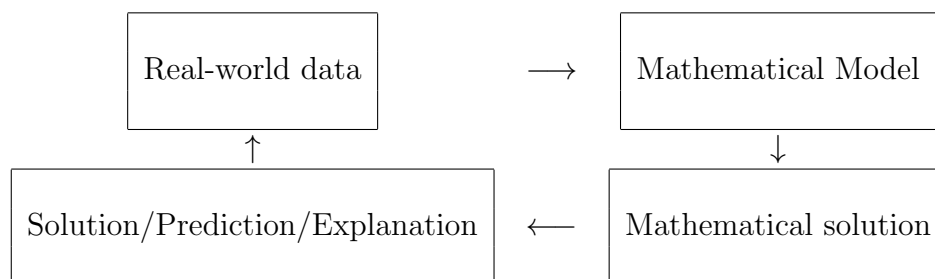


## Mathematical Modeling – Introduction and early examples

The process of mathematical modeling can be broken down into the following stages. In order to successfully perform each stage, it might be helpful to ask yourself the questions listed after each step.

1. **Identify the real problem. Identify the problem variables.** What do we need to find out? What is the problem asking for?
2. **Construct appropriate relations between the variables.** These relations, equations or functions, represent a mathematical model. Useful questions: What are we given? What do we already know? What is the relation between known and unknown quantities or data?
3. **Create a model that adequately describes the problem.** What is the simplest relation or equation that will model the data? Can diagrams help? Can unit consideration help? Finding the answers might require:
  - Taking measurements, collecting data.
  - Deciding on mathematical techniques to be used.
  - Using the appropriate technology.
  - Estimating the values of parameters within the model that cannot be measured or calculated.
4. **Obtain the mathematical solution.** What math techniques do I need? Do I need to write a program or computer simulation?
5. **Interpret the mathematical solution.** What does the mathematical answer mean in context of the original problem?
6. **Test the validity and the effectiveness of the model. Compare with reality.** Validity: Do your answers make sense? Do the predictions agree with real data? Do the values have correct sign? Correct units? Correct size? Do they increase or decrease when they should? Are they linear (non-linear, exponential, logarithmic etc) as they should? Effectiveness: Could a simpler model be used? Have I found a right balance between greater precision (i.e. greater complexity) and simplicity?
7. **Present your results. Write a report.** Who will be reading the report? how detailed I need to be? How can I get the right balance between including all the important facts and still be clear and concise? Keep in mind: a clear **introduction** is *essential*. In the **main part** of the report, explain your variables and methodology clearly. Clearly define the problem statement, variables, methods used and explain your mathematical model. Clearly interpret the solution and draw the appropriate **conclusions**.

The following diagram represents the above steps.



Things to keep in mind when learning about mathematical modeling:

1. Learning to apply mathematics is very different from learning mathematics. As “pure” mathematics is usually perceived as a list of specific procedures, techniques, theorems, and rules, “applied” mathematics is used for solving a wide range of problems, many of which do not seem mathematical in nature.
2. There are no precise set of rules on how to create a model.
3. Modeling can be learnt only by solving problems.

## Mock-up example

**Problem.** If there are 11 coins, nickels and dimes, valuing 70 cents, how much of each are there?

**Discussion.** First you try to identify the variables needed. Keeping in mind what the problem is asking you to find – that might be helpful when you decide on the relevant variables. This problem is asking for the number of nickels and the number of dimes. Thus, let  $x$  = number of nickels and  $y$  = number of dimes. The natural next step is to create a system of two equations with two unknowns: the first one describing the fact that there are 11 coins all together:  $x + y = 11$  and the second one describing the fact that the value of  $x$  many coins worth 5 cents each and the value of  $y$  many coins worth 10 cents each should add to the total value of 70 cents. Thus, the second equation is  $5x + 10y = 70$ .

**Solution.** Using the usual methods (elimination of variable or your other favorite method for solving a system of linear equations) gives us  $x = 8$  and  $y = 3$ . Interpreting this mathematical answer in context of the problem gives us the answer that there are 8 nickels and 3 dimes. Testing the validity: do 3 and 8 really add to 11? Does 8 times 5 cents plus 3 times 10 cent equal 70 cents?

## Modeling using ordinary differential equations

**Problem.** A bacteria culture starts with 500 bacteria and grows at a rate proportional to its size. After 3 hours there are 8000 bacteria. Find the number of bacteria after 4 hours.

**Discussion.** Identifying variables: let  $y$  stands for the bacteria culture and  $t$  stands for time passed. The first part of the problem “A bacteria culture starts with 500 bacteria..” tells us that  $y(0) = 500$ . The second part “... and grows at a rate proportional to its size.” is the key for getting the mathematical model. Recall that the rate is the derivative and that “...is

proportional to..“ corresponds to ” equal to constant multiple of..“ So, the equation relating the variables is  $\frac{dy}{dt} = ky$ . The solution of this differential equation is  $y = y_0e^{kt}$ . Since  $y_0 = 500$ , it remains to determine the proportionality constant  $k$ . From the condition ”After 3 hours there are 8000 bacteria.“ we obtain that  $8000 = 500e^{3k}$  which gives us that  $k = \frac{1}{3} \ln 16 = .924$ . Thus, the number of bacteria after  $t$  hours can be described by  $y = 500e^{.924t}$ .

**Solution.** Using the function we have obtained, we find the number of bacteria after 4 hours to be  $y(4) = 20159$  bacteria.