^{-6} coulomb and there is no initial current, find the charge Q as a function of t. Graph the solution and classify the type of motion the graph displays by noting what happens with the amplitude of the solution.

(f) Determine the values of γ for which the equation $u'' + \gamma u' + 9u = 0$ has solutions which are not overdamped.

Solutions

1. Homogeneous Equations.

(a) $y'' - 2y' + 5y = 0 \Rightarrow r^2 - 2r + 5 = 0 \Rightarrow r = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i \Rightarrow y_1 = e^x \cos 2x$ and $y_2 = e^x \sin 2x$. General solution $y = c_1 e^x \cos 2x + c_2 e^x \sin 2x$.

(b) $y''' - 2y'' + y' = 0 \Rightarrow r^3 - 2r^2 + r = 0 \Rightarrow r(r-1)^2 = 0 \Rightarrow r = 0$ is a zero and $r = 1$ is a double zero $\Rightarrow y_1 = e^{0x} = 1$, $y_2 = e^x$ and $y_3 = xe^x$. The general solution is $y = c_1 + c_2 e^x + c_3 x e^x$.

(c) $y^{(4)} - y = 0 \Rightarrow r^4 - 1 = 0 \Rightarrow (r^2 - 1)(r^2 + 1) = 0 \Rightarrow r = \pm 1$ and $r = \pm i \Rightarrow$ the general solution is $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$.
Alternatively, you can find the four solutions by considering $\sqrt[4]{1} = \sqrt[4]{1e^{0i}} = 1\, e^{\frac{2k\pi}{4} i} = e^{\frac{k\pi}{2} i}$ for $k = 0, 1, 2, 3$. Then $r_0 = 1, r_1 = i, r_2 = -1$ and $r_3 = -i$ yield the same general solution.

(d) $y^{(4)} - 5y'' - 36y = 0 \Rightarrow r^4 - 5r^2 - 36 = 0 \Rightarrow (r^2 - 9)(r^2 + 4) = 0 \Rightarrow r = \pm 3$ and $r = \pm 2i \Rightarrow$ the general solution is $y = c_1 e^{3x} + c_2 e^{-3x} + c_3 \cos 2x + c_4 \sin 2x$.

(e) $y^{(5)} - 32y = 0 \Rightarrow r^5 - 32 = 0 \Rightarrow r^5 = 32 = 32e^{0i} \Rightarrow r_k = \sqrt[5]{32} e^{\frac{0+2k\pi}{5} i} = 2e^{\frac{2k\pi}{5} i}$ for $k = 0, 1, \ldots, 4$. $r_0 = 2e^{0i} = 2$, $r_1 = 2e^{2\pi i/5} = 2(\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}) = 0.618 + 1.902i$, $r_2 = 2e^{4\pi i/5} = 2(\cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5}) = -1.618 + 1.176i$, $r_3 = 2e^{6\pi i/5} = 2(\cos \frac{6\pi}{5} + i \sin \frac{6\pi}{5}) = -1.618 - 1.176i$, $r_4 = 2e^{8\pi i/5} = 2(\cos \frac{8\pi}{5} + i \sin \frac{8\pi}{5}) = 0.618 - 1.902i$.
r_1 and r_4 are conjugated and r_2 and r_3 are conjugated. The general solution is $y = c_1 e^{2x} + c_2 e^{0.618x} \cos 1.902x + c_3 e^{0.618x} \sin 1.902x + c_4 e^{-1.618x} \cos 1.176x + c_5 e^{-1.618x} \sin 1.176x$.

(f) $y^{(5)} + 32y = 0 \Rightarrow r^5 + 32 = 0 \Rightarrow r^5 = -32 = 32e^{\pi i} \Rightarrow r_k = \sqrt[5]{32} e^{\frac{\pi+2k\pi}{5} i} = 2e^{\frac{(2k+1)\pi}{5} i}$ for $k = 0, 1, \ldots, 4$. $r_0 = 2e^{\pi i/5} = 2(\cos \frac{\pi}{5} + i \sin \frac{\pi}{5}) = 1.618 + 1.176i$, $r_1 = 2e^{3\pi i/5} = 2(\cos \frac{3\pi}{5} + i \sin \frac{3\pi}{5}) = -0.618 + 1.902i$, $r_2 = 2e^{\pi i} = 2(\cos \pi + i \sin \pi) = -2$, $r_3 = 2e^{7\pi i/5} = 2(\cos \frac{7\pi}{5} + i \sin \frac{7\pi}{5}) = -0.618 - 1.902i$, $r_4 = 2e^{9\pi i/5} = 2(\cos \frac{9\pi}{5} + i \sin \frac{9\pi}{5}) = 1.618 - 1.176i$.
r_0 and r_4 are conjugated and r_1 and r_3 are conjugated. The general solution is $y = c_1 e^{-2x} + c_2 e^{1.618x} \cos 1.176x + c_3 e^{1.618x} \sin 1.176x + c_4 e^{-0.618x} \cos 1.902x + c_5 e^{-0.618x} \sin 1.902x$.

2. Variation of Parameters. See more detailed solutions on the class handout.

(a) $y_h = c_1 e^{3x} + c_2 x e^{3x}$ and $y_p = \frac{1}{2} x^{-1} e^{3x}$. The general solution is $y = c_1 e^{3x} + c_2 x e^{3x} + \frac{1}{2} x^{-1} e^{3x}$.

(b) $y_h = c_1e^{2x} + c_2e^{3x}$ and $y_p = e^x$. The general solution is $y = c_1e^{2x} + c_2e^{3x} + e^x$.

(c) $y_h = c_1e^{-2x} + c_2xe^{-2x}$ and $y_p = -\ln x\ e^{-2x} - e^{-2x}$. The general solution is $y = (c_1 - 1)e^{-2x} + c_2xe^{-2x} - \ln x\ e^{-2x}$ which is the same as $y = c_1e^{-2x} + c_2xe^{-2x} - \ln x\ e^{-2x}$.

3. Undetermined Coefficients.

(a) The zeros of the characteristic equation are $r = 2$ and $r = 3$ and $y_h = c_1e^{2x} + c_2e^{3x}$. We have that $p = 2$ and $p_k(x) = 4$ so $y_p = x^sAe^{2x}$. Since 2 is a (single) zero of the characteristic equation, $s = 1$ and so $y_p = Axe^{2x}$. Find the derivatives $y_p' = Ae^{2x} + 2Axe^{2x}$ and $y_p'' = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x} = 4Ae^{2x} + 4Axe^{2x}$ and plug them and y_p the into the equation. Obtain that $4Ae^{2x} + 4Axe^{2x} - 5Ae^{2x} - 10Axe^{2x} + 6Axe^{2x} = 4e^{2x} \Rightarrow 4A + 4Ax - 5A - 10Ax + 6Ax = 4 \Rightarrow 4A - 5A = 4 \Rightarrow A = -4$. Thus, $y_p = -4xe^{2x}$ and the general solution is $y = c_1e^{2x} + c_2e^{3x} - 4xe^{2x}$.

(b) The zeros of the characteristic equation are $r = \pm 2i$ and so $y_h = c_1 \cos 2x + c_2 \sin 2x$. We have that $p = 1$ and $p_k(x) = 5x^2$ so $y_p = x^s(Ax^2 + Bx + C)e^x$. Since 1 is not a zero of the characteristic equation, $s = 0$ and so $y_p = (Ax^2 + Bx + C)e^x$. Find the derivatives $y_p' = (2Ax+B)e^x + (Ax^2+Bx+C)e^x = (2Ax+B+Ax^2+Bx+C)e^x$ and $y_p'' = (2A+2Ax+B)e^x + (2Ax+B+Ax^2+Bx+C)e^x$ and plug them and y_p the into the equation. Obtain that $(2A+2Ax+B)e^x + (2Ax+B+Ax^2+Bx+C)e^x + (4Ax^2+4Bx+4C)e^x = 5x^2e^x \Rightarrow 2A+2Ax+B+2Ax+B+Ax^2+Bx+C+4Ax^2+4Bx+4C = 5x^2$. Equating the terms with x^2, we obtain that $5A = 5$ so $A = 1$. Equating the terms with x, we obtain that $4A+5B = 0 \Rightarrow 5B = -4A = -4$ since $A = 1$. Hence, $B = -\frac{4}{5}$. Equating the terms with no x, we obtain that $= 2A+2B+5C = 0 \Rightarrow 5C = -2A-2B = -2+\frac{8}{5} = \frac{-2}{5}$ so $C = -\frac{2}{25}$. Thus, $y_p = (x^2 - \frac{4}{5}x - \frac{2}{25})e^x$ and the general solution is $y = c_1 \cos 2x + c_2 \sin 2x + (x^2 - \frac{4}{5}x - \frac{2}{25})e^x$.

(c) The characteristic equation is $r^2 - 2r + 1 = 0 \Rightarrow (r-1)(r-1) = 0$ so $r = 1$ is a double zero and $y_h = c_1e^x + c_2xe^x$. We have that $p = 1$ and $p_k(x) = 7x$ so $y_p = x^s(Ax+B)e^x$. Since 1 is a double zero of the characteristic equation $s = 2$ and so $y_p = x^2(Ax+B)e^x = (Ax^3+Bx^2)e^x$. Find the derivatives y_p' and y_p'' and plug them and y_p into the equation. Obtain that $A = \frac{7}{6}$, and $B = 0$ so $y_p = \frac{7}{6}x^3e^x$ and the general solution is $y = c_1e^x + c_2xe^x + \frac{7}{6}x^3e^x$.

(d) The zeros of the characteristic equation are $r = -3$ and $r = 1$ and so $y_h = c_1e^{-3x} + c_2e^x$. We have that $p + iq = 0 + 3i$ and $p_k(x) = 5$ so $y_p = x^sAe^{0x} \cos 3x + x^sBe^{0x} \sin 3x$. Since $3i$ is not a zero of the characteristic equation, $s = 0$ and so $y_p = A \cos 3x + B \sin 3x$. From here $y_p' = -3A \sin 3x + 3B \cos 3x$ and $y_p'' = -9A \cos 3x - 9B \sin 3x$ and the equation becomes $-9A \cos 3x - 9B \sin 3x - 6A \sin 3x + 6B \cos 3x - 3A \cos 3x - 3B \sin 3x = 5 \sin 3x$. Equating the terms with $\cos 3x$ and the terms with $\sin 3x$ obtain $-9A + 6B - 3A = 0$ and $-9B - 6A - 3B = 5$. From the first equation $B = 2A$ and from the second $-6A - 24A = 5 \Rightarrow -30A = 5 \Rightarrow A = -\frac{1}{6}$. Thus $B = -\frac{1}{3}$ and $y_p = -\frac{1}{6} \cos 3x - \frac{1}{3} \sin 3x$. The general solution is $y = c_1e^{-3x} + c_2e^x - \frac{1}{6} \cos 3x - \frac{1}{3} \sin 3x$.

(e) The roots of the characteristic equation $r^2 - 3r - 10 = (r-5)(r+2) = 0$ are 5 and -2 so the homogeneous solution is $y_h = c_1e^{5x} + c_2e^{-2x}$.

You have to consider functions $g_1(x) = 3xe^{2x}$ and $g_2(x) = 5e^{-2x}$ separately and obtain two separate particular solutions y_{p1} and y_{p2}.

For $g_1(x) = 3xe^{2x}$, $p = 2$ and $p_k(x) = 3x$ so $y_{p1} = x^s(Ax + B)e^{2x}$. Since 2 is not a solution of the characteristic equation, $s = 0$ and so $y_{p1} = (Ax + B)e^{2x}$.

For $g_2(x) = 5e^{-2x}$, $p = -2$ and $p_k(x) = 5$ so $y_{p2} = x^s Ce^{-2x}$. Since -2 is a (single) solution of the characteristic equation, $s = 1$ and so $y_{p2} = x^1 Ce^{-2x} = Cxe^{-2x}$.

The general solution has the form $y = c_1e^{5x} + c_2e^{-2x} + (Ax + B)e^{2x} + Cxe^{-2x}$.

(f) The characteristic equation is $r^2 - 8r + 16 = (r - 4)(r - 4) = 0$, so $r = 4$ is a double zero. The homogeneous solution is $y_h = c_1e^{4x} + c_2xe^{4x}$. Consider the functions $g_1(x) = 3x^2$ and $g_2(x) = -5e^{4x}$ separately and obtain two separate particular solutions y_{p1} and y_{p2}.

For $g_1(x) = 3x^2 = 3x^2e^{0x}$, $p = 0$ and $p_k(x) = 3x^2$ so $y_{p1} = x^s(Ax^2 + Bx + C)e^{0x}$. Since 0 is not a solution of the characteristic equation, $s = 0$ and so $y_{p1} = Ax^2 + Bx + C$.

For $g_2(x) = -5e^{4x}$, $p = 4$ and $p_k(x) = -5$ so $y_{p2} = x^s De^{4x}$. Since 4 is a double zero of the characteristic equation, $s = 2$, and so $y_{p2} = x^2De^{4x} = Dx^2e^{4x}$.

The general solution has the form $y = c_1e^{4x} + c_2xe^{4x} + Ax^2 + Bx + C + Dx^2e^{4x}$.

(g) The characteristic equation $r^2+4r+13 = 0$ has solutions $r = \frac{-4\pm\sqrt{16-52}}{2} = \frac{-4\pm 6i}{2} = -2\pm 3i$. So, the homogeneous solution is $y_h = c_1e^{-2x}\cos 3x + c_2e^{-2x}\sin 3x$.

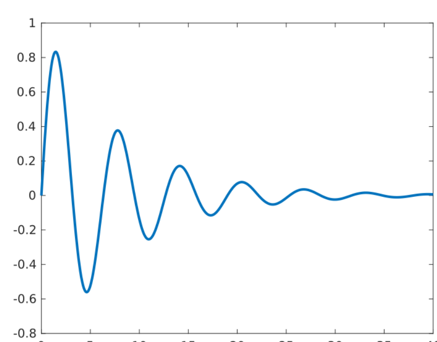
For $g_1(x) = -2\sin 3x = -2e^{0i}\sin 3x$, $p + iq = 0 + 3i$ and $p_k(x) = -2$ so $y_{p1} = x^sAe^{0x}\cos 3x + x^sBe^{0x}\sin 3x$. Since $0 + 3i$ is not a solution of the characteristic equation, $s = 0$ and so $y_{p1} = A\cos 3x + B\sin 3x$.

For $g_2(x) = e^{-2x}\cos 3x$, $p + iq = 0 + 3i$ and $p_k(x) = 1$ so $y_{p1} = x^sCe^{-2x}\cos 3x + x^sDe^{-2x}\sin 3x$. Since $-2 + 3i$ is a solution of the characteristic equation, $s = 1$ and so $y_{p2} = x(Ce^{-2x}\cos 3x + De^{-2x}\sin 3x) = Cxe^{-2x}\cos 3x + Dxe^{-2x}\sin 3x$.

The general solution has the form $y = c_1e^{-2x}\cos 3x + c_2e^{-2x}\sin 3x + A\cos 3x + B\sin 3x + Cxe^{-2x}\cos 3x + Dxe^{-2x}\sin 3x$.

4. Applications.

(a) (i) The characteristic equation is $r^2 + \frac{1}{4}r + 1 = 0$ and it has solutions $\frac{-\frac{1}{4}\pm\sqrt{\frac{1}{16}-4}}{2} = \frac{-1}{8} \pm \frac{\sqrt{\frac{-63}{16}}}{2} = \frac{-1}{8} \pm \frac{\sqrt{63}i}{8} - .125 \pm .992i$. Thus, the general solution is $u = c_1e^{-.125t}\cos .992t + c_2e^{-.125t}\sin .992t$. Since the object is set in motion from the equilibrium position, $u(0) = 0$. Since it is set in motion with an initial velocity of 1 m/s, $u'(0) = 1$. Use the condition $u(0) = 0$ (plug 0 for t and set u equal to 0), we have that $0 = c_1(1) + c_2(0) = c_1$. To use the condition $u'(0) = 1$, find u' first, then set it to 1 and plug 0 for t. As $u' = -.125c_1e^{-.125t}\cos .992t - .992c_1e^{-.125t}\sin .992t - .125c_2e^{-.125t}\sin .992t + .992c_2e^{-.125t}\cos .992t$ and $c_1 = 0$, we have that $1 = -.125(0)(1) - .992(0)(0) - .125c_2(0) + .992c_2(1) = .992c_2 \Rightarrow c_2 = \frac{1}{.992} = 1.008$. Hence, $u = 1.008e^{-.125t}\sin .992t$



(ii) The presence of sine and cosine in the solution means that the motion is not overdamped so there are oscillations. Since $e^{-.125t} \to 0$ for $t \to \infty$ and $e^{-.125t}$ is present in both terms, u converges to 0 meaning that the oscillations have a *decreasing amplitude*. Thus, this is an underdamped free oscillator.

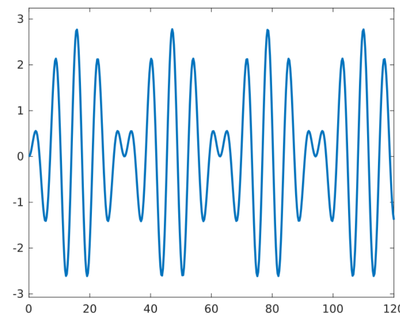
(b) (i) The general equation of motion is $mu'' + \gamma u' + ku = F(t)$. We are given that $m = 1, \gamma = 0, L = 9.8$, and $F(t) = \frac{1}{2}\cos 0.8t$. Compute k using the formula $k = \frac{mg}{L}$. Thus,

$k = \frac{1(9.8)}{9.8} = 1$. Hence the equation $u'' + u = \frac{1}{2}\cos 0.8t$ models this motion. (ii) The characteristic equation is $r^2 + 1 = 0$ and has solutions $r = \pm i$. Hence, the homogeneous solution is $u_h = c_1 \cos t + c_2 \sin t$. For $F(t) = \frac{1}{2}\cos 0.8t$, $p + iq = 0 + 0.8i$, $p_k(t) = \frac{1}{2}$ and so $u_p = t^s(A\cos 0.8t + B\sin 0.8t)$. Since $0.8i$ is not a solution of the characteristic equation, $s = 0$ and $u_p = A\cos 0.8t + B\sin 0.8t$. Find $u_p' = -.8A\sin 0.8t + .8B\cos 0.8t$ and $u_p'' = -.64A\cos 0.8t - .64B\sin 0.8t$ and plug them in the equation to have $-.64A\cos 0.8t - .64B\sin 0.8t + A\cos 0.8t + B\sin 0.8t = \frac{1}{2}\cos 0.8t \Rightarrow .36A\cos 0.8t + .36B\sin 0.8t = \frac{1}{2}\cos 0.8t$. Equating the terms with cosine, $.36A = \frac{1}{2} \Rightarrow A = \frac{25}{18} \approx 1.39$. Equating the terms with sine, $B = 0$. Thus $u = c_1\cos t + c_2 \sin t + \frac{25}{18}\cos 0.8t$.

The initial conditions are $u(0) = 1$ and $u'(0) = 0$. Using the first condition, we have that $0 = c_1(1) + c_2(0) + \frac{25}{18}(1) = c_1 + \frac{25}{18} \Rightarrow c_1 = -\frac{25}{18}$. As $u' = -c_1 \sin t + c_2 \cos t - 0.8\,\frac{25}{18}\sin 0.8t$, $u'(0) = 0$,

and $c_1 = -\frac{25}{18}$, we have that $0 = -c_1(0) + c_2(1) - 0.8\,\frac{25}{18}(0) = c_2 \Rightarrow c_2 = 0$. Hence, $u = -\frac{25}{18}\cos t + \frac{25}{18}\cos 0.8t$.

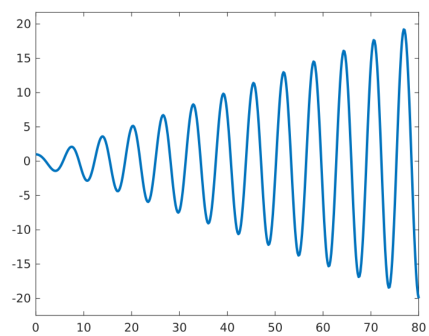
(iii) The presence of trigonometric functions indicate oscillations. The presence of two different frequencies 1 and 0.8 indicate oscillates with a *periodic amplitude*. So, the oscillations are with *beats*.



(c) (i) The equation is $u'' + u = \frac{1}{2}\cos t$. (ii) The homogeneous solution is $u_h = c_1 \cos t + c_2 \sin t$. For $F(t) = \frac{1}{2}\cos t$, $p + qi = 0 + 1i$, $p_k(t) = \frac{1}{2}$ and so $u_p = t^s(A\cos t + B\sin t)$. Since i is a solution of the characteristic equation, $s = 1$ and $u_p = At\cos t + Bt\sin t$. Find $u_p' = A\cos t - At\sin t + B\sin t + Bt\cos t$ and $u_p'' = -A\sin t - A\sin t - At\cos t + B\cos t + B\cos t - Bt\sin t$ and plug them in the equation to have $-A\sin t - A\sin t - At\cos t + B\cos t + B\cos t - Bt\sin t + At\cos t + Bt\sin t = \frac{1}{2}\cos t \Rightarrow -2A\sin t + 2B\cos t = \frac{1}{2}\cos t \Rightarrow -2A = 0$ and $2B = \frac{1}{2} \Rightarrow A = 0$ and $B = \frac{1}{4}$. Thus, $u = c_1\cos t + c_2\sin t + \frac{1}{4}t\sin t$. Since the object is set in motion at 1 meter from the equilibrium position, $u(0) = 1$. Since it is set in motion from rest, $u'(0) = 0$. Using $u(0) = 1$, $1 = c_1(1) + c_2(0) + \frac{1}{4}(0) = c_1 \Rightarrow c_1 = 1$.

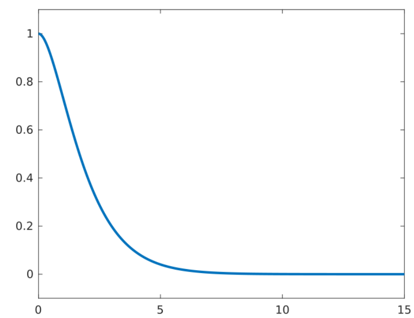
Find the derivative $u' = -c_1\sin t + c_2\cos t + \frac{1}{4}\sin t + \frac{1}{4}t\cos t$ and use the condition $u'(0) = 0$. Thus $0 = -c_1(0) + c_2(1) + 0 - 0 \Rightarrow c_2 = 0$. Hence, $u = \cos t + \frac{1}{4}t\sin t$.

(iii) The presence of trigonometric functions indicate oscillations. The presence of t in front of the sine function indicates an *increasing amplitude*. So, the oscillations are with a *resonance*.



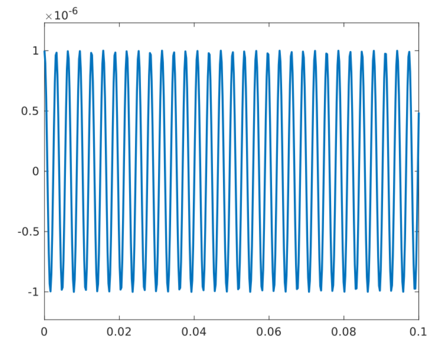
(d) The characteristic equation is $r^2 + 2r + 1 = 0 \Rightarrow (r + 1)(r + 1) = 0$ so -1 is a double zero and the general solution is $u = c_1 e^{-t} + c_2 te^{-t}$. Since the mass is set in motion from 1 meter from the equilibrium, $u(0) = 1$. Since the mass is set from resting, the initial velocity is zero and so $u'(0) = 0$. Using $u(0) = 1$, obtain $1 = c_1(1) + c_2(0) \Rightarrow c_1 = 1$.

As $u' = -c_1e^{-t} + c_2e^{-t} - c_2te^{-t}$, $u'(0) = 0$, and $c_1 = 1$, $0 = -1 + c_2(1) - c_2(0) = -1 + c_2 \Rightarrow c_2 = 1$. Thus, $u = e^{-t} + te^{-t}$. The absence of trigonometric functions indicates that there are *no oscillations*. Hence, this is the overdamped case. The mass returns to equilibrium position without oscillations.



(e) Note that $R = 0$, $\frac{1}{C} = 4 \cdot 10^6$, and there is no applied voltage so $E(t) = 0$. Thus, the general circuit equation $LQ'' + RQ' + \frac{1}{C}Q = E(t)$ becomes $Q'' + 4 \cdot 10^6 Q = 0$. The characteristic equation $r^2 + 4 \cdot 10^6 = 0$ has solutions $r = \pm 2000i$ and so the general solution is $Q = c_1 \cos 2000t + c_2 \sin 2000t$.

The initial conditions are $Q(0) = 10^{-6}$ and $Q'(0) = 0$. The first implies $10^6 = c_1(1) + c_2(0) = c_1$. As $Q' = -2000c_1 \sin 2000t + 2000c_2 \cos 2000t$, the second implies $0 = -2000c_1(0) + 2000c_2(1) = 2000c_2 \Rightarrow c_2 = 0$. Thus, $Q = 10^{-6} \cos 2000t$. This is an undamped free oscillator and the solution is a periodic function with a *constant amplitude*.



(f) The solutions are not overdamped if the characteristic equation has complex solutions (since just in this case the solution has periodic functions present). The characteristic equation is $r^2 + \gamma r + 9 = 0$. The solutions are $r = \frac{-\gamma \pm \sqrt{\gamma^2 - 36}}{2}$. Thus, the complex solutions are present just if the expression under the root is negative. So, $\gamma^2 - 36 < 0 \Rightarrow (\gamma - 6)(\gamma + 6) < 0$. This inequality has the solution $-6 < \gamma < 6$. In addition, since γ is nonnegative, this corresponds to the interval $0 \leq \gamma < 6$.