Differential Equations
Lia Vas

The Final Exam Review

Review of Systems of ODE

1. Find all the equilibrium points of the following systems.

   (a) $\frac{dx}{dt} = x - x^2 - xy \qquad \frac{dy}{dt} = 0.75y - y^2 - 0.5xy$

   (b) $\frac{dx}{dt} = x - x^2 - xy \qquad \frac{dy}{dt} = 0.5y - 0.25y^2 - 0.75xy$

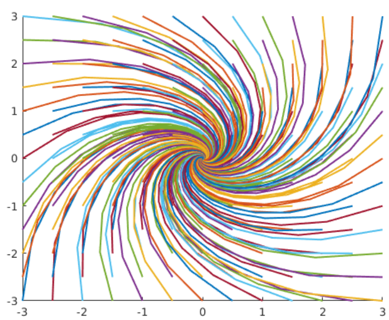
2. The point (0,0) is the only equilibrium point of the following systems. The graph on the left represents the trajectories in the phase plane and the graph on the right represents the graph of solutions with initial conditions $x(0) = 1$ and $y(0) = 3$. For each system, do the following.

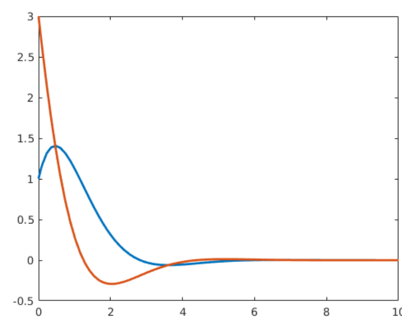
   i) Use the graphs to classify the equilibrium point (0,0) and determine its stability.

   ii) Determine the limiting values of the general solutions x and y for $t \to \infty$.

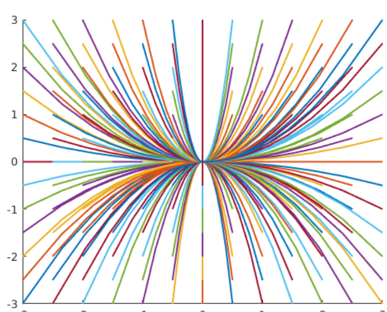
   iii) Determine the direction of the trajectories in the phase plane.

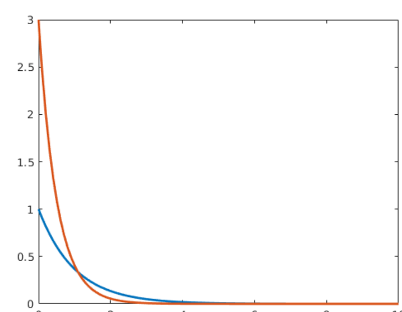
   (a) $\frac{dx}{dt} = -x + y$ and $\frac{dy}{dt} = -x - y$



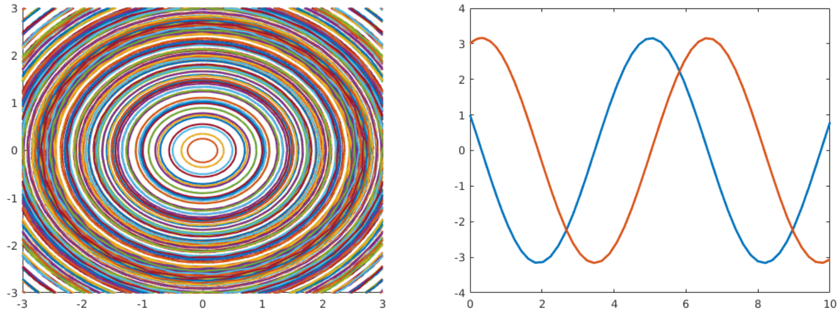


   (b) $\frac{dx}{dt} = -x$ and $\frac{dy}{dt} = -2y$

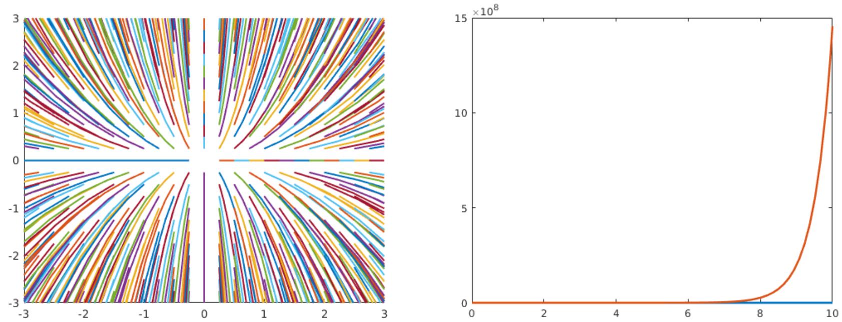




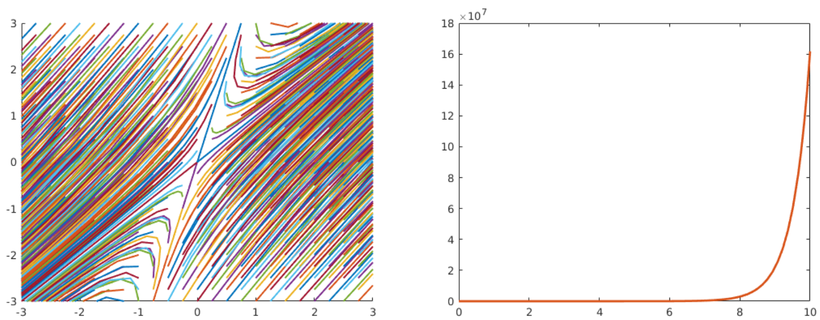
   (c) $\frac{dx}{dt} = -y$ and $\frac{dy}{dt} = x$



(d) $\frac{dx}{dt} = x$ and $\frac{dy}{dt} = 2y$



(e) $\frac{dx}{dt} = 3x - y$ and $\frac{dy}{dt} = 4x - 2y$.



3. Find the solution of the following systems using the Laplace Transform. Compare your answers with graphs and your conclusions for 2(a) and 2(e).

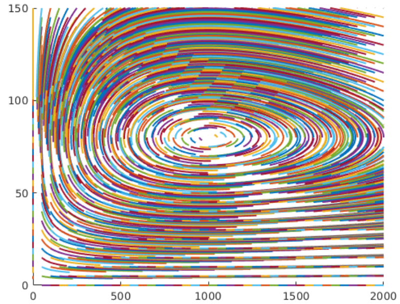
   (a) $\frac{dx}{dt} = -x + y \quad \frac{dy}{dt} = -x - y, \quad x(0) = 1, \; y(0) = 3.$

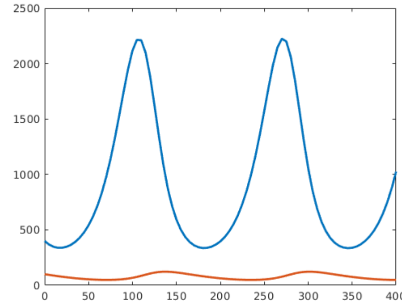
   (b) $\frac{dx}{dt} = 3x - y \quad \frac{dy}{dt} = 4x - 2y, \quad x(0) = 1, \; y(0) = 3.$

4. A population of rabbits R and wolves W is described by the predator-prey model $\frac{dx}{dt} = 0.08x - 0.001xy, \quad \frac{dy}{dt} = -0.02y + 0.00002xy.$

   (a) Find the equilibrium points.

   (b) Consider the following graphs of trajectories in the phase plane and the solutions with initial conditions $R(0) = 400$ and $W(0) = 100$ to classify the equilibrium points and determine their stability. Indicate the direction in which the curves in the phase place are traced as the parameter increases and discuss the long term tendencies of the system.





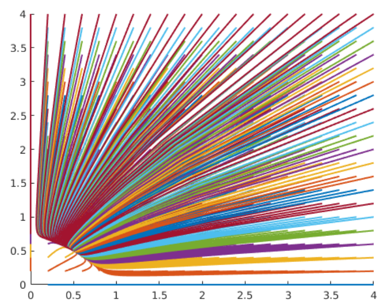
(c) Using the second graph, estimate the values in between which the solutions oscillate.

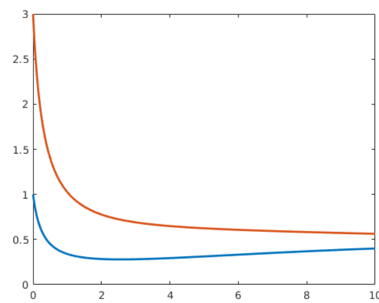
5. Suppose that there are two competing species in a closed environment. Let x and y denote the sizes of two populations at time t measured in thousands.

   (a) Assume that the rate of change of the populations is governed by the equations.

$$\frac{dx}{dt} = x - x^2 - xy \qquad \frac{dy}{dt} = 0.75y - y^2 - 0.5xy$$

   This system has phase plane given by the first graph below. The graph of the solutions with initial conditions $x(0) = 1$ and $y(0) = 3$ is given on the right.



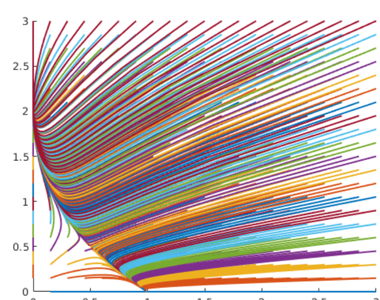


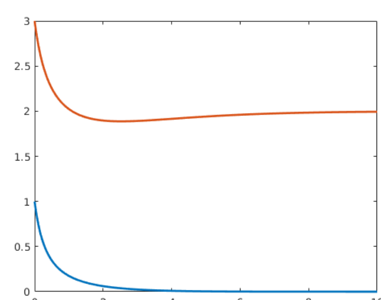
   Find the equilibrium points of the system and discuss their stability. Discuss the long term behavior and provide biological interpretation.

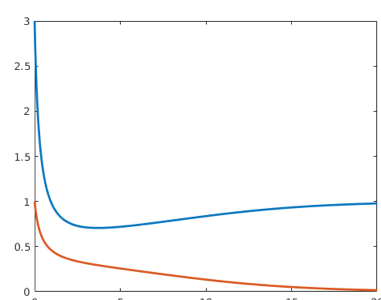
   (b) Assume that the changes in environment cause the coefficients in the second equation to change. The modified system becomes

$$\frac{dx}{dt} = x - x^2 - xy \qquad \frac{dy}{dt} = 0.5y - 0.25y^2 - 0.75xy$$

   This system has phase plane given by the first graph below. The graph of the solutions with initial conditions $x(0) = 1$ and $y(0) = 3$ is in the middle given and the graph of the solutions with initial conditions $x(0) = 3$ and $y(0) = 1$ is on the right.







   Find the equilibrium points of the system and discuss their stability as well as their *type*. Discuss the long term behavior and provide biological interpretation.

Review of the First Exam Material

1. **Checking if a given function is a solution.**

   (a) Check if $y = x^2$ and $y = 2 + e^{-x^3}$ are solutions of differential equation $y' + 3x^2 y = 6x^2$.

   (b) Determine all values of r for which $6y'' - 7y' - 3y = 0$ has a solution of the form $y = e^{rt}$.

   (c) Find value of constant A for which the function $y = Ae^{3x}$ is the solution of the equation $y'' - 3y' + 2y = 6e^{3x}$.

2. **Solving the first order equations.** Solve the following differential equations.

   (a) $y' - 2y = x$

   (b) $y' = y^2 x e^{2x}$

   (c) $x^3 y^4 + (x^4 y^3 + 2y) y' = 0$

   (d) $x^2 y' + 2xy = y^3$

   (e) $y' = \frac{x^2+xy+y^2}{x^2}$ (Hint: rewrite right side as $1 + \frac{y}{x} + (\frac{y}{x})^2$)

   (f) $xy' + 2y = x^3$

3. **Autonomous equations.** Find equilibrium solutions and determine their stability. Sketch the graph of solutions of the following equations.

   (a) $y' = y(2-y)^2(5-y)^3$

   (b) $y' = (y-a)(y-b)$

4. **Modeling with differential equations.**

   (a) A population of field mice inhabits a certain rural area. In the absence of predators, the mice population increases so that each month, the population increases by 50%. However, several owls live in the same area and they kill 15 mice per day. Find an equation describing the population size and use it to predict the long term behavior of the population. Find the general solution.

   (b) A glucose solution is administered intravenously into the bloodstream at a constant rate r. As the glucose is added, it is converted into other substances and removed from the bloodstream at a rate proportional to the amount present at that time.

   (a) Set up a differential equation that models this situation.

   (b) If $r = 4$ and the proportionality constant is 2, find the equilibrium solution, examine the stability and determine the amount of the glucose present after a long period of time. Sketch the graphs of general solutions.

   (c) If 1 mg is present initially, find the formula describing the amount present after t hours.

   (c) Suppose that a 10-kg object is *dropped* from the initial height. Assume that the drag is proportional to the velocity with the drag coefficient of 2 kg/sec. Formulate a differential equation describing the velocity of the object. Find the limiting velocity by analyzing the equilibrium solutions. Then find the formula describing the velocity of the object.

5. **Exact equations with parameters.** Find the value of parameter for which the equation $ay^2 e^{3x} + 2x^2 y + (4ye^{3x} + \frac{2}{3}x^3 + 12e^{4y})y' = 0$ is exact and solve it using those parameter value.

Review of the Second Exam Material

1. **Homogeneous equations with constant coefficients.** Solve the following equations.

   (a) $y''' - 2y'' + y' = 0$

   (b) $y^{(4)} - y = 0$

   (c) $y^{(4)} - 5y'' - 36y = 0$

   (d) $y^{(5)} - 32y = 0$.

2. **Non-homogeneous equations with constant coefficients. Variation of Parameters.** Solve the following differential equations. Note that they cannot be solved using Undetermined Coefficients method.

   (a) $y'' - 6y' + 9y = x^{-3}e^{3x}$.

   (b) $y'' + 4y' + 4y = x^{-2}e^{-2x}$

3. **Non-homogeneous equations with constant coefficients. Undetermined Coefficients.** Find general solution of problems (a)–(c). In problems (d) and (e), find the *form* of particular solutions and the general solutions. For (d) and (e), you **do not** have to solve for unknown coefficients in particular solutions.

   (a) $y'' - 5y' + 6y = 4e^{2x}$

   (b) $y'' + 4y = 5x^2e^x$

   (c) $y'' - 2y' + y = 7xe^x$

   (d) $y'' - 3y' - 10y = 3xe^{2x} + 5e^{-2x}$

   (e) $y'' + 4y' + 13y = -2\sin 3x + e^{-2x}\cos 3x$

4. **Applications of higher order differential equations.**

   (a) Consider a motion of an object modeled by the equation $u'' + \frac{1}{4}u' + u = 0$ where the position u (in meters) is a function of time (in seconds). Assume that the object is set in motion from equilibrium with an initial velocity of 1 meter per second. (i) Determine the position u as a function of time. (ii) Graph the solution and classify the type of motion the graph displays by noting what happens with the amplitude of the solution.

   (b) Consider a motion of a 1-kg mass which stretches a spring by 9.8 meters. Use the value of 9.8 m/sec^2 for g. Assume that there is no damping and that the mass is acted on by an external force of $\frac{1}{2}\cos 0.8t$ newtons. (i) Write down an equation which models the motion. (ii) Assume that the mass is set in motion from resting at its equilibrium position. Determine the position u as a function of time t. (iii) Graph the solution, and classify the type of motion the graph displays by noting what happens with the amplitude of the solution.

   (c) Assume that the force in the previous problem changes to $\frac{1}{2}\cos t$ newtons and that the initial conditions are determined by assuming that the mass is set in motion by pulling it 1 meter from the equilibrium position and then releasing it from rest. Do the parts (i) to (iii) of the previous problem in this case.

   (d) A series circuit has capacitor of $C = 0.25 \cdot 10^{-6}$ farad and inductor of $L = 1$ henry. If the initial charge on the capacitor is 10^{-6} coulomb and there is no initial current, find the charge Q as a function of t. Graph the solution and classify the type of motion the graph displays by noting what happens with the amplitude of the solution.

   (e) Determine the values of γ for which the equation $u'' + \gamma u' + 9u = 0$ has solutions which are not overdamped.

Review of the Third Exam Material

Use the Laplace transform to solve the following problems. In problems 1. to 4., assume that the given equation models the motion of an undamped harmonic oscillator where y is in meters and t is in seconds. For problems 1. to 4., besides finding solution, write your answer as a piecewise function, sketch its graph, and describe the motion.

1. $y'' + y = \begin{cases} 1, & 5 \le t < 20, \\ 0, & t < 5 \text{ and } t \ge 20. \end{cases}$ $y(0) = 0$, $y'(0) = 0$.

2. $y'' + 4y = \begin{cases} 0, & t < 5, \\ t-5, & 5 \le t < 10, \\ 5, & t \ge 10. \end{cases}$ $y(0) = 0$, $y'(0) = 0$. The function on the right side can be represented as $u_5(t)(t-5) - u_{10}(t)(t-10)$.

3. $y'' + 4y = \delta(t - 4\pi)$, $y(0) = 1$, $y'(0) = 0$.

4. $y'' + 3y' + 4y = \delta(t-3)$, $\quad y(0) = 0$, $\quad y'(0) = 0$.

5. $y(t) + \int_0^t (t-\tau)\, y(\tau)\, d\tau = t$.

6. $y'(t) + \int_0^t y(t-\tau)\, e^{-2\tau}\, d\tau = 1$, $y(0) = 1$.

7.
$$\begin{aligned} x' &= -x + y \qquad & x(0) &= 1 \\ y' &= -x - y \qquad & y(0) &= 2 \end{aligned}$$

Solutions (Systems)

Just the final answers of some of the problems are given.
Refer to the class handouts or earlier review sheets for step by step solutions.

1. (a) There are four equilibrium points (0,0), (0, 0.75), (1,0), and (0.5, 0.5).
   (b) There are four equilibrium points (0,0), (0, 2), (1,0), and (0.5, 0.5).

2. (a) (0,0) is an asymptotically stable spiral point. When $t \to \infty$, $x \to 0$ and $y \to 0$ regardless of initial conditions.
   (b) (0,0) is an asymptotically stable node. When $t \to \infty$, $x \to 0$ and $y \to 0$ regardless of initial conditions.
   (c) (0,0) is a stable (but not asymptotically stable) center. Both x and y do not converge to 0 as $t \to \infty$ but they stay bounded. The circles in the phase plane are traversed counter clock-wise.

(d) (0,0) is an unstable node. When $t \to \infty$, x and y do not stay bounded. The exact limit depends on the initial conditions: if the initial condition $(x(0), y(0))$ is in the first quadrant $x \to \infty$ and $y \to \infty$, if the $(x(0), y(0))$ is in the second quadrant $x \to -\infty$ and $y \to \infty$, $(x(0), y(0))$ is in the third quadrant $x \to -\infty$ and $y \to -\infty$, and if $(x(0), y(0))$ is in the fourth quadrant $x \to \infty$ and $y \to -\infty$.

(e) (0,0) is a saddle point (thus unstable). If $(x(0), y(0))$ is on the left of the separatrix, then $x \to -\infty$ and $y \to -\infty$ for $t \to \infty$. If $(x(0), y(0))$ is on the right of the separatrix, then $x \to \infty$ and $y \to \infty$ for $t \to \infty$.

3. (a) Let $X = \mathcal{L}[x]$ and $Y = \mathcal{L}[y]$. Taking $\mathcal{L}$ of both equations produces $sX - 1 = -X + Y$ and $sY - 3 = -X - Y$. From the first equation $Y = sX + X - 1$. Plugging that in the second gives you $s(sX+X-1)-3 = -X-(sX+X-1) \Rightarrow s^2X+2sX+2X = s+4 \Rightarrow X = \frac{s+4}{s^2+2s+2}$. Thus, $Y = \frac{s^2+4s+s+4-s^2-2s-2}{s^2+2s+2} = \frac{3s+2}{s^2+2s+2}$.
Complete the denominator to a sum of squares: $s^2+2s+2 = s^2+2s+1+1 = (s+1)^2+1$. Then $x = \mathcal{L}^{-1}[X] = \mathcal{L}^{-1}[\frac{s+1+3}{(s+1)^2+1}] = \mathcal{L}^{-1}[\frac{s+1}{(s+1)^2+1} + 3\frac{1}{(s+1)^2+1}] = e^{-t}\cos t + 3e^{-t}\sin t$ and $y = \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}[\frac{3s+2}{s^2+2s+2}] = \mathcal{L}^{-1}[\frac{3(s+1)-1}{(s+1)^2+1}] = 3e^{-t}\cos t - e^{-t}\sin t$.

(b) If $X = \mathcal{L}[x]$ and $Y = \mathcal{L}[y]$, taking $\mathcal{L}$ of both equations produces $sX - 1 = 3X - Y$ and $sY - 3 = 4X - 2Y$. From the first equation, $Y = -sX + 3X + 1$. Substitute that in the second equation. $s(-sX+3X+1) - 3 = 4X - 2(-sX+3X+1) \Rightarrow -s^2X + 3sX - 4X - 2sX + 6X = -s + 3 - 2 \Rightarrow -s^2X + sX + 2X = -s + 1 \Rightarrow -(s^2 - s - 2)X = -(s-1)$ $X = \frac{s-1}{s^2-s-2}$. Thus $Y = \frac{-s^2+s+3s-3+s^2-s-2}{s^2-s-2} = \frac{3s-5}{s^2-s-2}$. The partial fraction decomposition produces $X = \frac{s-1}{(s-2)(s+1)} = \frac{1/3}{s-2} + \frac{2/3}{s+1}$ and $Y = \frac{3s-5}{(s-2)(s+1)} = \frac{1/3}{s-2} + \frac{8/3}{s+1}$. Taking inverse Laplace transform, $x = \mathcal{L}^{-1}[X] = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$ and $y = \mathcal{L}^{-1}[Y] = \frac{1}{3}e^{2t} + \frac{8}{3}e^{-t}$.

4. (a) Two equilibrium points (0,0) and (1000, 80). (b) (1000, 80) is a center (thus stable but not asymptotically stable) and (0,0) is a saddle point (thus unstable). The solutions oscillate: x-values about 1000 and y-values about 80 in the counter clock-wise direction. (c) With 400 rabbits and 100 wolves initially, the number of rabbits oscillates between about 460 and 2200 and the number of wolves between about 50 and 200.

5. (a) Four critical points: three unstable (0,0), (0,0.75), and (1,0) and one stable node (0.5,0.5). The trajectory starting at $(1, 3)$ converges towards (0.5, 0.5) when $t \to \infty$. This supports the conclusion that (0.5,0.5) is an asymptotically stable node. The number of both population approach 500 members after some period of time provided that both populations have at least one member initially.

(b) Four critical points: two unstable (0,0) and saddle (0.5,0.5) and two stable nodes (0,2), (1,0). If the initial conditions are above separatrix passing (0.5,0.5), then $(x, y) \to (0, 2)$. If the initial conditions are below separatrix, then $(x, y) \to (1, 0)$. So the coexistence is possible just in the off chance scenario that the initial conditions are exactly on the separatrix. In any other case, one species eventually overwhelms the other resulting in the extinction of the other. If the initial conditions are such that the x's survive, the trajectory ends up at the node (1,0) resulting in 1000 members of the first and zero of the second population. If the initial conditions are such that the y's survive, the trajectory ends up at the node (0,2) resulting in 2000 members of the second and zero of the first population.

Solutions (The First Exam Material)

Just the final answers of most of the problems are given.
Refer to the class handouts or earlier review sheets for step by step solutions.

1. (a) $y = x^2$ is not a solution. $y = 2 + e^{-x^3}$ is a solution.

   (b) $r = \frac{-1}{3}$ and $r = \frac{3}{2}$.

   (c) $A = 3$ so $y = 3e^{3x}$.

2. (a) Linear equation. $y = \frac{-x}{2} - \frac{1}{4} + ce^{2x}$.

   (b) Separable. $y = \frac{1}{\frac{-1}{2}xe^{2x}+\frac{1}{4}e^{2x}+c}$.

   (c) Exact equation. $\frac{1}{4}x^4y^4 + y^2 + c = 0$.

   (d) Bernoulli's equation with $n = 3$. $y = (\frac{2}{5x} + cx^4)^{-1/2} = \frac{1}{\sqrt{\frac{2}{5x}+cx^4}}$.

   (e) Homogeneous. $y = x\tan(\ln|x| + c)$.

   (f) Linear. $y = \frac{\frac{x^5}{5}+c}{x^2} = \frac{x^3}{5} + \frac{c}{x^2}$.

3. (a) Three equilibrium solutions: $y = 0$ is unstable, $y = 2$ is semistable and $y = 5$ is stable.

   (b) You can consider the following cases: $a > b$, $a < b$ and $a = b$. In the first case, $y = b$ is stable and $y = a$ is unstable. In the second case, $y = a$ is stable and $y = b$ is unstable. If $a = b$, there is just one equilibrium solution and it is semistable.

4. (a) y = size of the mice population, t = time. Model is $\frac{dy}{dt} = 0.5y - 450$. One unstable equilibrium solution $y = 900$. The solution is $y = ce^{t/2} + 900$.

   (b) (a) y = amount present, t = time. Model is $y' = r - ky$. (b) $y' = 4 - 2y$ has one stable equilibrium solution $y = 2$. (c) $y = 2 - e^{-2t}$.

   (c) (a) $\frac{dv}{dt} = g - \frac{2}{m}v = 9.8 - \frac{v}{5}$. One stable equilibrium solution $v = 49$. (b) $v = 49 - 49e^{-t/5}$.

5. $a = 6$ and then the solution is $2y^2e^{3x} + \frac{2}{3}x^3y + 3e^{4y} = C$.

Solutions (The Second Exam Material)

Just the final answers of most of the problems are given.
Refer to the class handouts or earlier review sheets for step by step solutions.

1. Homogeneous Equations. (a) $y = c_1 + c_2e^x + c_3xe^x$.

   (b) $y = c_1e^x + c_2e^{-x} + c_3\cos x + c_4\sin x$. (c) $y = c_1e^{3x} + c_2e^{-3x} + c_3\cos 2x + c_4\sin 3x$.

   (d) $y = c_1e^{2x}+c_2e^{0.618x}\cos 1.902x+c_3e^{0.618x}\sin 1.902x+c_4e^{-1.618x}\cos 1.176x+c_5e^{-1.618x}\sin 1.176x$.

2. Variation of Parameters. (a) $y = c_1e^{3x} + c_2xe^{3x} + \frac{1}{2}x^{-1}e^{3x}$.

   (b) $y = (c_1 - 1)e^{-2x} + c_2xe^{-2x} - \ln x\; e^{-2x}$ same as $y = c_1e^{-2x} + c_2xe^{-2x} - \ln x\; e^{-2x}$.

3. Undetermined Coefficients.

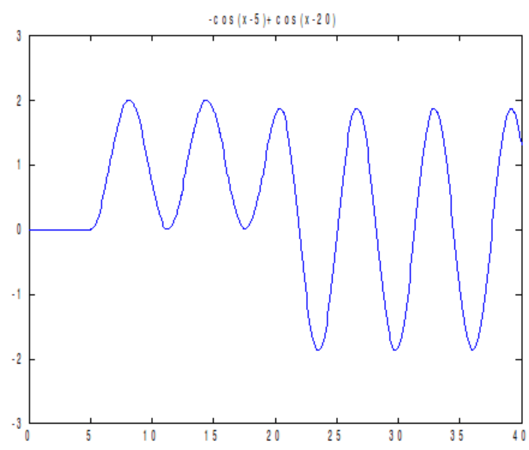
   (a) $y = c_1e^{2x} + c_2e^{3x} - 4xe^{2x}$.
   (b) $y = c_1 \cos 2x + c_2 \sin 2x + (x^2 - \frac{4}{5}x - \frac{2}{25})e^x$.
   (c) $y = c_1e^x + c_2xe^x + \frac{7}{6}x^3e^x$.
   (d) $y = c_1e^{5x} + c_2e^{-2x} + (Ax + B)e^{2x} + Cxe^{-2x}$.
   (e) $y = c_1e^{-2x} \cos 3x + c_2e^{-2x} \sin 3x + A \cos 3x + B \sin 3x + Cxe^{-2x} \cos 3x + Dxe^{-2x} \sin 3x$.

4. Applications.

   (a) The general solution is $u = c_1e^{-.125t} \cos .992t + c_2e^{-.125t} \sin .992t$. The initial conditions $u(0) = 0$ and $u'(0) = 1$ produce $c_1 = 0$ and $c_2 = 1.008$ so that $u = 1.008e^{-.125t} \sin .992t$. This is an underdamped free oscillator: u converges to 0 and so the oscillations have a *decreasing amplitude*.
   (b) The equation is $u'' + u = \frac{1}{2} \cos 0.8t$, $u_h = c_1 \cos t + c_2 \sin t$, and $u_p = \frac{25}{18} \cos 0.8t$. The initial conditions $u(0) = 1$ and $u'(0) = 0$ produce $c_1 = -\frac{25}{18}$ and $c_2 = 0$ and so $u = -\frac{25}{18} \cos t + \frac{25}{18} \cos 0.8t$. The presence of two different frequencies 1 and 0.8 indicate oscillates with a *periodic amplitude*. So, the oscillations are with *beats*.
   (c) The equation is $u'' + u = \frac{1}{2} \cos t$. $u_h = c_1 \cos t + c_2 \sin t$, and $u_p = \frac{1}{4}t \sin t$. The initial conditions $u(0) = 1$ and $u'(0) = 0$ produce $c_1 = 1$ and $c_2 = 0$ and so $u = \cos t + \frac{1}{4}t \sin t$. The presence of t in front of the sine function indicates an *increasing amplitude*. So, the oscillations are with a *resonance*.
   (d) The equation is $Q'' + 4 \cdot 10^6Q = 0$ with $Q(0) = 10^{-6}$ and $Q'(0) = 0$. The solution is $Q = 10^{-6} \cos 2000t$ which is a periodic function with a *constant amplitude*.
   (e) $\gamma^2 - 36 < 0 \Rightarrow -6 < \gamma < 6$. Since γ is nonnegative, $0 \leq \gamma < 6$.

Solutions (The Third Exam Material)

1. The function on the right side is a boxcar function given by $u_5(t) - u_{20}(t)$. Taking the Laplace transform of the equation with $\mathcal{L}[y] = Y$, we obtain $s^2Y + Y = \frac{e^{-5s}}{\ } - \frac{e^{-20s}}{\ }$. From here $Y =$ $(e^{-5s} - e^{-20s})\frac{1}{s(s^2+1)}$. Let $F(s) = \frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$. and $C = 0$ so that $F(s) = \frac{1}{s} - \frac{s}{s^2+1} \Rightarrow f(t) = \mathcal{L}^{-1}[F$

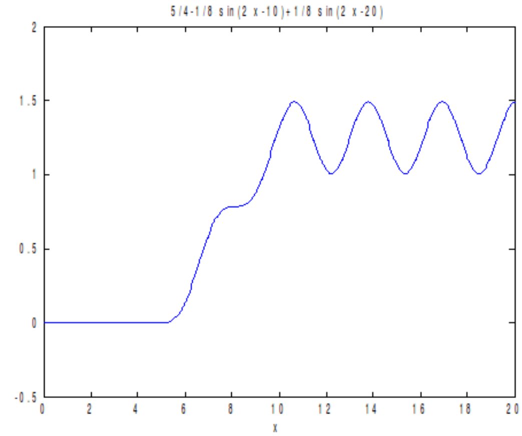


   Thus, the solution is $y = \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}[(e^{-5s}F(s) - e^{-20s}F(s)] = u_5(t)f(t-5) - u_{20}(t)f(t-20) = u_5(t)(1 - \cos(t-5)) - u_{20}(t)(1 - \cos(t-20))$.

$$y = \begin{cases} 0, & t < 5 \\ 1 - \cos(t-5), & 5 \leq t < 20, \\ -\cos(t-5) + \cos(t-20), & t \geq 20. \end{cases}$$

2. The Laplace transform of the equation is $s^2Y + 4Y = e^{-5s}\frac{1}{s^2} - e^{-10s}\frac{1}{s^2}$. Thus $Y = (e^{-5s} - e^{-10s})\frac{1}{s^2(s^2+4)}$. Let $F(s) = \frac{1}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}$. Determine that $A = C = 0$, $B = \frac{1}{4}$, and $D = \frac{-1}{4}$. So, $F(s) = \frac{1/4}{s^2} - \frac{1/4}{s^2+4}$ and $f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{4}t - \frac{1}{8}\sin 2t$. Thus,

$$y = \mathcal{L}^{-1}[Y] = u_5(t)f(t-5) - u_{10}(t)f(t-10) =$$
$$u_5(t)(\frac{1}{4}(t-5) - \frac{1}{8}\sin 2(t-5)) - u_{10}(t)(\frac{1}{4}(t-10) - \frac{1}{8}\sin 2(t-10)).$$



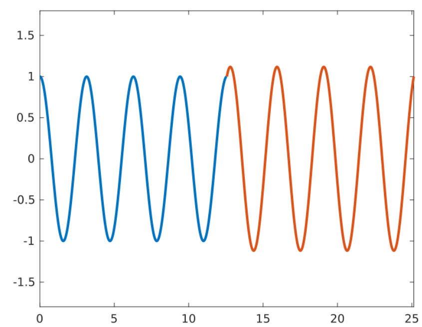
$$y = \begin{cases} 0, & 0 \le t < 5, \\ \frac{1}{4}(t-5) - \frac{1}{8}\sin 2(t-5), & 5 \le t < 10, \\ \frac{5}{4} - \frac{1}{8}\sin 2(t-5) + \frac{1}{8}\sin 2(t-10), & t \ge 10. \end{cases}$$

Thus, there are no oscillations before 5. Between 5 and 10 seconds, the mass oscillates about the line $\frac{1}{4}(t-5)$. After 10 seconds, the mass oscillates about $\frac{5}{4}$ with a constant amplitude.

3. The Laplace transform of the equation is

   $s^2Y - s + 4Y = e^{-4\pi s}$. Thus $Y = \frac{e^{-4\pi}+s}{s^2+4}$. Then $y = u_{4\pi}(t)\frac{1}{2}\sin 2(t-4\pi) + \cos 2t =$

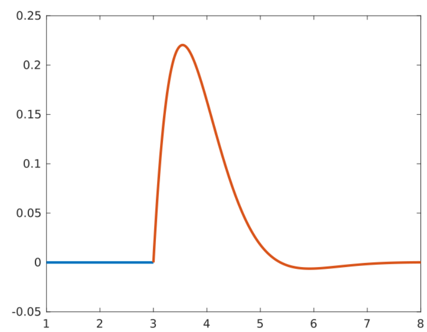
   $$\begin{cases} \cos 2t, & t < 4\pi, \\ \cos 2t + \frac{1}{2}\sin 2(t-4\pi) & t \ge 4\pi. \end{cases}$$



4. Let $Y = \mathcal{L}[y]$. Applying the Laplace transform to the equation $y'' + 3y' + 4y = \delta(t-3)$ with $y(0) = y'(0) = 0$ produces $s^2Y + 3sY + 4Y = e^{-3s}$. From here $Y = \frac{e^{-3s}}{s^2+3s+4}$. Complete the denominator of $F(s) = \frac{1}{s^2+3s+4}$ to a sum of squares. $s^2 + 3s + 4 = s^2 + 2s(\frac{3}{2}) + \frac{9}{4} + 4 - \frac{9}{4} = (s+\frac{3}{2})^2 + \frac{7}{4}$. Thus $f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}[\frac{1}{(s+\frac{3}{2})^2+\frac{7}{4}}] = \frac{2}{\sqrt{7}}\mathcal{L}^{-1}[\frac{\frac{\sqrt{7}}{2}}{(s+\frac{3}{2})^2+\frac{7}{4}}] = \frac{2}{\sqrt{7}}e^{-3t/2}\sin\frac{\sqrt{7}}{2}t$.

   $y = \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}[\frac{e^{-3s}}{s^2+3s+4}] = \mathcal{L}^{-1}[e^{-3s}F(s)] = u_3(t)\,f(t-3) = u_3(t)\,\frac{2}{\sqrt{7}}e^{-3(t-3)/2}\sin\frac{\sqrt{7}}{2}(t-3) =$

   $$\begin{cases} 0, & t < 3, \\ \frac{2}{\sqrt{7}}e^{-3(t-3)/2}\sin\frac{\sqrt{7}}{2}(t-3), & t \ge 3. \end{cases}$$



   Thus, the object starts oscillating only after 3 seconds. It oscillates with a decreasing amplitude given by $\frac{2}{\sqrt{7}}e^{-3(t-3)/2}$ converging to 0 and the oscillations become negligible in time.

5. $y + t * y = t \Rightarrow Y + \frac{1}{s^2}Y = \frac{1}{s^2} \Rightarrow Y = \frac{1}{s^2+1} \Rightarrow y = \sin t$.

6. $y' + y * e^{-2t} = 1 \Rightarrow sY - 1 + Y\frac{1}{s+2} = \frac{1}{s} \Rightarrow Y = \frac{s+2}{s(s+1)}$. Find the partial fraction decomposition to be $Y = \frac{2}{s} - \frac{1}{s+1}$. The solution is $y = 2 - e^{-t}$.

7. Taking $\mathcal{L}$ produces $sX - 1 = -X + Y$ and $sY - 2 = -X - Y$. $x = \mathcal{L}^{-1}[X] = \mathcal{L}^{-1}[\frac{s+1+2}{(s+1)^2+1}] = \mathcal{L}^{-1}[\frac{s+1}{(s+1)^2+1} + 2\frac{1}{(s+1)^2+1}] = e^{-t}\cos t + 2e^{-t}\sin t$ and $y = \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}[\frac{2s+1}{s^2+2s+2}] = \mathcal{L}^{-1}[\frac{2s+2-1}{(s+1)^2+1}] = \mathcal{L}^{-1}[2\frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}] = 2e^{-t}\cos t - e^{-t}\sin t$.