Differential Equations
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Systems of Differential Equations

A system of n first order differential equations has the form

$$y_1' = F_1(t, y_1, \ldots, y_n), \;\; y_2' = F_2(t, y_1, \ldots, y_n), \;\; \ldots, \;\; y_n' = F_n(t, y_1, \ldots, y_n),$$

This system is linear if functions $F_1, F_2, \ldots, F_n$ are linear functions of $y_1, y_2, \ldots, y_n$.

In particular, a system of two first order differential equations in two unknown functions x and y has the form

$$\frac{dx}{dt} = f(x, y, t), \qquad \frac{dy}{dt} = g(x, y, t).$$

The solution of such system is a parametric function $x = x(t)$ and $y = y(t)$. The curve $(x(t), y(t))$ is called a **trajectory**. It may be helpful to think of the independent variable t as the time and the depend variables x and y as the position (x, y) in xy-plane. In this case, xy-plane is referred to as the **phase plane**.

Autonomous Systems. A system of two equations is **autonomous** or **homogeneous** if it is of the form $\frac{dx}{dt} = f(x, y)$ and $\frac{dy}{dt} = g(x, y)$ (that is if the variable t does not appear on the right side). Just like in one-equation case $y' = f(y)$, one can analyze the limit of the solutions by setting the right side of the equation(s) to zero and solving for the dependent variable(s). Just as in one-equation case, these solutions are called the **equilibrium values or points** and are also referred to as the steady states of the system. Also just like in the one-equation case, we are interested in the **stability** of these equilibrium points: whether the solutions of the system remain close and converge to the equilibrium point or diverge from it. This analysis provides the insight in the long-term behavior of the system, the sensitivity to the initial conditions, the type of the solutions (whether the solutions are periodic functions, increasing or decreasing functions, etc) and the sensitivity to changes in parameters.

To find the equilibrium values of an autonomous system, set the equations equal to 0 and solve for points (x, y) which amounts to solving the equations

$$f(x, y) = 0 \quad \text{and} \quad g(x, y) = 0.$$

Note that this parallels the method for finding equilibrium solutions of the first order autonomous equation $y' = f(y)$. Assuming that the point (a, b) is a solution of the equations above, the point (a, b) is said to be

- **asymptotically stable** if $x \to a$ and $y \to b$ when $t \to \infty$ for any value of initial conditions.
- **stable** if a solution which starts close to the equilibrium point (a, b) stays close to it (but does not necessarily converge to it).
- **unstable** if it is not stable.

Note also that every autonomous system can be converted into a single first order differential equation. Dividing two autonomous differential equations with each other produces a single equation with derivative $\frac{dy}{dx}$ on the left side:

$$\frac{dx}{dt} = f(x,y) \quad \frac{dy}{dt} = g(x,y) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g(x,y)}{f(x,y)}$$

Although it may be possible to find solution $y = y(x)$ as a function of x, it is important to keep in mind that the formula for $y(x)$ is not an oriented curve: all the information related to parameter t is lost in this way.

In order to obtain the direction of parametric curves in the phase plane one can analyze the graph in the phase plane together with the graphs of $x(t)$ and $y(t)$ as functions of t. The next several examples illustrate this method as well as the analysis of asymptotic behavior of the system.

Example 1. Consider the system

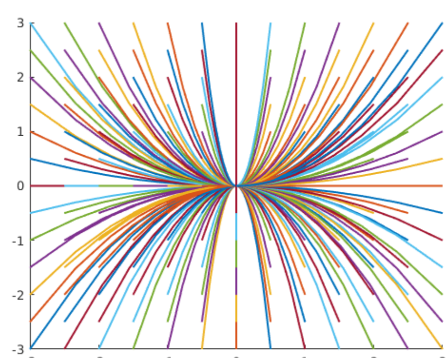
$$\frac{dx}{dt} = -x \quad \text{and} \quad \frac{dy}{dt} = -2y.$$

Its equilibrium point can be obtained from equations $-x = 0$ and $-2y = 0$. So (0,0) is the only equilibrium point.

Note that this system is very simple: it consists of two independent first order equations which can each be solved independently. By separating the variables, integrating and solving for x and y, one obtains the solutions $x = c_1 e^{-t}$ and $y = c_2 e^{-2t}$ (note that it is possible to do this by hand only because the system is very simple). When $t \to \infty$, $x \to 0$ and $y \to 0$ regardless of the values of c_1 and c_2. Hence, (0,0) is an asymptotically stable equilibrium point.

One can obtain graphs of several solutions in the phase plane relatively easily in this case, Because of more complex systems, let us examine how one would graph a few solutions in the phase plane using Matlab. To start, represent the right side of the system as a vector function of two variables by **f =@(t,y)[-y(1); -2*y(2)]**. Recall that **y(1)** represents x and **y(2)** represents y. The command **ode45(f,** $[t_0, t_1]$**,** $[a; b]$**)** plots the trajectory in the phase plane with initial conditions $x(t_0) = a$ and $y(t_0) = b$ for $t_0 \leq t \leq t_1$. Repeating this command for sufficiently many values of the initial conditions a and b creates a plot of solutions in the phase plane which can be used for the analysis of the system. For example, let us graph the solutions of the given system for $0 \leq t \leq 10$ and parameters a and b taking values -3, -2.5, -2, -1.5, ... 2, 2.5, 3. In this case the following script can be used after the right side is represented as **f** and the graph below is obtained.

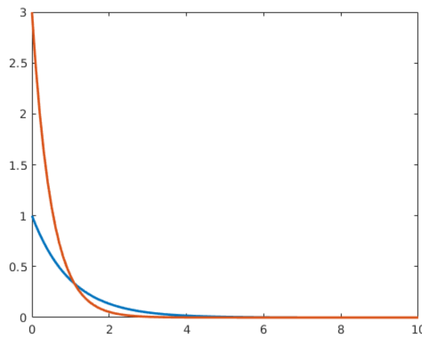
```
close all; hold on
for a = -3:0.5:3
  for b = -3:0.5:3
    [t, y] = ode45(f, 0:0.25:10, [a; b]);
    plot(y(:,1), y(:,2))
  end
end
hold off
axis([-3 3 -3 3])
```



Stable node: the phase plane graph

Since $x \to 0$ and $y \to 0$ for $t \to \infty$ for any value of initial conditions $(x(0), y(0))$, the orientation of the parametric curves on the graph is towards the origin. For more complex systems, one can determine the orientation by graphing one or more of them as functions of t.

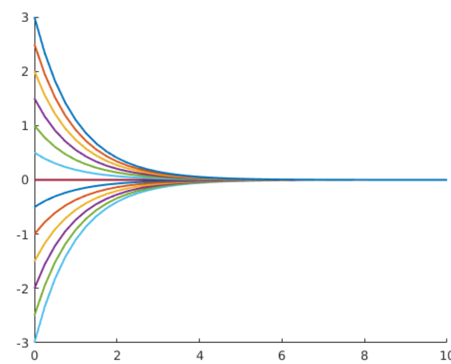
For example, for $x(0) = 1$ and $y(0) = 3$, one obtains the graph on the right by **[t, y]=ode45(f,[0,10],[1;3]); plot(t,y)**. We can see that $x \to 0$ and $y \to 0$ when $t \to \infty$ confirming our earlier conclusion.



An asymptotically stable point of the same type as (0,0) in this example is called a **stable node**.

One can easily modify the above script to obtain graphs of several solutions in tx-plane or ty-plane. For example, for the graph in tx-plane, change the command **plot(y(:,1), y(:,2))** to **plot(t, y(:,1))**. Similarly, for ty-plane, this should be replaced by **plot(t, y(:,2))**. Change also the **axis** command to match the t and x values.

```
close all; hold on
for a = -3:0.5:3
  for b = -3:0.5:3
    [t, y] = ode45(f, 0:0.25:10, [a; b]);
    plot(t, y(:,1))
  end
end
hold off
axis([0 10 -3 3])
```

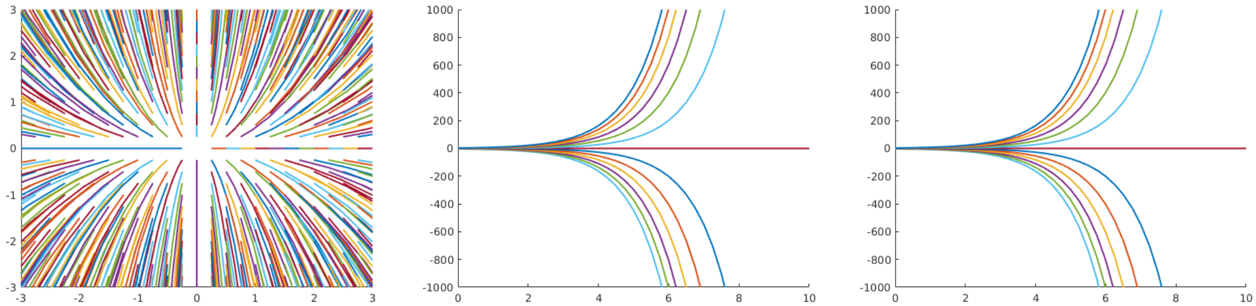


Stable node: the tx-plane graph

The graph in the ty-plane is very similar. Note that these are familiar graphs of a stable equilibrium solution of a first order autonomous equation.

One can also obtain the formula for the trajectories as $y = y(x)$. Dividing the second equation by the first, we obtain $\frac{dy}{dx} = \frac{-2y}{-x} = 2\frac{y}{x}$. Note that this is a separable differential equation $\frac{dy}{y} = 2\frac{dx}{x}$. The general solutions of this equation has the form $y = cx^2$. Note that this last formula contains no information on the direction of the parametric curves when parametrized by t.

Example 2. Consider now the system $\frac{dx}{dt} = x$ and $\frac{dy}{dt} = 2y$. The solutions are $x = c_1 e^t$ and $y = c_2 e^{2t}$, and, dividing the second equation by the first and solving for y in terms of x produces the same equation $\frac{dy}{dx} = 2\frac{y}{x}$ and the parabolas $y = cx^2$ as in the previous example. However, in this case, the trajectories have the opposite direction than in the previous example which can be determined on the same way as in the previous example: you can use the same script to obtain a plot with several solutions in the phase plane and then graph one solution to examine stability. The graphs below contain several solutions in the phase plane, in the tx-plane and in the ty-plane. From the second and the third graph we can conclude that the solutions diverge away from the equilibrium point (0,0). Thus, (0,0) is **unstable**. This type of equilibrium point is called an **unstable node**.

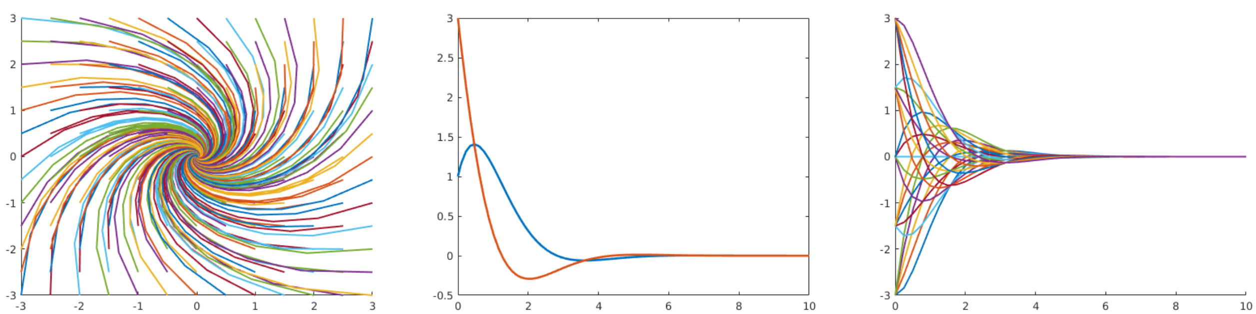


Unstable node: the phase plane, the tx-plane, and the ty-plane graphs

In particular, the limit of x and y for $t \to \infty$ critically depends on the location of the initial condition $(x(0), y(0))$. In this example, if $(x(0), y(0))$ is in the first quadrant, both x and y converge to positive infinity and if $(x(0), y(0))$ is in the third quadrant, both x and y converge to negative infinity. If $(x(0), y(0))$ is in the second quadrant, x converges to negative and y to positive infinity and if $(x(0), y(0))$ is in the fourth quadrant, vice versa.

Example 3. Consider the system $\frac{dx}{dt} = -x + y$ and $\frac{dy}{dt} = -x - y$. The equations $-x + y = 0$ and $-x - y = 0$ have a single solution $x = 0$ and $y = 0$ so the system has just one equilibrium point (0,0). To graph the trajectories in the phase plane, represent the right side of the system as a vector function by **f =@(t,y) [-y(1)+y(2);-y(1)-y(2)]** and use the same script as before. The output is the first graph on the figure below. The point (0,0) is called a **spiral point** in this case.

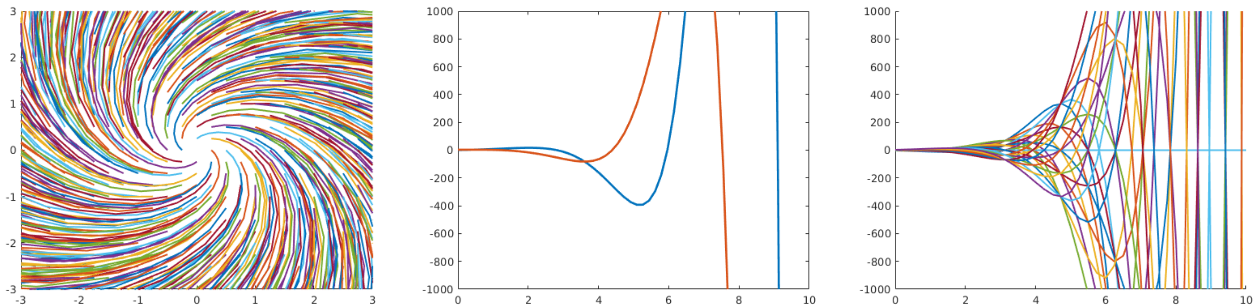
To figure out the directions of the trajectories, graph one solution or more solutions in tx and ty-planes as functions of t. For example, for $x(0) = 1$ and $y(0) = 3$, you get the second graph below by using **[t, y]=ode45(f,[0,10],[1;3]); plot(t,y)**). From this graph, we can see that $x \to 0$ and $y \to 0$ when $t \to \infty$. Graphing more initial conditions if necessary, as it has been done in the tx-plane on the third graph below, you can see that the trajectories are approaching the equilibrium point $(0, 0)$ when $t \to \infty$ for every value of the initial conditions $(x(0), y(0))$. Thus, the spiral point (0,0) is **asymptotically stable**.



Stable spiral point: the phase plane, a single (x, y) solution, and the tx-plane graphs

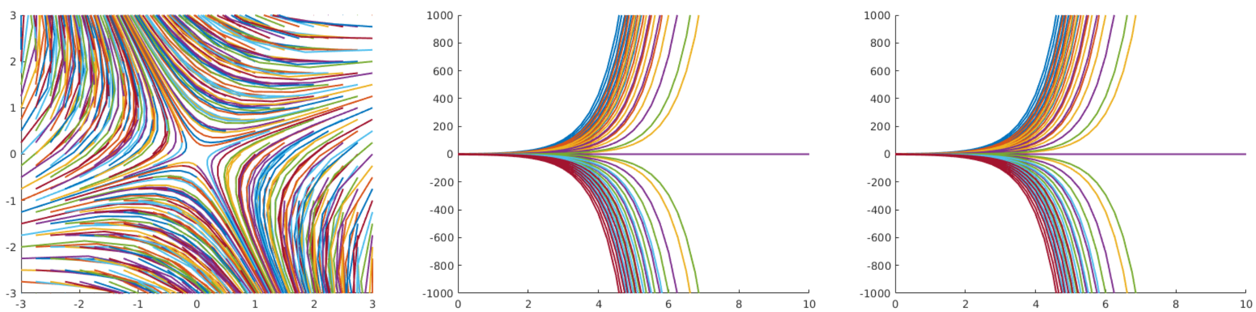
If you need to obtain the explicit formulas of the solutions x and y you can use Matlab command **dsolve** or, in this case, you can use Laplace Transform as well. The solutions of this system are $x = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t$ and $y = -c_1 e^{-t} \sin t + c_2 e^{-t} \cos t$. The fact that the terms e^{-t} converge to zero for $t \to \infty$ and, as a consequence $(x, y) \to (0, 0)$ agrees with our earlier analysis.

Example 4. The system $\frac{dx}{dt} = x + y$ and $\frac{dy}{dt} = -x + y$ is an example of the system with **unstable spiral point.** The formulas of the solution looks similar as in previous example except that the coefficients in the exponent of e are positive. The graphs in the phase plane, the graph of solutions with initial conditions $x(0) = 1$ and $y(0) = 3$, and the graph in the tx-plane are below. The graph in ty-plane looks similar to the graph in the tx-plane. From the graphs, we conclude that the phase plane curves are traced away from (0,0). Thus, (0,0) is an unstable point. In this case the limits of x and y for $t \to \infty$ do not exist.



Unstable spiral point: the phase plane, a single (x, y) solution, and the tx-plane graphs

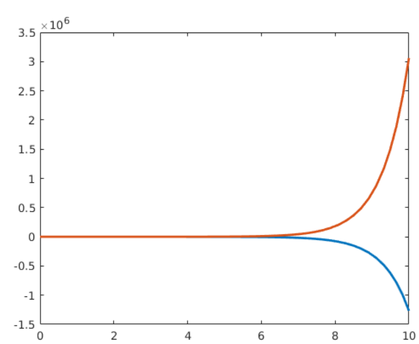
Example 5. The system $\frac{dx}{dt} = -x - y$ and $\frac{dy}{dt} = -x + y$ has one equilibrium point (0,0). The solutions turn out to be linear combination of $e^{\sqrt{2}t}$ and $e^{-\sqrt{2}t}$ (you can find explicit formulas using **dsolve** or using the Laplace transform). Graphing the phase plane and the tx and ty-planes produces the graphs below. The solutions in the phase plane are hyperbolas. The point (0,0) is called a **saddle point** in this case.



Saddle point: the phase plane, the tx-plane, and the ty-plane graphs

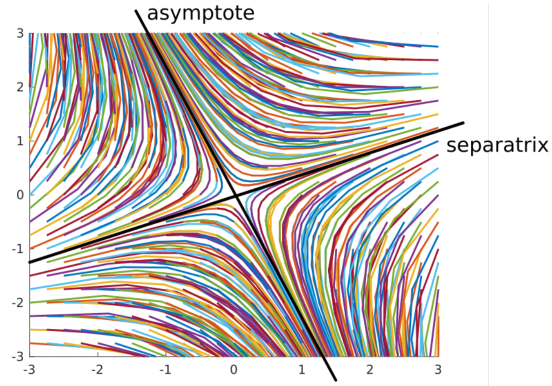
By the second and the third graph, the point (0,0) is **unstable** which is the case for every point of this type.

However, the direction of the curves in the phase plane may not be clear, nor the limits of x and y when $t \to \infty$. To figure out the directions of the parametric curves on this graph, graph one of them as function of t. For example, for $x(0) = 1$ and $y(0) = 3$, you get the graph on the right. We can see that $x \to -\infty$ and $y \to \infty$ when $t \to \infty$.



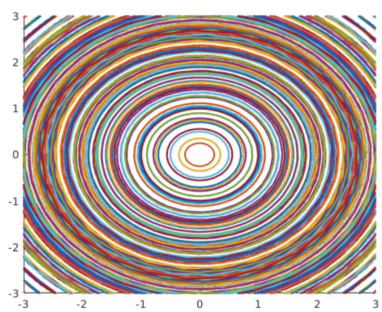
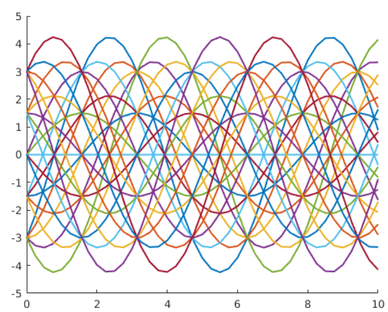
Graphing more initial conditions if necessary and analyzing the hyperbolas in the phase plane, you can see that they lie in four "quadrants" formed by two lines, one called the **asymptote** and the other called the **separatrix** of the solutions.

The solution with $x(0) = 1$ and $y(0) = 3$ and any other solution with $(x(0), y(0))$ above separatrix has the direction towards the upper half of the asymptote. So, in this case $x \to -\infty$ and $y \to \infty$ when $t \to \infty$. On the other hand, if the initial point $(x(0), y(0))$ is below the separatrix, the solutions converge towards the lower half of the asymptote and so $x \to \infty$ and $y \to -\infty$ when $t \to \infty$.



Thus, the separatrix "separates" two types of behavior of solutions: converging towards the upper half of the asymptote and converging towards the lower half of the asymptote. Just like in the case of an unstable node, this system is critically sensitive to the values of the initial conditions.

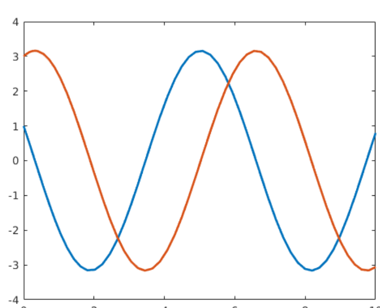
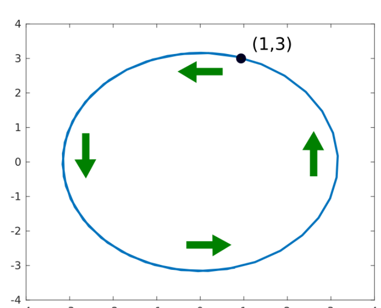
Example 6. Consider now the system $\frac{dx}{dt} = -y$ and $\frac{dy}{dt} = x$. Its only equilibrium point is (0,0). Using Laplace transform (or **dsolve** in Matlab), you can see that the solutions are linear combinations of $\cos t$ and $\sin t$. To obtain the formula for y as a function of x, you can divide the second equation by the first, obtain $\frac{dy}{dx} = \frac{x}{-y} \Rightarrow ydy = -xdx \Rightarrow y^2 = -x^2 + 2c \Rightarrow x^2 + y^2 = C$. So, the solutions are circles with the center at (0,0). In this case (or similar case when the solutions are ellipses), the equilibrium point is called a **center**.

Center point: the phase plane and the tx-plane graphs

The first graph contains solutions in the phase plane and the second graph several solutions in the tx-plane. The graphs in the ty-plane look very similar.

You can use the graph of single solution to determine the directions of the parametric curves in the phase plane. For example, graph the solution with $x(0) = 1$ and $y(0) = 3$. The first graph are x and y as functions of t and the second graph below is the graph of $(x(t), y(t))$ in the phase plane.

Considering the x-curve (in blue) for example, one can conclude that the x-values are decreasing. So, starting at (1,3) in the phase plane and considering the circle passing this point, the direction should be such that the x-values are decreasing and this happens if this circle is traversed counter clock-wise. Similarly, considering y-values, the first graph indicates that the y-values first increase for a bit and then decrease. This produces the same conclusion: the direction is counter clock-wise.

Regarding the stability, we can conclude that both x and y do not converge to 0 as $t \to \infty$. However, they do not diverge to ∞ or $-\infty$ either but stay bounded: starting close to (0,0), the solutions remain close to 0 also as t increases. Thus, a center point is an example of stable equilibrium point which is not asymptotically stable.

Summary of the types. To summarize, let us consider the following homogeneous system with constant coefficients.

$$\frac{dx}{dt} = ax + by \qquad \frac{dy}{dt} = cx + dy$$

The determinant

$$\begin{vmatrix} a-r & b \\ c & d-r \end{vmatrix} = (a-r)(d-r) - bc = r^2 - (a+d)r + ad - bc$$

takes over the role of the characteristic equation of a second order equation. Say that r_1 and r_2 are two solutions (more about this in your Linear Algebra course: in particular, r_1 and r_2 are the *eigenvalues* of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of the system). If 0 is not a solution of the above equation in r, the following cases exhaust all possibilities. Note also that 0 is a solution exactly when (0,0) is not the only critical point of the system. In this case, the system has infinitely many critical points and we consider such systems in the last section.

1. r_1 and r_2 are real and negative.
2. r_1 and r_2 are real and positive.
3. r_1 is real and positive and r_2 is real and negative.
4. $r_1 = p + iq$ (thus $r_2 = p - iq$) and p is negative.
5. $r_1 = p + iq$ (thus $r_2 = p - iq$) and p is positive.
6. $r_1 = iq$ (thus $r_2 = iq$) that is, p is zero.

In the first three cases, the solutions x and y are a sum of constant multiples of $e^{r_1 t}$ and $e^{r_2 t}$ (or $e^{r_1 t}$ and $te^{r_1 t}$ in case $r_1 = r_2$). The constants in front of these functions can be found using the material of Linear Algebra course (in fact by finding the *eigenvectors* of r_1 and r_2).

Let us consider these three cases first. The absence of trigonometric functions in the solutions indicates that (0,0) is not a spiral or a center point. Hence, the first three cases correspond to stable and unstable nodes and a spiral point.

1. r_1 and r_2 are real and negative. In this case, $e^{r_1 t} \to 0$ and $e^{r_2 t} \to 0$ when $t \to \infty$ so any combination of constant multiples of those two function converges to zero. Because of this $x \to 0$ and $y \to 0$ when $t \to \infty$ for any initial value of the initial conditions. Thus, (0,0) is asymptotically stable and so it cannot be an unstable node or a spiral. Thus, (0,0) is a **stable node** in this case.

2. r_1 and r_2 are real and positive. In this case, $e^{r_1 t} \to \infty$ and $e^{r_2 t} \to \infty$ when $t \to \infty$ so any combination of constant multiples of those two function depends on the sign of these multiples and converges to either ∞ or $-\infty$. This indicates sensibility to the initial conditions so (0,0) is unstable. When $t \to -\infty$, $e^{r_1 t} \to 0$ and $e^{r_2 t} \to 0$ which is not the case with a spiral point. So (0,0) is an **unstable node** in this case.

3. r_1 is real and positive and r_2 is real and negative. In this case, $e^{r_1 t} \to \infty$ and $e^{r_2 t} \to 0$ when $t \to \infty$ and, because of the first limit, (0,0) is unstable. We also have that $e^{r_1 t} \to 0$ and $e^{r_2 t} \to \infty$ when $t \to -\infty$. Thus, for large negative values of t, the terms with $e^{r_2 t}$ dominate and for large positive values of t, the terms with $e^{r_1 t}$ dominate. The dependence on the initial conditions and the fact that both for $t \to \infty$ and for $t \to -\infty$ x and y are not finite, implies that (0,0) is a **saddle point**. The t-value when the domination of $e^{r_2 t}$ ends and the domination of $e^{r_1 t}$ is the point on the graph of a solution which is closest to (0,0).

Let us now consider the cases when r_1 and r_2 are complex. If $r_1 = p + iq$ (thus $r_2 = p - iq$), the solutions x and y are a sum of constant multiples of $e^{pt} \cos qt$ and $e^{pt} \sin qt$. The presence of trigonometric functions in the solution indicates that the system has a stable spiral, an unstable spiral or a center point.

4. $r_1 = p + iq$ (thus $r_2 = p - iq$) and p is negative. In this case, $e^{pt} \to 0$ when $t \to \infty$ so both $e^{pt} \cos qt$ and $e^{pt} \sin qt$ converge to 0 also. As a result, any combination of constant multiples of those two function converges to zero. Because of this $x \to 0$ and $y \to 0$ when $t \to \infty$ for any initial value of the initial conditions. Thus, (0,0) is asymptotically stable and so it has to be a **stable spiral** (an unstable spiral and a center point are not asymptotically stable.

5. $r_1 = p + iq$ (thus $r_2 = p - iq$) and p is positive. In this case, $e^{pt} \to \infty$ when $t \to \infty$ so the limit of both $e^{pt} \cos qt$ and $e^{pt} \sin qt$ is not defined (the values oscillate from $-\infty$ to ∞). Thus, (0,0) is neither stable nor asymptotically stable and so (0,0) is an **unstable spiral**.

6. $r_1 = iq$ (thus $r_2 = iq$) that is, p is zero. In this case, the solutions x and y are a sum of constant multiples of $\cos qt$ and $\sin qt$. Hence, the values of x and y do not converge to 0 but also do not diverge to ∞ or $-\infty$. This makes (0,0) not asymptotically stable and also not unstable. Hence, (0,0) is a **center point**.

If a system of differential equations is not linear, the system can have more than one equilibrium point and the graphs in phase plane can be a combination of the types listed above. We study further examples of both linear and nonlinear systems by considering systems that model situations studied in biology and population dynamics.

Applications of Systems of Differential Equations

When studying the first order differential equations, we have seen several examples that model the population growth of a single species. If two species are interacting and their growth is govern by the outcome of such interaction, their size can be modeled by a system of differential equations. The most widely used models are **competitive hunter model** (in which two species compete for common resources) and **predator-prey model** in which one population acts as a predator and the other as pray. Let $x(t)$ and $y(t)$ denote the sizes of two species at time t.

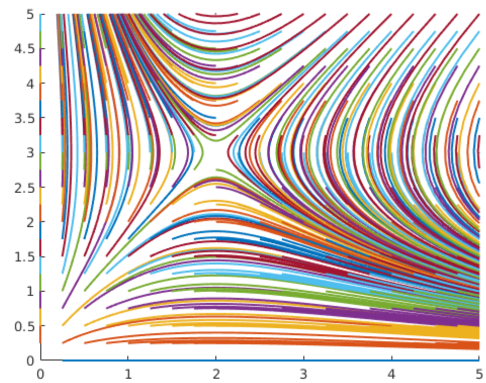
Competitive Hunter Model. In this model, x is growing at a rate proportional to the size of x but is decaying at a rate proportional to the number of interactions between the two species. Same is true for y so the system describing the sizes of the two species is given by

$$\frac{dx}{dt} = ax - bxy \qquad \frac{dy}{dt} = cy - dxy$$

Note that the first two terms on the right side of the equations mean that in the absence of the other population, the remaining population increases.

The equilibrium points of this system are $(0,0)$ and $(\frac{c}{d}, \frac{a}{b})$. Note that along the horizontal line $y = \frac{a}{b}$, $\frac{dx}{dt} = 0$ and so x is constant in time. Along the vertical line $x = \frac{c}{d}$, $\frac{dy}{dt} = 0$ and so y is constant in time.

If $y < \frac{a}{b}$ and $x < \frac{c}{d}$ both derivatives are positive and so the trajectory is increasing towards $(\frac{a}{b}, \frac{c}{d})$ and away from (0,0). If $y < \frac{a}{b}$ and $x > \frac{c}{d}$, x is increasing and y deceasing. If $y > \frac{a}{b}$ and $x < \frac{c}{d}$, x is decreasing and y increasing. Finally, if $y > \frac{a}{b}$ and $x > \frac{c}{d}$, both derivatives are negative so both x and y are decreasing.



Competitive Hunter System

The graph above displays several trajectories in the phase plane of one typical competitive hunter model. The equilibrium point (0,0) is an unstable node and $(\frac{c}{d}, \frac{a}{b})$ is a saddle point. Thus, there is no stable equilibrium value meaning that the system is very dependent on the initial conditions. There is a separatrix approximately close to a line connecting two equilibrium points. It distinguishes two types of behavior: if the initial conditions are such that $x(0)$ is much larger than $y(0)$ (more precisely on the left side of the separatrix) the species x survives and y dies out. Otherwise, y survives and x dies out. Both species survive only if the initial condition point lies exactly on the separatrix. In this case, the (x, y)-values approach the saddle point and so $x \to \frac{c}{d}$ and $y \to \frac{a}{b}$ when $t \to \infty$. This off chance is the only case when the two species coexists together. So, in vast majority of cases, one of the two species dies out. This phenomenon is called the **principle of competitive exclusion**.

Predator-Prey Model. Let us consider the situation in which two species are such that one prays on the other. This leads to a predator-prey model. Let x denote the size of prey and y denote the size of predator population at time t. Then we can assume that x is growing at a rate proportional to the size of x but is decaying at a rate proportional to the number of interactions xy between the two species. The rate of y on the other hand, is increasing proportionally to the number of interactions xy and is decreasing proportionally to the size (because the more predators there are, less food to support all of them there will be). Thus the system describing the sizes of the two species is given by

$$\frac{dx}{dt} = ax - bxy \qquad \frac{dy}{dt} = -cy + dxy$$

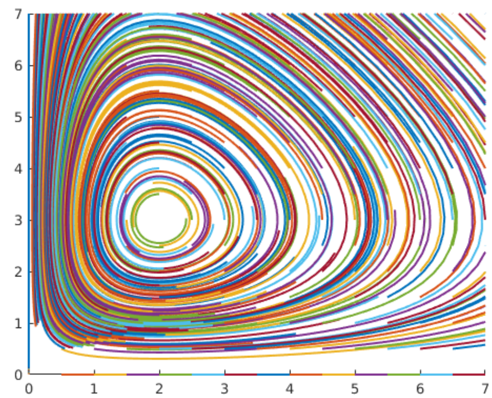
Note that the first two terms on the right side of the equations mean that

- In the absence of the predators, prey grows at a rate proportional to the size.

- In the absence of the prey, the predator dies out – thus the size is decreasing at a rate proportional to the size.

The equilibrium points of this system are $(0,0)$ and $(\frac{c}{d}, \frac{a}{b})$. Note that along the horizontal line $y = \frac{a}{b}$, $\frac{dx}{dt} = 0$ and so x is constant in time. Along the vertical line $x = \frac{c}{d}$, $\frac{dy}{dt} = 0$ and so y is constant in time.

If $y < \frac{a}{b}$ and $x < \frac{c}{d}$, x is increasing and y deceasing. If $y < \frac{a}{b}$ and $x > \frac{c}{d}$, both derivatives are positive so the both species are increasing in size. If $y > \frac{a}{b}$ and $x < \frac{c}{d}$, both derivatives are negative so the trajectory is decreasing. Finally, if $y > \frac{a}{b}$ and $x > \frac{c}{d}$, x is decreasing and y increasing. This gives us that the trajectories in the phase plane revolve about $(\frac{c}{d}, \frac{a}{b})$ so this equilibrium point is a center. (0,0) is a saddle point. Thus, (0,0) is unstable and $(\frac{c}{d}, \frac{a}{b})$ is stable but not asymptotically stable (i.e. there is not a single x and y value towards which the trajectories converge when $t \to \infty$).



Predator-Prey System

The graph above displays several trajectories in the phase plane of one typical predator-prey model. The equilibrium point (0,0) is a saddle point and $(\frac{c}{d}, \frac{a}{b})$ is a center point. The existence of a center guarantees that no species becomes extinct: an increase in x causes y's to increase. As a consequence, x's are hunted more and they decrease. This causes a decrease of y's also because the decrease in the food supplies. The decrease of y's causes x's to be hunted less and they start increasing again and so the cycle continues. This periodicity reflects the fact that the graphs of solutions in tx and ty planes are periodic curves. The initial conditions impacts the size of the amplitude and the horizontal and vertical shifts of x and y curves. Thus the species coexist regardless of the initial conditions.

The predator-prey model is also known as **Lotka-Volterra model** in honor of its creators Lotka and Volterra. The basic model can be modified depending on any additional assumptions. Let us consider two such scenarios.

1. **External factors diminishing the growth**. If the prey is being hunted by humans or another species so that its rate is decreasing proportionally to the population size, we can add the term $-kx$ to the right side of the first equation. This decrease in the size of prey results in a decrease in the size of the predators as well and it is feasible to assume that the predator size decreases also proportionally to the population size. So, the term $-ry$ can be added to the right side of the second equation.

   With those assumptions, we arrive to the modified model

$$\frac{dx}{dt} = ax - bxy - kx \qquad \frac{dy}{dt} = -cy + dxy - ry$$

   The additional terms result in the shift of the center point from $(\frac{c}{d}, \frac{a}{b})$ to $(\frac{c+r}{d}, \frac{a-k}{b})$. Note that this means that a moderate hunt of the prey in fact increases the average level of the prey and decreases the average level of the predator population. This conclusion is known as Volterra's principle.

The above model also applies to the cases when both predator and prey population are decreasing as a result of an external force (for example pollution, insecticides etc).

2. **Internal competitiveness of the prey.** One criticism of the Lotka-Volterra models is that in the absence of predators, prey is increasing without bound. This can be corrected by getting the logistic instead of exponential solution for x when $y = 0$. In this case, the equilibrium point becomes asymptotically stable (either a node or a spiral). This can be achieved when taking into consideration the level of internal competitiveness within prey population. In this case, we can add the term $-kx^2$ to the first equation. Thus the model becomes $\frac{dx}{dt} = ax - bxy - kx^2 \quad \frac{dy}{dt} = -cy + dxy$.

   In this case, the center point shifts from $(\frac{c}{d}, \frac{a}{b})$ to $(\frac{c}{d}, \frac{a}{b} - \frac{kc}{bd})$. Note although the added term diminish the derivative of x, the long term effect is that the y-coordinate of the equilibrium point decreases.

3. **Internal competitiveness of both populations.** Another criticism of the basic model is that the system is unrealistic because it is not asymptotically stable whereas most observed natural systems tend to converge to equilibrium values. One model that does exhibit oscillations but has asymptotically stable equilibrium value takes into consideration the level of internal competitiveness within *both* species. In this case, we can add the terms $-kx^2$ and $-ry^2$ to the equations respectively. Thus the model becomes

$$\frac{dx}{dt} = ax - bxy - kx^2 \qquad \frac{dy}{dt} = -cy + dxy - ry^2.$$

Similar modification of the basic hunter model is possible as well. If we incorporate the internal competitiveness and assume that the rate of decrease is proportional to the number of the encounters of members of the same species, the basic model $\frac{dx}{dt} = ax - bxy$ and $\frac{dy}{dt} = cy - dxy$ is modified to become

$$\frac{dx}{dt} = ax - bxy - kx^2 \qquad \frac{dy}{dt} = cy - dxy - ry^2.$$

Practice Problems.

1. Find all the equilibrium points of the following systems.

   (a) $\frac{dx}{dt} = x - x^2 - xy \qquad \frac{dy}{dt} = 0.75y - y^2 - 0.5xy$

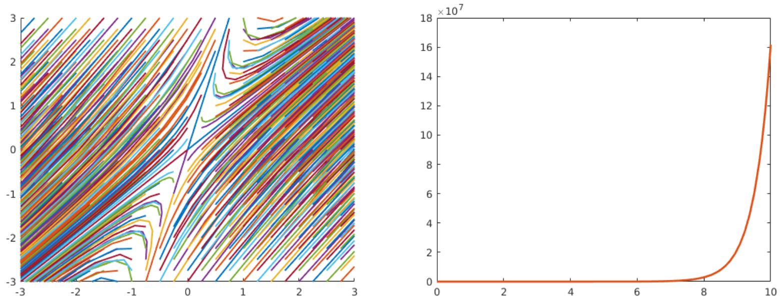
   (b) $\frac{dx}{dt} = x - x^2 - xy \qquad \frac{dy}{dt} = 0.5y - 0.25y^2 - 0.75xy$

2. Recall that the total charge Q of a simple series circuit with a resistance R, capacitance C, and an inductance L can be described using Kirchhoff's second law as a differential equation of the second order $LQ'' + RQ' + \frac{1}{C}Q = E(t)$. where $E(t)$ is the impressed voltage at time t. Write this equation as a system of two first order differential equations.

3. Consider the system $\frac{dx}{dt} = 3x - y$ and $\frac{dy}{dt} = 4x - 2y$.

(a) Find the solution of this system with initial conditions $x(0) = 1$ and $y(0) = 3$ using the Laplace Transform.

(b) The point (0,0) is the only equilibrium point of the system. The first graph below represents the trajectories in the phase plane and the second graph represents the graph of solutions with initial conditions $x(0) = 1$ and $y(0) = 3$.

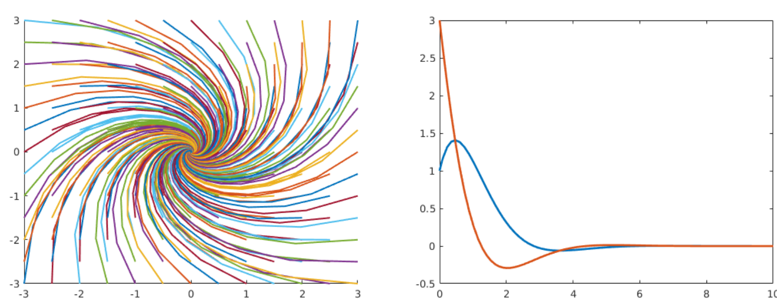


Use the graphs to classify the equilibrium point (0,0), determine its stability and the direction of the trajectories in the phase plane. Use your conclusions to determine the limiting values of the general solutions x and y for $t \to \infty$ with the initial conditions $x(0) = 1$ and $y(0) = 3$.

4. Consider the system $\frac{dx}{dt} = -x + y$ and $\frac{dy}{dt} = -x - y$

(a) Find the solution of this system with initial conditions $x(0) = 1$ and $y(0) = 3$ using the Laplace Transform.

(b) The point (0,0) is the only equilibrium point of the system. The first graph below represents the trajectories in the phase plane and the second graph represents the graph of solutions with initial conditions $x(0) = 1$ and $y(0) = 3$.

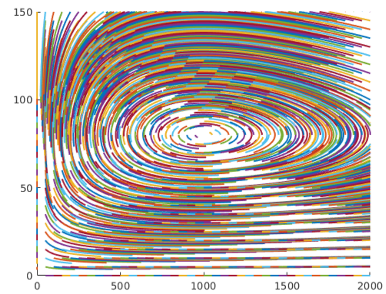


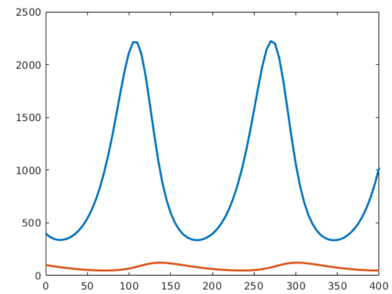
Use the graphs to classify the equilibrium point (0,0), determine its stability and the direction of the trajectories in the phase plane. Compare your conclusions with the answers of part (a) to ensure they agree.

5. The sizes R and W of a population of rabbits and a population of wolves are described using the predator-prey model with $(a, b, c, d) = (0.08, 0.001, 0.02, 0.00002)$.

(a) Find the equilibrium points.

(b) Consider the following graphs of trajectories in the phase plane and the solutions with initial conditions $R(0) = 400$ and $W(0) = 100$ to classify the equilibrium points and determine their stability. Indicate the direction in which the curves in the phase place are traced as the parameter increases and discuss the long term tendencies of the system.





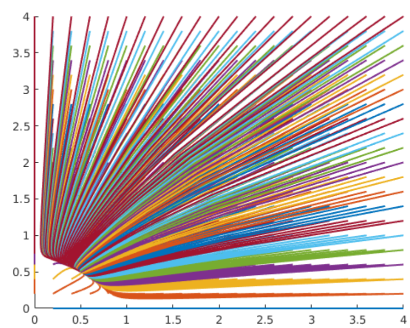
(c) Using the second graph, estimate the values in between which the solutions oscillate.

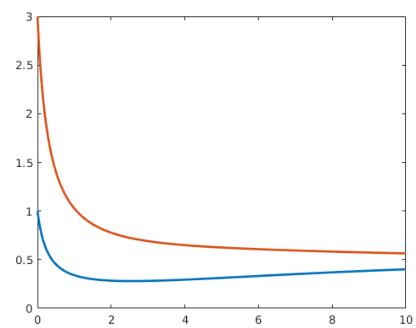
6. Suppose that there are two competing species in a closed environment. Let x and y denote the sizes of two populations at time t measured in thousands.

   (a) Assume that the rate of change of the populations is governed by the equations.

$$\frac{dx}{dt} = x - x^2 - xy \qquad \frac{dy}{dt} = 0.75y - y^2 - 0.5xy$$

   The first graph below contains a multitude of solutions in the phase plane. The graph of the solutions with initial conditions $x(0) = 1$ and $y(0) = 3$ is given on the right.



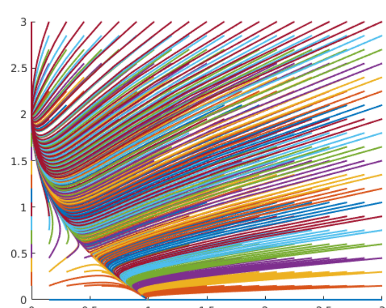


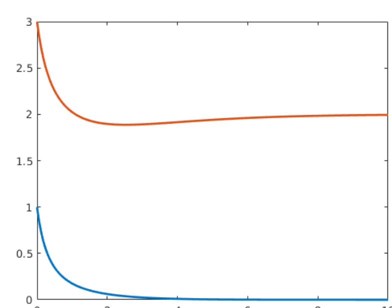
   Find the equilibrium points of the system and discuss their stability. Discuss the long term behavior and provide biological interpretation.

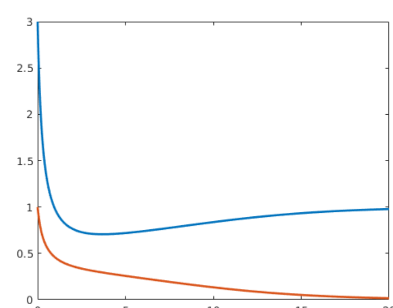
   (b) Assume that the changes in environment cause the coefficients in the second equation to change. The modified system becomes

$$\frac{dx}{dt} = x - x^2 - xy \qquad \frac{dy}{dt} = 0.5y - 0.25y^2 - 0.75xy$$

   The first graph below contains a multitude of solutions in the phase plane. The graph of the solutions with initial conditions $x(0) = 1$ and $y(0) = 3$ is in the middle given and the graph of the solutions with initial conditions $x(0) = 3$ and $y(0) = 1$ is on the right.







Find the equilibrium points of the system and discuss their stability as well as their *type*. Discuss the long term behavior and provide biological interpretation.

Solutions.

1. (a) The first equation $x - x^2 - xy = 0$ factors as $x(1-x-y) = 0$. In the first case, $x = 0$ and in the second $x = 1 - y$.

   In the first case ($x = 0$), the second equation becomes $0.75y - y^2 = 0$. This has two solutions $y = 0$ and $y = 0.75$. Thus, the first case yield two equilibrium points (0,0) and (0, 0.75).

   In the second case ($x = 1-y$), the second equation becomes $0.75y - y^2 - 0.5(1-y)y = 0 \Rightarrow y(0.75 - y - 0.5 + 0.5y) = 0 \Rightarrow y(0.25 - 0.5y) = 0 \Rightarrow y = 0$ and $y = 0.5$. Since $x = 1 - y$, the two corresponding x-values are $x = 1$ and $x = 0.5$. Thus, the second case produces two equilibrium points (1,0) and (0.5, 0.5).

   So, there are four equilibrium points total (0,0), (0, 0.75), (1,0), and (0.5, 0.5).

   (b) The first equation $x - x^2 - xy = 0$ factors as $x(1-x-y) = 0$. In the first case, $x = 0$ and in the second $x = 1 - y$.

   In the first case ($x = 0$), the second equation becomes $0.5y - 0.25y^2 = 0 \Rightarrow \frac{1}{4}y(2-y) = 0 \Rightarrow y = 0$ and $y = 2$. Thus, the first case yield two equilibrium points (0,0) and (0, 2).

   In the second case ($x = 1-y$), the second equation becomes $0.5y - 0.25y^2 - 0.75(1-y)y = 0 \Rightarrow \frac{1}{4}y(2 - y - 3 + 3y) = 0 \Rightarrow y(2y-1) = 0 \Rightarrow y = 0$ and $y = 0.5$. Since $x = 1 - y$, the two corresponding x-values are $x = 1$ and $x = 0.5$. Thus, the second case produces two equilibrium points (1,0) and (0.5, 0.5).

   So, there are four equilibrium points total (0,0), (0, 2), (1,0), and (0.5, 0.5).

2. Putting $y_1 = Q$ and $y_2 = Q'$, the differential equation becomes a system: $y_1' = y_2$ and $y_2' = -\frac{R}{L}y_2 - \frac{1}{LC}y_1 - \frac{E(t)}{L}$.

3. (a) Let $X = \mathcal{L}[x]$ and $Y = \mathcal{L}[y]$. Taking $\mathcal{L}$ of both equations gives you $sX - 1 = 3X - Y$ and $sY - 3 = 4X - 2Y$. From the first equation $Y = -sX + 3X + 1$. Plugging that in the second gives you $s(-sX + 3X + 1) - 3 = 4X - 2(-sX + 3X + 1) \Rightarrow s - 3 + 2 = s^2X - 3sX + 4X + 2sX - 6X \Rightarrow X = \frac{s-1}{s^2-s-2} = \frac{s-1}{(s-2)(s+1)}$. Thus $Y = \frac{(-s+3)(s-1)+s^2-s-2}{s^2-s-2} = \frac{3s-5}{s^2-s-2} = \frac{3s-5}{(s-2)(s+1)}$. Using partial fraction decomposition $X = \frac{s-1}{(s-2)(s+1)} = \frac{1/3}{s-2} + \frac{2/3}{s+1}$ and $Y = \frac{3s-5}{(s-2)(s+1)} = \frac{1/3}{s-2} + \frac{8/3}{s+1}$. Taking inverse Laplace transform, $x = \mathcal{L}^{-1}[X] = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$ and $y = \mathcal{L}^{-1}[Y] = \frac{1}{3}e^{2t} + \frac{8}{3}e^{-t}$.

   (b) From the first graph, we can conclude that (0,0) is a saddle point (thus unstable). There is a separatrix (line with positive slope passing the origin). The solutions (x, y) with initial conditions on the left of the separatrix converge to $(-\infty, -\infty)$. The solutions (x, y) with initial conditions on the right of the separatrix converge to (∞, ∞). Since (1,3) is on the right of separatrix, $x \to \infty$ and $y \to \infty$. The second graph supports this conclusion.

4. (a) Let $X = \mathcal{L}[x]$ and $Y = \mathcal{L}[y]$. Taking $\mathcal{L}$ of both equations produces $sX - 1 = -X + Y$ and $sY - 3 = -X - Y$. From the first equation $Y = sX + X - 1$. Plugging that in the second gives you $s(sX+X-1) - 3 = -X - (sX+X-1) \Rightarrow s^2X + 2sX + 2X = s + 4 \Rightarrow X = \frac{s+4}{s^2+2s+2}$. Thus, $Y = \frac{s^2+4s+s+4-s^2-2s-2}{s^2+2s+2} = \frac{3s+2}{s^2+2s+2}$.

Complete the denominator to a sum of squares: $s^2+2s+2 = s^2+2s+1+1 = (s+1)^2+1$. Then $x = \mathcal{L}^{-1}[X] = \mathcal{L}^{-1}[\frac{s+1+3}{(s+1)^2+1}] = \mathcal{L}^{-1}[\frac{s+1}{(s+1)^2+1} + 3\frac{1}{(s+1)^2+1}] = e^{-t}\cos t + 3e^{-t}\sin t$ and $y = \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}[\frac{3s+2}{s^2+2s+2}] = \mathcal{L}^{-1}[\frac{3(s+1)-1}{(s+1)^2+1}] = 3e^{-t}\cos t - e^{-t}\sin t$.

(b) From the first graph, we can conclude that (0,0) is an asymptotically stable spiral point. The second graph supports this conclusion. Hence, $x \to 0$ and $y \to 0$ when $t \to \infty$ regardless of initial conditions. The answers in part (a) agree with this conclusion since $e^{-t} \to 0$ for $t \to \infty$ so $x \to 0$ and $y \to 0$.

5. The system is $\frac{dR}{dt} = 0.08R - 0.001RW$ $\frac{dW}{dt} = -0.02W + 0.00002RW$.

(a) Set the equations to zero. Multiply the first by 1000 to avoid decimals. Get $80R - RW = 0 \Rightarrow R(80-W) = 0 \Rightarrow R = 0$ or $W = 80$. In the first case, the second equation is $-0.02W = 0 \Rightarrow W = 0$ giving you the first critical point (0,0).

In the second case, the second equation is $-1.6 + 0.0016R = 0 \Rightarrow -16000 + 16R = 0 \Rightarrow R = \frac{16000}{16} = 1000$. So, (1000, 80) is the second critical point.

(b) From the phase plane graph, we can see that the equilibrium point (1000, 80) is a center. Thus, it is stable but not asymptotically stable. (0,0) is a saddle point and it is unstable. The solutions oscillate: x-values about 1000 and y-values about 80. The amplitude of a solution depends on the initial conditions. The second graph also enables us to determine the direction of the trajectories: since starting with 400, the x values decrease a bit at first but then increase and the y-values decrease first starting at 100, we can conclude that the curves in the xy-plane are traversed in the counter clock-wise direction.

(c) With 400 rabbits and 100 wolves initially, the number of rabbits oscillates between about 460 and 2200 and the number of wolves between about 50 and 200.

6. (a) The system is the same as the one in problem 1 (a). Solving on the same way, we obtain four critical points (0,0), (0,0.75), (1,0) and (0.5,0.5). The first three involve the extinction of at least one of the two species and the last one corresponds to co-existence of both. The trajectories accumulate at (0.5, 0.5) indicating this to be a stable node.

From the second graph, we see the trajectory starting at $(1,3)$ converges towards (0.5, 0.5) when $t \to \infty$. This supports the conclusion that (0.5,0.5) is an asymptotically stable node. The other three points are unstable (in fact (1, 0) and (0, 0.75) are saddle points and (0,0) is an unstable node). This means that any trajectory with positive initial x and y-values approaches (0.5, 0.5). In context of the problem, this means that the number of both population approach 500 members after some period of time provided that both populations have at least one member initially.

(b) The system is the same as the one in problem 1 (b). Solving on the same way, we obtain four critical points (0,0), (0,2), (1,0) and (0.5,0.5).

The graph indicates certain clustering around both (0,2) and (1,0). This indicates that some trajectories converge to (0,2) and some other to (1,0). These two points are asymptotically stable nodes. The points (0,0) and (0.5, 0.5) are unstable. (0.5, 0.5) is a saddle points and there is a separatrix passing (0.5, 0.5) separating two possible outcomes: if the initial conditions are above separatrix, then $(x,y) \to (0,2)$. If the initial conditions are below separatrix, then $(x,y) \to (1,0)$. The point (0,0) is an unstable node.

The second two graphs confirm this hypothesis: if $x(0) = 1$ and $y(0) = 3$, then (1,3) is above the separatrix and the solutions converge x to 0 and y to 2, and if $x(0) = 3$ and $y(0) = 1$, then (3,1) is below the separatrix and the solutions converge x to 1 and y to 0.

These conclusions have biological interpretation too: the coexistence is possible just in the off chance scenario that the initial conditions are exactly on the separatrix. In any other case, one species eventually overwhelms the other resulting in the extinction of the other. Which species survives depends on the initial conditions. If the initial conditions are such that the x's survive, the trajectory ends up at the node (1,0) resulting in 1000 members of the first and zero of the second population. On the other hand, if the initial conditions are such that the y's survive, the trajectory ends up at the node (0,2) resulting in 2000 members of the second and zero of the first population.

Systems with infinitely many equilibrium points

In some cases, two equations with infinitely many solutions could be obtained when finding the equilibrium solutions. For example, this happens when the coefficients of one equation are proportional to the coefficients of the other equation. Note that in this case $r = 0$ is a solution of the quadratic equation in r which takes over the role of the characteristic equation (see the section summarizing the types).

If there are infinitely many critical points, there is no equilibrium point, but an equilibrium line or every point in the plane is an equilibrium point. The following example illustrates the situation with an equilibrium line.

Example 1. A car rental company has distributors in Orlando and Tampa. The company specializes in catering to travel agents who want to arrange tourist activities in both cities. Consequently, a traveler may rent a car in one city and return it in another. The company wants to determine how much to charge for this drop-off convenience. Let us assume that 60% of the cars rented in Orlando are returned there and 70% rented in Tampa are returned there. If there are 7000 cars total to be distributed between the two cities, determine the number of cars in either city in the following two scenarios: (1) There are 2000 cars in Orlando and 5000 cars in Tampa initially. (2) There are no cars in Orlando and all 7000 in Tampa initially.

Solution. To model this situation, consider the following variables. Let t denote the time, O the number of cars in Orlando at time t, and T the number of cards in Tampa at time t. The mathematical model of the system of two differential equations in O and T could be obtained by considering the following daily tendencies.

- Orlando is loosing 40% cars to Tampa and is gaining 30% of the number of cars from Tampa.

- Tampa is loosing 30% cars to Orlando and is gaining 40% of the number of cars from Orlando.

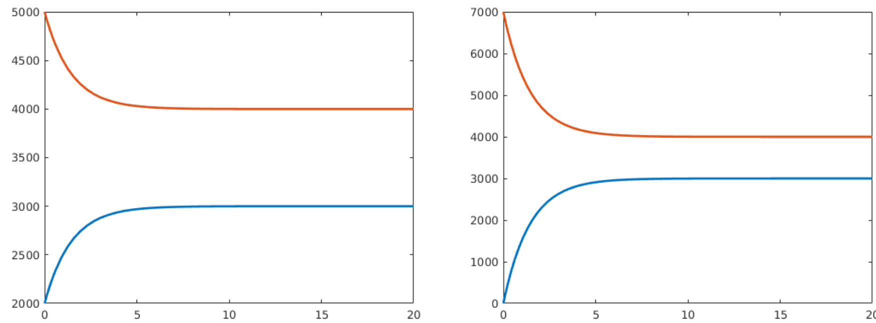
Thus

$$\frac{dO}{dt} = -0.4O + 0.3T \qquad \text{and} \qquad \frac{dT}{dt} = 0.4O - 0.3T$$

When we set the equations to zero, we obtain a system that does not have a unique solution but infinitely many solutions described by the condition $O = \frac{3}{4}T$. This condition tells us the optimal ratio of cars in the two cities.

Thus, if the total number of cars is 7000, $O+T=7000 \longrightarrow \frac{3}{4}T+T=7000 \longrightarrow \frac{7}{4}T=7000 \longrightarrow T=4000$ and $O=\frac{3}{4}400=3000$. (More generally, if m is the total number of cars, from $O+T=m$ we obtain that the limiting values of O and T would be $\frac{3}{7}m$ and $\frac{4}{7}m$ respectively.)

Thus, we can conclude that in both scenarios $O(0)=2000, T(0)=5000$ and $O(0)=0, T(0)=7000$, the distributions of the cars on the long run is $O=3000$ and $T=4000$. The graphs of the two solutions corresponding to two scenarios are on the figure below.



Two Scenarios in Orlando-Tampa Example

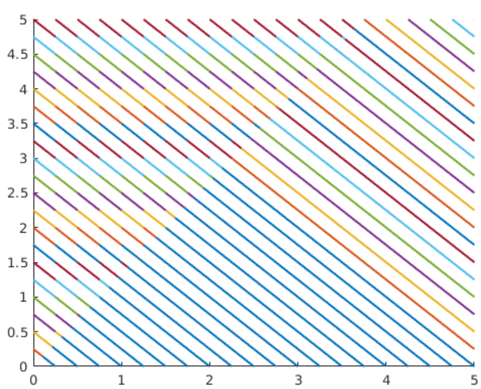
If this system of differential equations is solved, for example using Matlab, one obtains the solutions $O=3c_1+c_2e^{-.7t}$ and $T=4c_1-c_2e^{-.7t}$. When $t\to\infty$, $O\to 3c_1$ and $T\to 4c_1$. Since $O(0)=3c_1$ and $T(0)=4c_1$, $m=7c_1$ is the total number of cars. Thus $O\to\frac{3}{7}m$ and $T\to\frac{4}{7}m$ when $t\to\infty$.

Note also that dividing the two equations would yield

$$\frac{dO}{dT}=\frac{\frac{dO}{dt}}{\frac{dT}{dt}}=\frac{-0.4O+0.3T}{0.4O-0.3T}=\frac{-(0.4O-0.3T)}{0.4O-0.3T}=-1$$

Separating the variables produces $dO=-dT$ and integrating both sides produces $O=-T+c$. Using again that m is the total number of cars, we obtain that $c=m$.

Thus, the trajectories in the phase plane are parallel line segments with slope -1, presented on the graph on the right. The direction of the trajectories is towards the point $(\frac{3}{7}m,\frac{4}{7}m)$ which is the intersection of the line $O=-T+m$ with the line $O=\frac{3}{4}T$. For any given value of initial conditions $(T(0),O(0))$ the graph of the particular solution is a line segment on $O=-T+m$ ending at the intersection of $O=-T+m$ and $O=\frac{3}{4}T$.



Example 2. Consider a three party system with Republicans, Democrats, and Independents. Assume that in the next election 75% of those that voted Republican again vote Republican, 20% vote Democrat and 5% vote Independent. Of those that voted Democrat, 80% vote Democrat again, 10% vote Republican and 10% vote Independent. Of those that voted Independent, 60% vote Independent again, 10% vote Republican and 30% vote Democrat. Assuming that these tendencies continue from election to election and that no voters leave the system, estimate the long term tendencies.

Solution. Let R, D and I stands for the number of Republican, Democrat and Independent voters respectively at the time t. Note that

1. The rate of change of R : decreases by 25%R and increases by 10% D and by 10% I.
2. The rate of change of D : decreases by 20% D and increases by 20% R and 30% I.
3. The rate of change of I : decreases by 40% I, and increases by 5%R and 10% D.

The system of differential equations that models this is:

$$\begin{array}{rcrrr} \frac{dR}{dt} & = & -0.25R & +0.1D & +0.1I \\ \frac{dD}{dt} & = & 0.20R & -0.2D & +0.3I \\ \frac{dI}{dt} & = & 0.05R & +0.1D & -0.4I \end{array}$$

Getting a system of three ordinary equations by setting the right side to zero and solving it for (R, D, I) we obtain $(\frac{10}{6}I, \frac{19}{6}I, I)$. Note that if we assume that $R + D + I = 100\%$ (we assume that the number of voters remains the same for many elections), we have that $(\frac{10+19+6}{6})I = \frac{35}{6}I = 1$ and so $I = \frac{6}{35} = 17.14\%$ $R = \frac{10}{35} = 28.57\%$ and $D = \frac{19}{35} = 54.29\%$.

Practice Problems.

1. Assume that two lakes are connected by a water flow. Suppose also that the measurement of the pollution indicated that 10% of the pollution of the first lake comes from the other lake. For the second lake, assume that 65% of the pollution comes from the first lake. Represent this situation with a system of differential equations. Find the equilibrium values of the system and discuss the long term behavior.

2. There are three delivery restaurants near a university called Pizza Paradise, Quick Burger and Noodles Unlimited. They are trying to get as much customers out of 3000 university undergraduates as possible. A survey conducted showed that 80% of those that ordered pizzas in the past again order pizzas, 15% switch to burgers and 5% to noodles. Of those that ordered burgers, 60% order burgers again, 10% order pizzas and 30% order noodles. Of those that ordered noodles, 70% order noodles again, 10% switch to burgers and 20% to pizzas. Assuming that these tendencies continue and that the number of students remains constant, estimate the long term tendencies.

Solutions.

1. Let a and b be the total amounts of pollution in two lakes respectively after time t. System: $a' = -0.65a + 0.10b$ and $b' = 0.65a - 0.10b$. From here, $b = \frac{13}{2}a$. Assume that there is no new pollution added to either lake (thus $a + b$ is constant). If we denote the total amount of pollutant with $a + b = c$, the long term stable state of the system are $a = \frac{2}{15}c$ and $b = \frac{13}{15}c$. Thus about 13% of the pollutants ends up in the first and 87% in the second lake.

2. Let P, B and N denote the number of students that order pizzas, burgers and noodles respectively. The system becomes: $P' = -0.20P + 0.1B + 0.2N$, $B' = 0.15P - 0.4B + 0.1N$, and $N' = 0.05P + 0.3B - 0.3N$. From here, $P = \frac{18}{13}N$ and $B = \frac{10}{13}N$. Substituting that in $P + B + N = 3000$ and solving for N gives us $N = 951$. Then $P = 1317$ and $B = 732$.