Differential Equations
Lia Vas

The Third Exam Review

1. Use the definition of the Laplace transform to show that $\mathcal{L}[t] = \frac{1}{s^2}$.

2. Find the Laplace transform of the following functions.

   (a) $t^4 e^{-2t} + \cos 5t - 7$ (b) $\int_0^t \tau^3\, e^{t-\tau}\, d\tau$ (c) $\int_0^t \sin(2\tau)\, \cos(2t - 2\tau)\, d\tau$

3. Find the inverse Laplace transform of the following functions.

   (a) $\frac{5}{s^2+4}$, (b) $\frac{s^2}{(s+1)^3}$, (c) $\frac{10}{s^2+3s-4}$, (d) $\frac{s+4}{s^2+2s+5}$, (e) $\frac{5s^2+3s-2}{s^3+2s^2}$ (f) $\frac{3s^2-4s+5}{(s-1)(s^2+1)}$

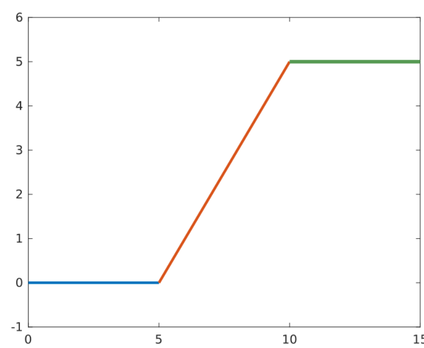
4. Use the Laplace transform to solve the equation $y'' - 6y' + 5y = 2$, $y(0) = 0$, $y'(0) = -1$.

5. Assume that an undamped harmonic oscillator is described by the following differential equation where y is in cm and t is in seconds. Find the solution, write your answer as a piecewise function, sketch its graph and describe the motion.

   (a) $y'' + y = \begin{cases} 1, & 5 \leq t < 20, \\ 0, & t < 5 \text{ and } t \geq 20. \end{cases}$ $y(0) = 0$, $y'(0) = 0$.

   (b) $y'' + 4y = \begin{cases} 0, & t < 5, \\ t-5, & 5 \leq t < 10, \\ 5, & t \geq 10. \end{cases}$ $y(0) = 0$, $y'(0) = 0$

   The function on the right side of the equation is known as **ramp loading** and can be represented as $u_5(t)(t-5) - u_{10}(t)(t-10)$.



   (c) $y'' + 4y = \delta(t - 4\pi)$, $y(0) = 1$, $y'(0) = 0$.

   (d) $y'' + 3y' + 4y = \delta(t - 3)$, $y(0) = 0$, $y'(0) = 0$.

6. Solve the following equations.

   (a) $y(t) + \int_0^t (t-\tau)\, y(\tau)\, d\tau = t$ (b) $y'(t) + \int_0^t y(t-\tau)\, e^{-2\tau}\, d\tau = 1$, $y(0) = 1$

7. Solve the following systems.

   (a) $x' = -x + y \;\; y' = -x - y$, $x(0) = 1$, $y(0) = 2$

   (b) $x' = -x - 3y \;\; y' = -x + y$, $x(0) = 1$, $y(0) = 0$

Solutions

1. $\mathcal{L}[t] = \int_0^\infty te^{-st}dt$ To evaluate this integral, use the integration by parts with $u = t$ and $dv = e^{-st}$. We have $\frac{-t}{s}e^{-st} - \frac{1}{s^2}e^{-st}|_0^\infty$. The limit of the first term for $t \to \infty$ is 0 (you may use L'Hopital's rule to see that). The limit of the second terms is also zero for $t \to \infty$ since $e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0$. At $t = 0$, the antiderivative is $\frac{-1}{s^2}$. So, $\mathcal{L}[t] = 0 - \frac{-1}{s^2} = \frac{1}{s^2}$.

2. (a) $\frac{24}{(s+2)^5} + \frac{s}{s^2+25} - \frac{7}{s}$ (b) The function is the convolution of t^3 and e^t. Thus the Laplace transform is $\mathcal{L}[t^3]\mathcal{L}[e^t] = \frac{6}{s^4}\frac{1}{s-1} = \frac{6}{s^4(s-1)}$. (c) The function is the convolution of $\sin 2t$ and $\cos 2t$. Thus the Laplace transform is $\mathcal{L}[\sin 2t]\mathcal{L}[\cos 2t] = \frac{2}{s^2+4}\frac{s}{s^2+4} = \frac{2s}{(s^2+4)^2}$.

3. (a) $\mathcal{L}^{-1}[\frac{5}{s^2+4}] = \frac{5}{2}\mathcal{L}^{-1}[\frac{2}{s^2+4}] = \frac{5}{2}\sin 2t$ (b) $\mathcal{L}^{-1}\left[\frac{s^2}{(s+1)^3}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+1} + \frac{-2}{(s+1)^2} + \frac{1}{(s+1)^3}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{(s+1)^3}\right] = e^{-t} - 2te^{-t} + \frac{1}{2}t^2e^{-t}$

   (c) Since s^2+3s-4 factors as $(s+4)(s-1)$, in order to find the Laplace transform, we need to find the partial fractions $\frac{A}{s+4}+\frac{B}{s-1}$. We obtain $A=-2$, $B=2$. So, $\mathcal{L}^{-1}[\frac{-2}{s+4}+\frac{2}{s-1}] = -2e^{-4t}+2e^t$.

   (d) s^2+2s+5 cannot be factored in a product of two linear real terms, so you need to write it as sum of squares as $s^2+2s+5 = (s+2s+1)+4 = (s+1)^2+2^2$. Then $\frac{s+4}{s^2+2s+5} = \frac{s+1+3}{(s+1)^2+2^2} = \frac{s+1}{(s+1)^2+2^2} + \frac{3}{2}\frac{2}{(s+1)^2+2^2}$. Hence the inverse Laplace is $e^{-t}\cos 2t + \frac{3}{2}e^{-t}\sin 2t$.

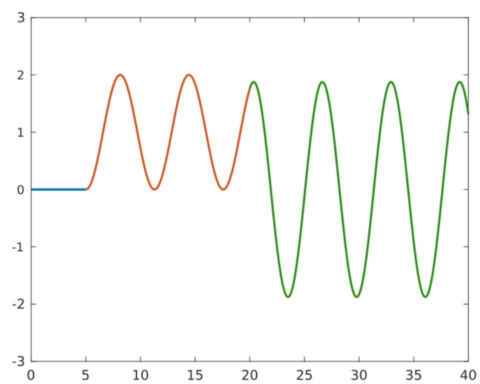
   (e) Using the partial fractions, $\frac{5s^2+3s-2}{s^3+2s^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} = \frac{2}{s} - \frac{1}{s^2} + \frac{3}{s+2}$. $\mathcal{L}^{-1}$ is $2 - t + 3e^{-2t}$.

   (f) Using the partial fractions, $\frac{3s^2-4s+5}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1} = \frac{2}{s-1} + \frac{s-3}{s^2+1}$ $\mathcal{L}^{-1}$ is $2e^t + \cos t - 3\sin t$.

4. The Laplace transform of the equation is $s^2Y + 1 - 6sY + 5Y = \frac{2}{s} \Rightarrow Y(s^2-6s+5) = \frac{2}{s} - 1 \Rightarrow Y(s-1)(s-5) = \frac{2-s}{s} \Rightarrow Y = \frac{2-s}{s(s-5)(s-1)}$. The partial fraction decomposition is $Y = \frac{2}{5s} - \frac{3}{20(s-5)} - \frac{1}{4(s-1)}$. Thus $y = \frac{2}{5} - \frac{3}{20}e^{5t} - \frac{1}{4}e^t$.

5. (a) The function on the right side is a boxcar function given by $u_5(t) - u_{20}(t)$. Taking the Laplace transform of the equation with $\mathcal{L}[y] = Y$, we obtain $s^2Y + Y = \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}$. From here $Y = (e^{-5s} - e^{-20s})\frac{1}{s(s^2+1)}$. Let $F(s) = \frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$. Find the coefficients to be $A=1, B=-1$, and $C=0$ so that $F(s) = \frac{1}{s} - \frac{s}{s^2+1} \Rightarrow f(t) = \mathcal{L}^{-1}[F(s)] = 1 - \cos t$.

   Thus, the solution is $y = \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}[(e^{-5s}F(s) - e^{-20s}F(s)] = u_5(t)f(t-5) - u_{20}(t)f(t-20) = u_5(t)(1-\cos(t-5)) - u_{20}(t)(1-\cos(t-20))$. See the handout for more details on getting the piecewise representation and the graph.



$$y = \begin{cases} 0, & t < 5 \\ 1-\cos(t-5), & 5 \le t < 20, \\ -\cos(t-5)+\cos(t-20), & t \ge 20. \end{cases}$$

   (b) The Laplace transform of the equation is $s^2Y + 4Y = e^{-5s}\frac{1}{s^2} - e^{-10s}\frac{1}{s^2}$. Thus $Y = (e^{-5s} - e^{-10s})\frac{1}{s^2(s^2+4)}$. Let $F(s) = \frac{1}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}$. Determine that $A = C = 0$, $B = \frac{1}{4}$, and $D = \frac{-1}{4}$. So, $F(s) = \frac{1/4}{s^2} - \frac{1/4}{s^2+4}$ and $f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{4}t - \frac{1}{8}\sin 2t$. Thus,

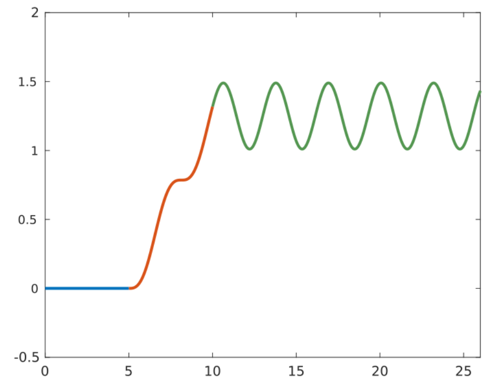
$$y = \mathcal{L}^{-1}[Y] = u_5(t)f(t-5) - u_{10}(t)f(t-10) =$$

$$u_5(t)(\frac{1}{4}(t-5) - \frac{1}{8}\sin 2(t-5)) - u_{10}(t)(\frac{1}{4}(t-10) - \frac{1}{8}\sin 2(t-10)).$$

   See the handout for more details on getting the piecewise representation and the graph.

$$y = \begin{cases} 0, & 0 \le t < 5, \\ \frac{1}{4}(t-5) - \frac{1}{8}\sin 2(t-5), & 5 \le t < 10, \\ \frac{5}{4} - \frac{1}{8}\sin 2(t-5) + \frac{1}{8}\sin 2(t-10), & t \ge 10. \end{cases}$$

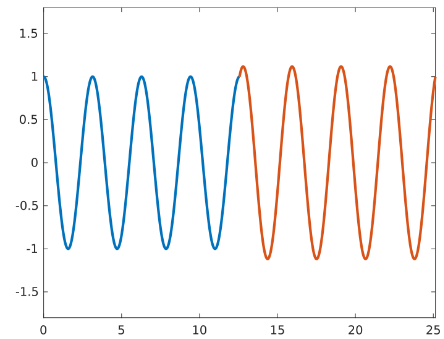
Thus, there are no oscillations before 5. Between 5 and 10 seconds, the mass oscillates about the line $\frac{1}{4}(t-5)$. After 10 seconds, the mass oscillates about $\frac{5}{4}$ with a constant amplitude.



(c) The Laplace transform of the equation is

$s^2Y - s + 4Y = e^{-4\pi s}$. Thus $Y = \frac{e^{-4\pi}+s}{s^2+4}$. Then $y = u_{4\pi}(t)\frac{1}{2}\sin 2(t-4\pi) + \cos 2t =$

$$\begin{cases} \cos 2t, & t < 4\pi, \\ \cos 2t + \frac{1}{2}\sin 2(t-4\pi) & t \ge 4\pi. \end{cases}$$

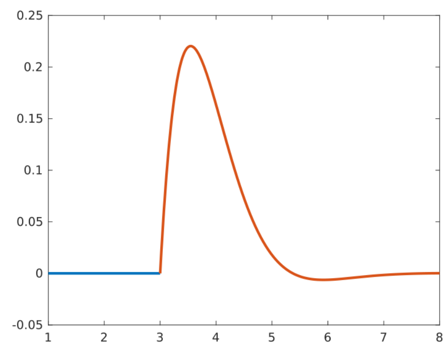


(d) Let $Y = \mathcal{L}[y]$. Applying the Laplace transform to the equation $y'' + 3y' + 4y = \delta(t-3)$ with $y(0) = y'(0) = 0$ produces $s^2Y + 3sY + 4Y = e^{-3s}$. From here $Y = \frac{e^{-3s}}{s^2+3s+4}$. Complete the denominator of $F(s) = \frac{1}{s^2+3s+4}$ to a sum of squares. $s^2 + 3s + 4 = s^2 + 2s(\frac{3}{2}) + \frac{9}{4} + 4 - \frac{9}{4} = (s+\frac{3}{2})^2 + \frac{7}{4}$. Thus $f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}[\frac{1}{(s+\frac{3}{2})^2+\frac{7}{4}}] = \frac{2}{\sqrt{7}}\mathcal{L}^{-1}[\frac{\frac{\sqrt{7}}{2}}{(s+\frac{3}{2})^2+\frac{7}{4}}] = \frac{2}{\sqrt{7}}e^{-3t/2}\sin\frac{\sqrt{7}}{2}t$.

$y = \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}[\frac{e^{-3s}}{s^2+3s+4}] = \mathcal{L}^{-1}[e^{-3s}F(s)] = u_3(t)\, f(t-3) = u_3(t)\,\frac{2}{\sqrt{7}}e^{-3(t-3)/2}\sin\frac{\sqrt{7}}{2}(t-3) =$

$$\begin{cases} 0, & t < 3, \\ \frac{2}{\sqrt{7}}e^{-3(t-3)/2}\sin\frac{\sqrt{7}}{2}(t-3), & t \ge 3. \end{cases}$$

Thus, the object starts oscillating only after 3 seconds. It oscillates with a decreasing amplitude given by $\frac{2}{\sqrt{7}}e^{-3(t-3)/2}$ converging to 0 and the oscillations become negligible in time.



6. (a) The equation is $y + t * y = t$. Taking $\mathcal{L}$, obtain $Y + \frac{1}{s^2}Y = \frac{1}{s^2} \Rightarrow Y = \frac{1}{s^2+1} \Rightarrow y = \sin t$.

(b) The equation is $y' + y * e^{-2t} = 1$. Thus $sY - 1 + Y\frac{1}{s+2} = \frac{1}{s} \Rightarrow Y(s + \frac{1}{s+2}) = \frac{1}{s} + 1 \Rightarrow Y\frac{s(s+2)+1}{s+2} = \frac{1+s}{s} \Rightarrow Y = \frac{(1+s)(s+2)}{s(s^2+2s+1)} = \frac{(1+s)(s+2)}{s(s+1)^2} = \frac{s+2}{s(s+1)}$. Find the partial fraction decomposition to be $Y = \frac{2}{s} - \frac{1}{s+1} \Rightarrow y = 2 - e^{-t}$.

7. (a) Let $X = \mathcal{L}[x]$ and $Y = \mathcal{L}[y]$. Taking $\mathcal{L}$ of both equations produces $sX - 1 = -X + Y$ and $sY - 2 = -X - Y$. From the first equation $Y = sX + X - 1$. Plugging that in the second gives

you $s(sX+X-1)-2=-X-(sX+X-1) \Rightarrow s^2X+2sX+2X=s+3 \Rightarrow X=\frac{s+3}{s^2+2s+2}$. Thus $Y=\frac{s^2+3s+s+3-s^2-2s-2}{s^2+2s+2}=\frac{2s+1}{s^2+2s+2}$.

Then $x=\mathcal{L}^{-1}[X]=\mathcal{L}^{-1}[\frac{s+1+2}{(s+1)^2+1}]=\mathcal{L}^{-1}[\frac{s+1}{(s+1)^2+1}+2\frac{1}{(s+1)^2+1}]=e^{-t}\cos t+2e^{-t}\sin t$ and $y=\mathcal{L}^{-1}[Y]=\mathcal{L}^{-1}[\frac{2s+1}{s^2+2s+2}]=\mathcal{L}^{-1}[\frac{2s+2-1}{(s+1)^2+1}]=\mathcal{L}^{-1}[2\frac{s+1}{(s+1)^2+1}-\frac{1}{(s+1)^2+1}]=2e^{-t}\cos t-e^{-t}\sin t$.

(b) If $X=\mathcal{L}[x]$ and $Y=\mathcal{L}[y]$, taking $\mathcal{L}$ of both equations produces $sX-1=-X-3Y$ and $sY=-X+Y$. From the second equation, $X=Y-sY$. Substitute that in the first equation. $s(Y-sY)-1=-(Y-sY)-3Y \Rightarrow sY-s^2Y-1=sY-4Y \Rightarrow -1=(s^2-4)Y \Rightarrow Y=\frac{-1}{s^2-4}$. Thus $X=Y-sY=\frac{s-1}{s^2-4}$. The partial fractions decomposition produces $Y=\frac{-1/4}{s-2}+\frac{1/4}{s+2}$ and $X=\frac{1/4}{s-2}+\frac{3/4}{s+2}$. Thus, $x=\mathcal{L}^{-1}[X]=\frac{1}{4}e^{2t}+\frac{3}{4}e^{-2t}$ and $y=\mathcal{L}^{-1}[Y]=\frac{-1}{4}e^{2t}+\frac{1}{4}e^{-2t}$.