Mathematical Modeling
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Modeling with Systems of Difference and Differential Equations

If we are monitoring how two quantities are changing in time simultaneously, a mathematical model for this situation may require a **system** of two or more equations. The informal formula describing the change *change = future value − present value* may be modified into a system of equations in cases like this.

If we monitor the values during discrete periods (for example, discrete time intervals), we obtain a system of difference equation. If the independent variable varies continuously (for example, time increasing continuously), we arrive to a system of differential equations.

Let us consider systems of difference equations first. As in the single variable case, such system corresponds to a system of coupled recursive equations. Conversely, two or more coupled recursive equations determine a system of difference equations.

$$\begin{array}{lcl} a_{n+1} = f(a_n, b_n) & \Longleftrightarrow & \Delta a_n = f(a_n, b_n) - a_n \\ b_{n+1} = g(a_n, b_n) & \Longleftrightarrow & \Delta b_n = g(a_n, b_n) - b_n \end{array}$$

Similarly as with single difference equation, when dealing with systems of difference equations, we are interested in equilibrium values and their stability. Letting $n \to \infty$ in the above formula and denoting limit of a_n by a and limit of b_n by b, we obtain two equations from which we obtain the equilibrium solutions (steady states of the system).

$$\begin{array}{lcl} a = f(a, b) & \Longleftrightarrow & 0 = f(a, b) - a \\ b = g(a, b) & \Longleftrightarrow & 0 = g(a, b) - b \end{array}$$

After determining the equilibrium points, we are interested in their stability. The equilibrium values provide the insight in the long-term behavior of the system and demonstrate if the system has periodic behavior or not, if there are oscillations, if the numerical solutions are sensitive on the initial conditions or not and how sensitive is the system to the changes of parameters in the model.

Example 1. Consider the following dynamical system

$$a_{n+1} = \frac{1}{1 + b_n}, \quad b_{n+1} = \frac{1}{4 + a_n}$$

Find its steady states and discuss its behavior for any positive value of initial values.

Solution. To find the equilibrium solutions, solve the equations $a = \frac{1}{1+b}$, $b = \frac{1}{4+a}$ for a and b. Get $a = \frac{1}{1+\frac{1}{4+a}} = \frac{4+a}{5+a} \longrightarrow 0 = \frac{4+a-5a-a^2}{5+a} = \frac{4-4a-a^2}{5+a} \longrightarrow a = 0.828$ and $a = -4.828$. By analyzing the sign of $\frac{4-4a-a^2}{5+a}$, we conclude that 0.828 is stable (-4.828 is unstable state). The state $a = 0.828$ corresponds to $b = \frac{1}{4+0.828} = 0.207$. Thus, for any positive value of initial conditions, $a_n \to 0.828$ and $b_n \to 0.207$.

Example 2. When monitoring the variations of the price of a product, it is observed that a high price for the product in the market attracts more suppliers thus increasing the quantity of items sold. In particular, the price of above 50 dollars, attracts more suppliers increasing the quantity sold for 0.5 items per dollar. However, increasing the quantity of the product tends to drive the price down. In particular, producing more than 200 items decreases the price by 0.1 dollars per item.

If the current price is 30 dollars and 100 items are produced, write down a model that can be used to predict the price and quantity in subsequent years. Then find the steady states of the system and explain it behavior.

Solution. In this situation, there is an interaction between price and quantity over time. If n denotes the number of years, P_n denotes the price at year n and Q_n denotes the quantity produced at year n, the following system of difference equations can be used to model this situation.

$$\Delta P_n = -0.1(Q_n - 200) \qquad \Delta Q_n = 0.5(P_n - 50)$$

The steady state of the system is $P = 50$ and $Q = 200$. Depending on the initial values, the values of the price oscillate about the steady state value of 50 dollars and the values of quantity oscillate about the steady state value of 200 items: too large quantity will drive the price down, low price will cause low numbers of suppliers, this will cause the quantity to decrease. Lower quantity will then drive the price up. High price will cause high number of suppliers and that will cause an increase of the quantity and this trend will continue periodically.

The price and quantity in subsequent years can be predicted by dynamical system corresponding to the difference equations system.

$$P_{n+1} - P_n = -0.1(Q_n - 200) \qquad Q_{n+1} - Q_n = 0.5(P_n - 50) \quad \longrightarrow$$

$$P_{n+1} = P_n - 0.1(Q_n - 200) \qquad Q_{n+1} = Q_n + 0.5(P_n - 50)$$

With initial values $P_0 = 30$ and $Q_0 = 100$, we obtain the values $P_1 = 30-0.1(100-200) = 40$ and $Q_1 = 100-0.5(30-50) = 110$, $P_2 = 40-0.1(110-200) = 40+9 = 49$ and $Q_2 = 110+0.5(40-50) = 115$, $P_3 = 49 - 0.1(115 - 200) = 49 + 8.5 = 57.5$ and $Q_3 = 115 + 0.5(49 - 50) = 114.5$, etc.

In some cases, two equations with infinitely many solutions could be obtained when finding the equilibrium solutions. In those cases, we do not have a single equilibrium state but an equilibrium line. In those cases, we obtain specific solution depending on the initial conditions.

Example 3. Assume that two lakes are connected by a water flow (for example, consider lakes Ontario and Erie). Suppose also that the measurement of the pollution indicated that 10% of the pollution of the first lake comes from the second lake. For the second lake, the measurements indicate that 65% of the pollution comes from the first lake. Represent this situation with a system of difference equations. Find the equilibrium values of the system and discuss the long term behavior.

Solution. To model this situation, consider the following variables. Let n denote the number of years, Let a_n and b_n be the total amounts of pollution in two lakes respectively after n years. In this case

$$a_{n+1} = 0.35a_n + 0.10b_n \qquad \text{and} \qquad b_{n+1} = 0.65a_n + 0.90b_n$$

The equilibrium values of this system would represent the amount of pollutant that would remain the lakes on the long run. If a denotes the limit of a_n (and a_{n+1}) and b denotes the limit of b_n (and b_{n+1}), we can find the equilibrium solutions from

$$a = 0.35a + 0.10b \qquad \text{and} \qquad b = 0.65a + 0.90b$$

Solving for a and b, we obtain the condition $b = \frac{65}{10}a = 6.5a$. This is the relation between the steady states that, in this case, lies on a line. The relation determines the limiting ratio of pollutant in the two lakes.

Assuming that there is no new pollution added to either lake (thus $a + b$ is constant) we can let 100% represent the total amount of pollutant. In this case, $a + b = 1 \longrightarrow a + 6.5a = 1 \longrightarrow 7.5a = 1 \longrightarrow a = 0.133 = 13\%$ and $b = 6.5a = .8667 \approx 87\%$. Thus, about 13% of the pollutants will end up in the first and 87% in the second lake.

More generally, if m denotes the total amount of pollutant, $a + b = m \longrightarrow 7.5a = m \longrightarrow a = 0.13m$ and $b = .87m$.

Probabilistic Modeling. Markov Chains. Note that the coefficients of the system in previous example vary in a probabilistic manner. If a system involves a finite number of states such that the sum of the probabilities for the transition from a present state is 1 for each state, such system is called a **Markov chain.** In a Markov chain, the system may move from one state to another, one for each step, and there is a probability associated with each transition for each possible outcome. The sum of probabilities of transitioning from present state to the next state is 1. For example, in the example considered above, the probabilities that pollution in the first lake comes from that lake (90%) and that it comes from the second lake (10%) add up to 1.

Thus, a Markov chain can be represented as a dynamical system such that the coefficients of the variables on the right side form a matrix with columns that add up to 1.

Difference versus Differential Equations. The previous problem could be modeled also using systems of differential equations. Noticing that

- The first lake looses 65% of its pollution to the second lake and is gaining 10% of the pollution from the second lake,

- The second lake looses 10% of its pollution to the first lake and is gaining 65% of the pollution from the first lake yearly,

we obtain the difference equations

$$\Delta a_n = -0.65a_n + 0.10b_n \qquad \text{and} \qquad \Delta b_n = 0.65a_n - 0.10b_n$$

or differential equations

$$\frac{da}{dt} = -0.65a + 0.10b \qquad \text{and} \qquad \frac{db}{dt} = 0.65a - 0.10b$$

Here t denotes the time and $a = a(t)$ and $b = b(t)$ denote the amounts of pollutant in the first and second lake at time t respectively. Setting the right sides to zero yields again the relation $b = 6.5a$ from example 3. The exact solution of the system of differential equations can be found using Matlab. They turn out to be $a = 2c_1 + c_2e^{-.75t}$ and $T = 13c_1 - c_2e^{-.75t}$. When $t \to \infty$, $a \to 2c_1$ and $b \to 13c_1$. Since $2c_1 + 13c_1 = m$ the total amount of pollutant, we have that $m = 15c_1 \longrightarrow c_1 = \frac{1}{15}m$. Thus $a \to \frac{2}{15}m = .13m$ and $b \to \frac{13}{16}m = .87m$ when $t \to \infty$.

Although finding exact functions representing how a and b depend on the time t might be challenging, sometimes it can be easier to find how the two solutions depend on each other. Namely, dividing two differential equations with each other will give us an equation with derivative $\frac{db}{da}$ on the left side. Namely,

$$\frac{db}{da} = \frac{\frac{db}{dt}}{\frac{da}{dt}} = \frac{0.65a - 0.10b}{-0.65a + 0.10b} = \frac{0.65a - 0.10b}{-(0.65a - 0.10b)} = -1$$

This gives us the solutions $b = -a + c$. Using again that m is the total amount of pollutant, we obtain $c = m$. Finding the intersection of the lines $b = -a + m$ and $b = 6.5a$ gives us the values $a = .13m$ and $b = .87m$.

Example 4. There are three delivery restaurants near a university called Pizza Paradise, Quick Burger and Noodles Unlimited. They are trying to get as much customers out of 3000 university undergraduates as possible. A survey conducted showed that 80% of those that ordered pizzas in the past month order pizzas again in the next month, 15% switch to burgers and 5% to noodles. Of those that ordered burgers, 60% order burgers again, 10% order pizzas and 30% order noodles. Of those that ordered noodles, 70% order noodles again, 10% switch to burgers and 20% to pizzas. Assuming that these tendencies continue and that the number of students remains constant, estimate the long term tendencies.

Solution. Let P_n, B_n and N_n denote the number of students that order pizzas, burgers and noodles respectively after n months. We can model the situation above by the following system

$$\begin{array}{rcl} P_{n+1} & = & 0.80P_n \quad +0.1B_n \quad +0.2N_n \\ B_{n+1} & = & 0.15P_n \quad +0.6B_n \quad +0.1N_n \\ N_{n+1} & = & 0.05P_n \quad +0.3B_n \quad +0.7N_n \end{array}$$

Note that all the columns of the matrix of coefficients on the right side add to 1. Thus, this system is a Markov chain. Finding the limiting values P, B and N of the system gives us

$$\begin{array}{rcl} P & = & 0.80P \quad +0.1B \quad +0.2N \\ B & = & 0.15P \quad +0.6B \quad +0.1N \\ N & = & 0.05P \quad +0.3B \quad +0.7N \end{array}$$

Solving the system for P, B, and N gives us $P = \frac{18}{13}N$, and $B = \frac{10}{13}N$. Note that if we assume that $P + B + N = 3000$, we have that $(\frac{18+10+13}{13})N = \frac{41}{13}N = 3000$ and so $N = 951$. Then $P = 1317$ and $B = 732$.

Alternatively, you can consider the following system of difference equations

$$\begin{array}{rcl} \Delta P_n & = & -0.20P_n \quad +0.1B_n \quad +0.2N_n, \\ \Delta B_n & = & 0.15P_n \quad -0.4B_n \quad +0.1N_n, \\ \Delta N_n & = & 0.05P_n \quad +0.3B_n \quad -0.3N_n. \end{array}$$

and find the equilibrium solutions by setting the right side 0.

Systems of Differential Equations. In general a system of first order differential equations in two unknown functions x and y has the form

$$\frac{dx}{dt} = f(x, y, t) \qquad \frac{dy}{dt} = g(x, y, t).$$

It might be helpful to think that the independent variable t denotes the time and the depend variables x and y denote the position (x, y) in xy-plane. In this case, xy-plane is referred to as a **phase plane**. The solution of the system is a parametric function $x = x(t)$ and $y = y(t)$. The curve $(x(t), y(t))$ is called a **trajectory**.

A system of two equations is **autonomous** or **homogeneous** if it is of the form $\frac{dx}{dt} = f(x, y)$ and $\frac{dy}{dt} = g(x, y)$ (that is if the variable t does not appear on the right side). When dealing with systems of autonomous differential equations, we are interested if the values of the system will remain close and approach the equilibrium value (steady state of the system) or not. The equilibrium values provide the insight in the long-term behavior of the system and demonstrate if the system has periodic behavior or not, if there are oscillations, if the numerical solutions are sensitive on the initial conditions or not and how sensitive is the system to the changes of parameters in the model.

System of differential equations in Matlab.

Symbolic Solutions. You can find the symbolic solutions of a system of differential equations by using the command **dsolve**. For example to solve the system $\frac{dx}{dt} = 2x - y$, $\frac{dy}{dt} = 3x - 2y$ you can use

[x,y] = dsolve('Dx = 2*x - y', 'Dy = 3*x - 2*y', 't')

If you have the initial conditions, $x(0) = 1$ and $y(0) = 2$, you can use

[x,y] = dsolve('Dx = 2*x - y', 'Dy = 3*x - 2*y', 'x(0) = 1', 'y(0) = 2', 't')

To graph these two solutions on the interval $0 \leq t \leq 20$, you can use

ezplot(x, [0,20]) **hold on** **ezplot(y, [0,20])** **hold off**

Numeric Solutions. Finding symbolic solutions might be very limiting. For example, many systems of differential equations cannot be solved explicitly in terms of elementary functions. For those equations or systems of equations, numerical methods are used in order to get the approximate solution. To find numeric solutions, you can use the command **ode45**. In order to use it, the system needs to be in the form $x' = f(x, y, t)$ and $y' = g(x, y, t)$ and the right sides of the equations should be represented as a vector using the command **inline** first. The function x can be represented as **y(1)** and the function y as **y(2)**. The first entry of the inlined function **f** is the right side of the first equation and the second entry of **f** is the right side of the second equation.

Example 5. Consider the system

$$\frac{dx}{dt} = 2x - x^2 - xy \qquad \frac{dy}{dt} = xy - y$$

with the initial conditions $x(0) = 1$ and $y(0) = 2$.

a) Find numerical solution for $0 \leq t \leq 20$.

b) Graph the solution for $0 \leq t \leq 20$.

c) Plot sufficiently many solutions of this system in the phase plane to determine the type of the equilibrium point (0,0).

Solution. First, we inline the right side of equation to be a function of independent variable t and the unknown functions x and y that are represented by **y(1)** and **y(2)** respectively. Thus,
f = inline('[2*y(1)-y(1)^2-y(1)*y(2); y(1)*y(2)-y(2)]','t','y');

a) The command **[t,y]=ode45(f,[0,20],[1;2])** creates a table of t, x and y values for $0 \leq t \leq 20$ starting at $t = 0$, $x = 1$ and $y = 2$ (note the initial conditions $x(0) = 1$ and $y(0) = 2$). Note that here **y** is a vector whose entries will be the values of **y(1)** and **y(2)**.

b) The command **ode45(f,[0,20],[1;2])** graphs the two solutions on the same plot as functions of t. The function x is graphed in blue and y in green. Note that both x and y approach 0 for large values of t. This may be relevant when determining the stability of the equilibrium point (0,0).

c) The following M-file can be used to graph the trajectories in the phase plane for $x(0)$ and $y(0)$ taking integer initial values between 0 and 5.

```
close all; hold on
for a = 0:1:5
   for b = 0:1:5 (modify these values to change the density and position of the curves)
      [t, y] = ode45(f, 0:0.2:20, [a; b]);
      plot(y(:,1), y(:,2)) (plots the entries of the vector y, i.e. the solutions x and y)
   end
end
hold off
axis([0 2 0 2]) (modify these values to change the window)
```

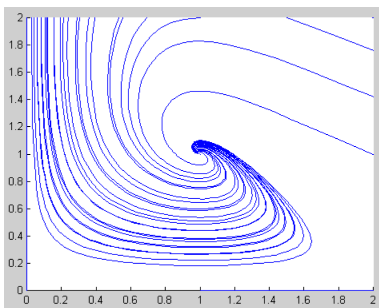

Figure 1: Phase plane graph

Stability of solutions. To find the equilibrium values of an autonomous system, set derivatives to 0 and solve for (x, y) (note that the same method was used when solving one first order autonomous equation $y' = f(y)$). Thus, finding equilibrium solutions would require solving

$$f(x, y) = 0 \qquad \text{and} \qquad g(x, y) = 0$$

Assuming that the point (a, b) is a solution of the equations above, it is call

- **asymptotically stable** if $x \to a$ and $y \to b$ when $t \to \infty$. Examples include: **node** and **spiral point**.
- **stable** if a trajectory that starts close to the equilibrium point stays close to it (but does not necessarily end up in it). For example, a **center** is stable but not asymptotically stable.
- **unstable** if it is not stable. Examples include **saddle point**, unstable node and unstable spiral point.

Every autonomous system can be converted into a single first order differential equation. Dividing two autonomous differential equations

$$\frac{dx}{dt} = f(x, y) \qquad \frac{dy}{dt} = g(x, y)$$

with each other will give us a single equation with derivative $\frac{dy}{dx}$ on the left side:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g(x, y)}{f(x, y)}$$

Although it may be possible to find solution $y = y(x)$ as a function of x, it is important to keep in mind that the formula for $y(x)$ is not an oriented curve: all the information related to parameter t is lost in this way.

In order to obtain the direction of parametric curves in the phase plane one can superimpose the vector field of the ordinary differential equation $\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$ with the curves in the phase plane or to analyze the graph in phase plane together with the graphs of $x(t)$ and $y(t)$ as functions of t. The next several systems illustrate both methods.

Example 6.

1. Consider the system $\frac{dx}{dt} = -x$ and $\frac{dy}{dt} = -2y$. Its equilibrium point can be obtained from equations $-x = 0$ and $-2y = 0$. So (0,0) is the only equilibrium point. In this case, both equations of the system are separable. Solving them produces $x = c_1e^{-t}$ and $y = c_2e^{-2t}$. (in Matlab, you can use **[x,y]=dsolve(’Dx=-x’, ’Dy=-2*y’, ’t’)**). You can graph the trajectories in the phase plane by inlining the right side of the equation by **f = inline(’[-y(1); -2*y(2)]’,’t’,’y’);** and then using the following M-file (which is the M-file from previous example with axis commands slightly modified). The output is the first graph on Figure 2.

```
close all; hold on
for a = -3:0.5:3
     for b = -3:0.5:3
          [tsol, ysol] = ode45(f, 0:0.5:10, [a; b]);
          plot(ysol(:,1), ysol(:,2))
     end
end
hold off
axis([-3 3 -3 3])
```

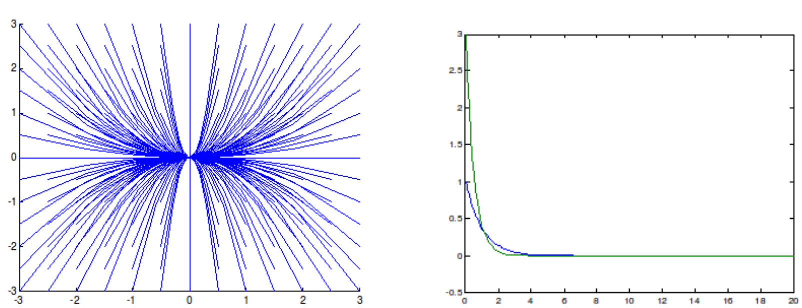

Figure 2: Stable Node

To figure out the directions of the parametric curves on this graph, graph one of them as function of t. For example, for $x(0) = 1$ and $y(0) = 3$, you get the graph on the right side of figure 2 using **[t, y]=ode45(f,[0,20],[1;3]); plot(t,y)**. We can see that $x \to 0$ and $y \to 0$ when $t \to \infty$. Graphing more initial conditions if necessary, you can see all trajectories are approaching the equilibrium point $(0, 0)$ when $t \to \infty$. Thus, the node (0,0) is **asymptotically stable** in this case.

To obtain the formula for the trajectories in the form $y = y(x)$, you can divide the first from the second equation: $\frac{dy}{dx} = \frac{-2y}{-x} = 2\frac{y}{x}$. Note that this is a separable differential equation $\frac{dy}{y} = 2\frac{dx}{x}$. The general solutions of this equation has the form $y = cx^2$.

2. Consider now the system $\frac{dx}{dt} = x$ and $\frac{dy}{dt} = 2y$. The solutions are $x = c_1 e^t$ and $y = c_2 e^{2t}$, and dividing the second equation by the first and solving for y in terms of x produces the same equation $\frac{dy}{dx} = 2\frac{y}{x}$ and the parabolas $y = cx^2$. However, in this case, the trajectories have the opposite direction than in the previous example. Thus, if c is positive $x \to \infty$ and $y \to \infty$ when $t \to \infty$. In that case (0,0) is an **unstable node**.

   Figure 3 displays the trajectories in the phase plane and the graph of solution with initial conditions $x(0) = 1$ and $y(0) = 3$ that can be used to establish stability.

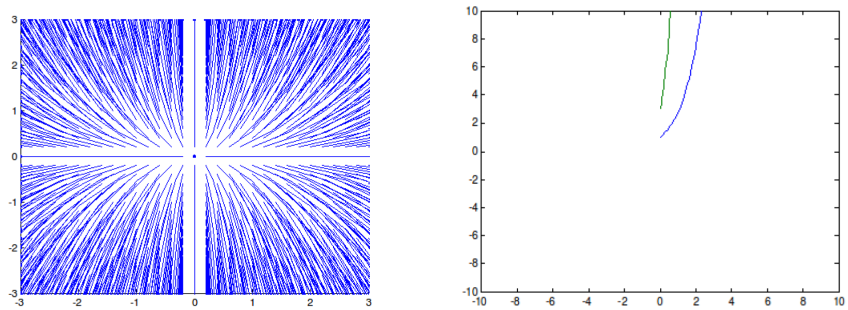

Figure 3: Unstable Node

3. Consider the system $\frac{dx}{dt} = -x+y$ and $\frac{dy}{dt} = -x-y$. The equations $-x+y = 0$ and $-x-y = 0$ have a single solution $x = 0$ and $y = 0$ so the system has just one equilibrium point (0,0). The solutions are $x = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t$ and $y = -c_1 e^{-t} \sin t + c_2 e^{-t} \cos t$ (you can obtain these formulas using **dsolve**). To graph the trajectories in the phase plane, inline the right side of the system using **f = inline('[-y(1)+y(2); -y(1)-y(2)]','t','y');** and use the same M-file as before. The output is displayed on Figure 4. The point (0,0) is called a **spiral point** in this case.

   To figure out the directions of the trajectories, graph one solution as function of t. For example, for $x(0) = 1$ and $y(0) = 3$, you get the graph on the right side of figure 4 (again using **[t, y]=ode45(f,[0,20],[1;3]); plot(t,y)**). From this graph, we can see that $x \to 0$ and $y \to 0$ when $t \to \infty$. Graphing more initial conditions if necessary, you can see that the trajectories are approaching the equilibrium point $(0, 0)$ when $t \to \infty$. Thus, the spiral point (0,0) is **asymptotically stable** in this case.

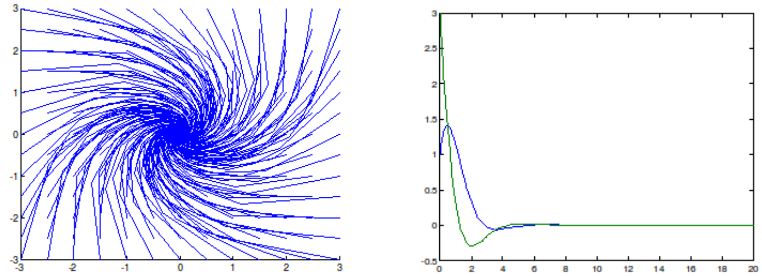

Figure 4: Spiral Point

4. The system $\frac{dx}{dt} = x + y$ and $\frac{dy}{dt} = -x + y$ is an example of the system with **unstable spiral point.** The formulas of the solution looks similar as in previous example except that the coefficients in the exponent of e are positive. The trajectories look similar as in the previous example except that the direction is away from (0,0).

5. The system $\frac{dx}{dt} = -x - y$ and $\frac{dy}{dt} = -x + y$ has one equilibrium point (0,0). The solutions turn out to be linear combination of $e^{\sqrt{2}t}$ and $e^{-\sqrt{2}t}$ (you can find explicit formulas using **dsolve**). To graph the trajectories, use **f = inline('[-y(1)-y(2); -y(1)+y(2)]','t','y');** and the same M-file as before. The output is displayed on Figure 5. We can see that the graphs of solutions are hyperbolas. The point (0,0) is called a **saddle point** in this case.

   To figure out the directions of the parametric curves on this graph, graph one of them as function of t. For example, for $x(0) = 1$ and $y(0) = 3$, you get the graph on the right side of figure 5. We can see that $x \to -\infty$ and $y \to \infty$ when $t \to \infty$. Graphing more initial conditions if necessary, you can see that none of the solutions is approaching the equilibrium point $(0, 0)$ when $t \to \infty$. Thus, the saddle point (0,0) is **unstable**.

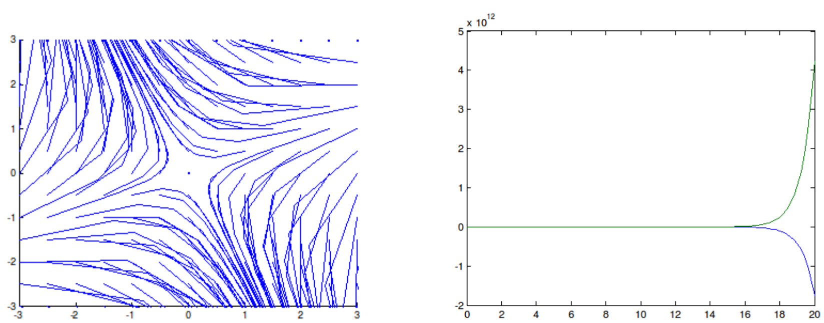

Figure 5: Saddle Point

6. Consider now the system $\frac{dx}{dt} = -y$ and $\frac{dy}{dt} = x$. Its only equilibrium point is (0,0). Using **dsolve**, you can see that the solutions are linear combinations of $\cos t$ and $\sin t$. To obtain the formula for y as a function of x, you can divide the second equation by the first, obtain $\frac{dy}{dx} = \frac{x}{-y} \Rightarrow ydy = -xdx \Rightarrow y^2 = -x^2 + 2c \Rightarrow x^2 + y^2 = C$. So, the solutions are circles with the center at (0,0). In this case (or similar case when the solutions are ellipsis), the equilibrium point is called a **center**.

To figure out the directions of the parametric curves in the phase plane, graph one of them as function of t. Using $x(0) = 1$ and $y(0) = 3$ again, we get the graph on the right side of figure 6. Considering the circle passing (1,3) in the phase plane and the fact that the periodic curve $x(t)$ decreases as t increases starting at $t = 0$, we can conclude that the circles in the phase plane are traversed counter clock-wise.

Regarding the stability, we can conclude that both x and y do not converge to 0 as $t \to \infty$. However, they not diverge to ∞ or $-\infty$ either but stay bounded (if you start close to (0,0), you remain close to it as t increases). Thus, a center point is an example of stable equilibrium point that is not asymptotically stable.

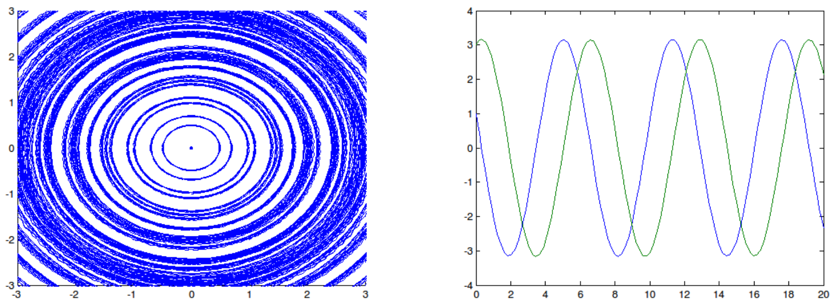

Figure 6: Center Point

If a system of differential equations is not linear, you can have more than one critical point and the graphs in phase plane can be a combination of the cases we have seen in the examples above. We shall study further examples of both linear and nonlinear systems by considering systems that model situations studied in biology (population dynamics in particular) and in examples coming from military practice.

In the section on differential equations we have seen several examples that model the population growth of a single species. If two species are interacting and their growth is govern by the outcome of such interaction, their size can be modeled by a system of differential equations. The most widely used models are competitive hunter model (in which two species compete for common resources, covered in more details in Differential Equations course) and predator-prey model in which one population acts as a predator and the other as pray. Let $x(t)$ and $y(t)$ denote the sizes of two species at time t.

Predator-Prey model. Let us consider the situation in which two species are such that one prays on the other. Let x denote the size of prey and y denote the size of predator population at time t. Then we can assume that x is growing at a rate proportional to the size of x but is decaying at a rate proportional to the number of interactions xy between the two species. The rate of y on the other hand, is increasing proportionally to the number of interactions xy and is decreasing proportionally to the size (because the more predators there are, less food to support all of them will there be).

Thus the system describing the sizes of the two species is given by

$$\frac{dx}{dt} = ax - bxy \qquad \frac{dy}{dt} = -cy + dxy$$

Note that the first two terms on the right side of the equations mean that

- In the absence of the predators, prey grows at a rate proportional to the size.
- In the absence of the prey, the predator dies out – thus the size is decreasing at a rate proportional to the size.

The equilibrium points of this system are $(0,0)$ and $(\frac{c}{d}, \frac{a}{b})$. Note that along the horizontal line $y = \frac{a}{b}$, $\frac{dx}{dt} = 0$ and so x is constant in time. Along the vertical line $x = \frac{c}{d}$, $\frac{dy}{dt} = 0$ and so y is constant in time. If $y < \frac{a}{b}$ and $x < \frac{c}{d}$, x is increasing and y deceasing. If $y < \frac{a}{b}$ and $x > \frac{c}{d}$, both derivatives are positive so the both species are increasing in size. If $y > \frac{a}{b}$ and $x < \frac{c}{d}$, both derivatives are negative so the trajectory is decreasing. Finally, if $y > \frac{a}{b}$ and $x > \frac{c}{d}$, x is decreasing and y increasing. This gives us that the trajectories in the phase plane revolve about $(\frac{c}{d}, \frac{a}{b})$ so this equilibrium point is a center. (0,0) is a saddle point. Thus, (0,0) is unstable and $(\frac{c}{d}, \frac{a}{b})$ is stable but not asymptotically stable (i.e. there is not a single x and y value towards which the trajectories converge when $t \to \infty$).

The existence of a center guarantees that no species will become extinct: an increase in x causes y's to increase. As a consequence, x's are hunted more and they decrease. This causes a decrease of y's also because the decrease in the food supplies. The decrease of y's causes x's to be hunted less and they start increasing again and so the cycle continues. This explains the fact that the graphs of solutions in xt and yt planes are periodic curves. The initial conditions will influence the size of the amplitude and the horizontal and vertical shifts of x and y curves. Thus the species coexist regardless of the initial conditions.

The predator-prey model is also known as **Lotka-Volterra model** in honor of its creators Lotka and Volterra.

Example 7. A population of rabbits R and wolves W is described using the predator-prey model with $(a, b, c, d) = (0.08, 0.001, 0.02, 0.00002)$. Write down the system of differential equations that models this situation. Find the differential equation for $\frac{dW}{dR}$. Find the equilibrium points of the system. Then, graph the trajectories in phase planes and examine the different types of equilibrium points. Discuss the long term behavior and provide biological interpretation.

Solution. The system is $\frac{dR}{dt} = 0.08R - 0.001RW$ $\frac{dW}{dt} = -0.02W + 0.00002RW$. Dividing the first equation from the second produces the differential equation for $\frac{dW}{dR}$. It is $\frac{dW}{dR} = \frac{-0.02W+0.00002RW}{0.08R-0.001RW}$.

Solving the right side of the system for zeros produces the two equilibrium points: (0,0) and (1000,80). Graph the system using **f = inline('[0.08*y(1)- 0.001*y(1)*y(2); -0.02*y(2)+ 0.00002*y(1)*y(2)]', 't', 'y');** and the M-file as in previous examples. To fit the values of this problem, modify the file. For example, you can use **a=0:50:1800**, **b=0:5:150** and **axis([0 1800 0 150]**).

Graph also a single trajectory as function of t. For example, with initial conditions $x(0) = 400$ and $y(0) = 100$, using **[t, y]=ode45(f,[0,200],[400;100]); plot(t,y)**, you obtain the graph on the left below.

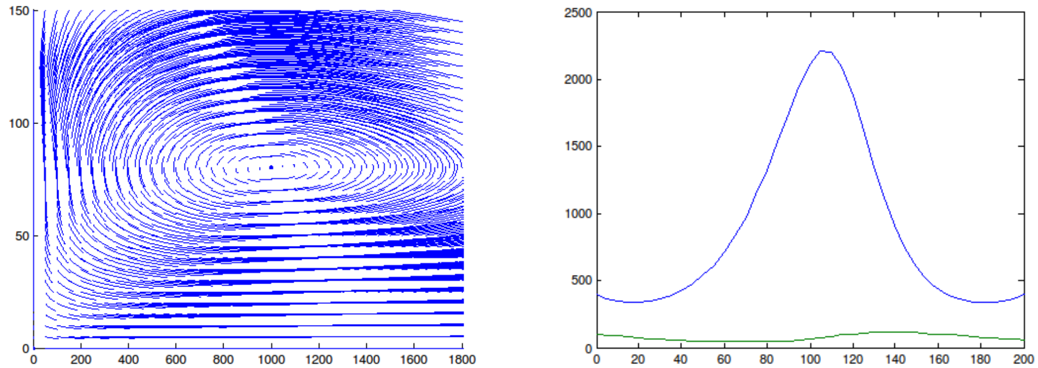

Figure 7: Predator-Prey Model

From the phase plane graph, we can see that the equilibrium point (1000, 80) is a center. Thus, it is stable but not asymptotically stable. (0,0) is a saddle point and it is unstable. Thus, the solutions oscillate: x-values about 1000 and y-values about 80. The amplitude of a solution depends on the initial conditions. The second graph also enables us to determine the direction of the trajectories: since starting with 400, the x values decrease a bit at first but then increase and the y-values decrease first starting at 100, we conclude that, we can conclude that the curves in the xy-plane are traversed in counter clock-wise direction.

Modeling a battle. Consider two military forces X and Y about to be engaged in a battle. Assuming that that at any given time, the troops on both side are either alive and fighting or are dead. Also, assume that the kill rate for X soldiers is a (i.e. that a is the number of X soldiers killed by one Y soldier at each time unit) and b is the kill rate for Y. Develop a mathematical model that can be used to (1) predict which army will win depending on the initial sizes and kill rates, (2) estimate the size of the winning side at the end and (3) predict how long will the battle last.

Use the model to predict the winner in a battle of 5000 with 10000 troops if the smaller army has superior military equipment that makes each its soldier 1.5 times more effective than a soldier from the other army. Assume that the kill rate for the other army is 0.1. Find the size of the winning side and predict the length of the battle.

Solution. Let $x(t)$ denote the number of X soldiers and $y(t)$ denote the number of Y soldiers. The system that models the battle is

$$\frac{dx}{dt} = -ay \qquad \frac{dy}{dt} = -bx$$

This model is known as **Lanchaster Combat Model.**

In this case, we can obtain the equation of curves in the phase plane by dividing one equation with the other.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-bx}{-ay} = \frac{bx}{ay}.$$

This is a separable differential equation. Separating the variables obtain $aydy = bxdx$. Integrating both sides and multiplying by 2 gives you $ay^2 = bx^2 + c$. Thus the solutions are hyperbolas.

Analyzing the trajectories in parametric form (or simply observing that at no point the size of the armies increase), we conclude that the direction is downwards.

The only equilibrium point is (0,0). From the graphs in the phase plane that you can obtain on the same way as in previous examples, we conclude that it is a **saddle point**. There is a **separatrix** dividing the solutions into two groups: the solutions with initial conditions below the line are such that $y \to 0$ so X wins, and the solutions with initial conditions above the line are such that $x \to 0$ so Y wins. Just in the case that initial conditions are exactly on separatrix, both x and y converge to 0 so this represents the mutual annihilation.

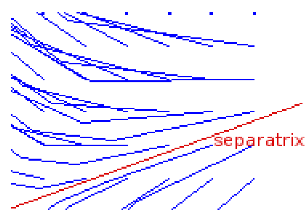

Figure 8: A phase plane with saddle point at the origin and a separatrix

If initial conditions are denoted by x_0 and y_0, the integration constant can be represented as $c = ay_0^2 - bx_0^2$. Thus, the solutions are given by

$$ay^2 = bx^2 + ay_0^2 - bx_0^2.$$

The exact formula of the separatrix can be obtained for $c = 0$. In that case $y^2 = \frac{b}{a}x^2$. The equation of separatrix is $y = \sqrt{\frac{b}{a}}\,x$.

Winning criterion. The side Y wins if $ay^2 - bx^2 = ay_0^2 - bx_0^2 > 0 \longrightarrow \frac{y_0^2}{x_0^2} > \frac{b}{a}$.

The side X wins if $ay^2 - bx^2 = ay_0^2 - bx_0^2 < 0 \longrightarrow \frac{y_0^2}{x_0^2} < \frac{b}{a}$.

The battle is a draw if the initial conditions lie on the separatrix i.e. if $ay_0^2 - bx_0^2 = 0 \longrightarrow \frac{y_0^2}{x_0^2} = \frac{b}{a}$.

Analysis of the proportions $\frac{y_0^2}{x_0^2}$ and $\frac{b}{a}$, enables one to strategize. For example, if Y has more troops initially, the commander of X can calculate the factor by which the military machinery of X should be superior in order for X to win.

Size of the winning army. In case that X wins, the size of X at the end can be found as x-intercept. Solving $0 = bx^2 + ay_0^2 - bx_0^2$ for x, we have $x = \sqrt{\frac{1}{b}(bx_0^2 - ay_0^2)}$.

In case that Y wins, the size of Y at the end can be found as y-intercept. Solving $ay^2 = 0 + ay_0^2 - bx_0^2$ for y, we have $y = \sqrt{\frac{1}{a}(ay_0^2 - bx_0^2)}$.

Length of the battle. The length of the battle can be found when solving the formulas for $x(t)$ or $y(t)$ for zeros. These formulas can be obtained using Matlab. Alternatively, these formulas could be obtained analytically. Namely, differentiate the second equation and obtain $\frac{d^2y}{dt^2} = -b\frac{dx}{dt}$. Substitute that in the first equation and get a single second order differential equation

$$\frac{-1}{b}\frac{d^2y}{dt^2} = -ay \longrightarrow \frac{d^2y}{dt^2} - aby = 0.$$

This equation has constant coefficients so its general solution is determined by the solutions of its characteristic equation $r^2 - ab = 0$. From here $r = \pm\sqrt{ab}$ and so the solution is $y = c_1 e^{\sqrt{ab}t} + c_2 e^{-\sqrt{ab}t}$. The term $\sqrt{ab}$ will impact the length of the battle.

To find the exact formula for the length of the battle, differentiate the formula for y and equate the derivative y' with $y' = -bx$. Solve for x and get $x = -\sqrt{\frac{a}{b}} c_1 e^{\sqrt{ab}t} + \sqrt{\frac{a}{b}} c_2 e^{-\sqrt{ab}t}$. The constants c_1 and c_2 can be found from the initial conditions $x(0) = x_0$ and $y(0) = y_0$ to be $c_1 = \frac{1}{2}(y_0 - \sqrt{\frac{b}{a}} x_0)$ and $c_2 = \frac{1}{2}(y_0 + \sqrt{\frac{b}{a}} x_0)$.

In case that X wins, when solving y for zero, you have that

$$t = \frac{1}{2\sqrt{ab}} \ln\left(\frac{\sqrt{b}x_0 + \sqrt{a}y_0}{\sqrt{b}x_0 - \sqrt{a}y_0}\right).$$

In case that Y wins, when solving x for zero, you have that

$$t = \frac{1}{2\sqrt{ab}} \ln\left(\frac{\sqrt{a}y_0 + \sqrt{b}x_0}{\sqrt{a}y_0 - \sqrt{b}x_0}\right).$$

Let us turn to example with $x_0 = 5000$, $y_0 = 10000$, The kill rate of the larger army Y is 0.1 meaning that $\frac{dx}{dt} = -0.1y$. The kill rate of the smaller army X is 1.5 times larger and so it is 0.15. Thus, the second equation is $\frac{dy}{dt} = -0.15x$. Thus, $a = 0.1$ and $b = 0.15$. In this case $\frac{y_0^2}{x_0^2} = 4$ and $\frac{b}{a} = 1.5$, Thus, $\frac{y_0^2}{x_0^2} > \frac{b}{a}$ and so Y wins despite of larger military power of X.

The size of the winning army Y is $y = \sqrt{\frac{1}{a}(ay_0^2 - bx_0^2)} \approx 7906$ troops at the end.

The battle ends after $t = 4.082 \ln \frac{5098.77}{1225.79} = 5.819$ hours.

Modification of the Lanchaster model. Various modifications of the basic Lanchaster model are possible. For example, assuming that the kill rates depend on time as it may be the case when artillery is changing location.

Another modification may include assuming that derivatives are decreasing proportionally not to the number of troops but to number of contacts between the troops. In this case, the equations become

$$\frac{dx}{dt} = -axy \qquad \frac{dy}{dt} = -bxy$$

This model is convenient for assuming that the battle involves two unconventional, guerrilla forces. The rate of decrease is proportional to the number of interactions between the forces. In this case, enemy fire targets the whole guerrilla area and the more guerrillas there are on the area, the more they will be annihilated. Also, the more enemy forces are there, the more guerrilla forces will be killed. This explains why the rate of decrease of X is larger when both size of X and size of Y are increasing.

To find the solution, divide the equations again, just like in the case of conventional model. Obtain that $\frac{dy}{dx} = \frac{b}{a}$. Thus, the solutions lie on lines $y = \frac{b}{a}x + c$. From initial conditions $c = y_0 - \frac{b}{a}x_0$. Thus the solution is

$$y = \frac{b}{a}x + y_0 - \frac{b}{a}x_0$$

The winning criterion is obtained by comparing $\frac{y_0}{x_0}$ and $\frac{b}{a}$. The size of the winning army is determined by x or y intercepts of lines representing solutions again. The formula for the length of the battle can be obtained similarly as for the Lanchaster model.

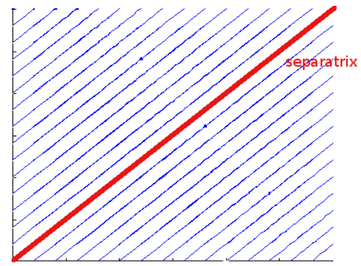

Figure 9: Two guerrilla forces

Divide-and-conquer strategy. Let us consider two armies of equal kill rate. In this case, clearly the larger initial size will determine a winner. However, considering the following strategy known as divide-and-conquer, the smaller army can win. The idea is that X uses the full force to attack just a portion of army Y that is small enough to guarantee victory of X but as large enough as possible. Then X continues to attack a portion of remaining Y troops with surviving forces in the second battle. The size of Y troops can again be chosen to be small enough to guarantee victory of X but as large enough as possible. Continuing on this manner, X may be victorious after a sequence of battles.

This strategy was used by Lord Nelson at the **Battle of Trafalgar** in 1805. Lord Nelson was in charge of British fleet of 27 ships. Napoleon, on the other side, was in charge of French-Spanish fleet of 33 ships. The kill rate for each side was 0.1.

Napoleon's fleet was arranged in a line of ships separated in 3 groups of 17, 3 and 13 ships respectively. Nelson decided to divide his army into 2 groups of 13 and 14 ships each. The first group will attack the smallest Napoleon group first, defeat them and then combine with remaining 14 ships again the larger Napoleon fleet of 17 ships. The surviving ships of that battle will finally be able to defeat the remaining 13 Napoleon's ships.

The Lanchaster model predicts the following outcome:

	Battle 1	Battle 2	Battle 3
Napoleon	3	17	13
Nelson	13	12+14	15
Winner	Nelson	Nelson	Nelson
Ships remaining	12	15	7

Thus, after 3 battles, Nelson should be victorious with 7 ships still remaining.

Historical data supports the accuracy of the model – Nelson did indeed win the first two battles. During the third battle the French disengaged and returned to France with 13 ships. Lord Nelson was killed during the battle but the British were victorious.

Example 8. Let us consider another example in which this strategy can be used. Consider two armies with $x_0 = 5000$, $y_0 = 7500$, and $a = 0.1$ and $b = 0.15$. Here Y is larger but X has larger military power. Using the Lanchaster model, Y wins since $\frac{y_0^2}{x_0^2} = 2.25 > \frac{b}{a} = 1.5$. For a win or at least a draw, the commander of X considers using divide-and-conquer strategy and fight portions of army Y at the time. The size of army Y that will result in the victory of X or a draw can be found from the condition $\frac{y_0^2}{5000^2} \leq \frac{b}{a} = 1.5$. From this, the commander obtains 6123 and decides to attacks just 5000 troops of Y in the first battle.

X wins the first battle with $x = \sqrt{\frac{1}{b}(bx_0^2 - ay_0^2)} = 2886$ troops. Then the commander attacks the remaining 2500 Y troops and wins having 2040 troops left. Thus, he conquers army Y in two battles.

	Battle 1	Battle 2
X	5000	2886
Y	5000	7500-5000
Winner	X	X
troops remaining	2886	2040

Maximizing the survivors. The strategy can be improved when noting the following: if a smaller number of Y troops were attacked in the first battle, there will be more survivors and, thus, larger fighting power in the second battle and more survives at the end. For example, if just 4000 of Y troops were attacked in the first battle, X would win with 3785 troops. If those troops were used to attack remaining 3500 Y troops in the second battle, X will win the war with as many as 2481 survivors (compare with 2040 in the first scenario).

	Battle 1	Battle 2
X	5000	3785
Y	4000	7500-4000
Winner	X	X
troops remaining	3785	2481

The number of survivors could be increased further following Nelson's idea more i.e. not engaging the whole army in the first battle. For example, if just 3000 troops of X will fight a smaller portion of Y troops, say 2000, in the first battle, and then join forces with remaining 2000 troops in the second battle to fight larger remaining portion of Y troops, the survival size of X will be even larger as the following table indicates.

	Battle 1	Battle 2	Battle 3
X	3000	2516+2000	3118
Y	2000	4000	7500-6000
Winner	X	X	X
troops remaining	2516	3118	2733

Practice Problems.

1. A car rental company has distributors in Orlando and Tampa. The company specializes in catering to travel agents who want to arrange tourist activities in both cities. Consequently, a traveler may rent a car in one city and return it in another. The company wants

to determine how much to charge for this drop-off convenience. Let us assume that 60% of the cars rented in Orlando are returned there and 70% rented in Tampa are returned there. If there are 7000 cars total to be distributed between the two cities, determine the trend for the number of cars in either city in the following two scenarios

- There are 2000 cars in Orlando and 5000 cars in Tampa initially.
- There are no cars in Orlando and all 7000 in Tampa initially.

2. Consider a three party system with Republicans, Democrats, and Independents. Assume that in the next election 75% of those that voted Republican again vote Republican, 20% vote Democrat and 5% vote Independent. Of those that voted Democrat, 80% vote Democrat again, 10% vote Republican and 10% vote Independent. Of those that voted Independent, 60% vote Independent again, 10% vote Republican and 30% vote Democrat. Assuming that these tendencies continue from election to election and that no voters leave the system, estimate the long term tendencies.

3. Consider the system $\frac{dx}{dt} = 3x - y$ and $\frac{dy}{dt} = 4x - 2y$. (a) Graph the particular solution of the initial value problem with $x(0) = 1$ and $y(0) = 2$. (b) Plot sufficiently many solutions in the phase plane to determine the type of the equilibrium point (0,0). Discuss the stability of (0,0).

4. Model the situation from Example 3 (with two lakes and pollutant) using differential equations. Graph the solutions in the phase plane and discuss the long term behavior for any value of the initial amount of the pollutant.

5. Determine (a) the winning criterion, and (b) size of the winning army for a battle of two guerrilla forces X and Y whose sizes $x(t)$ and $y(t)$ at time t are modeled by the system

$$\frac{dx}{dt} = -axy \qquad \frac{dy}{dt} = -bxy$$

with initial values $x(0) = x_0$ and $y(0) = y_0$.

Solutions.

1. **Difference equations model**. To model this situation, consider the following variables. Let n denote the number of days, O_n the number of cars in Orlando at the end of day n, and T_n the number of cards in Tampa at the end of day n. In this case

$$O_{n+1} = 0.6O_n + 0.3T_n \qquad \text{and} \qquad T_{n+1} = 0.4O_n + 0.7T_n$$

The equilibrium values of this system would be the number of cars that would remain the system in an optimal balance. If O denotes the limit of O_n (and O_{n+1}) and T denotes the limit of T_n (and T_{n+1}), we can find the equilibrium solution from

$$O = 0.6O + 0.3T \qquad \text{and} \qquad T = 0.4O + 0.7T$$

Solving from O and T, we obtain the condition $O = \frac{3}{4}T$. The relation determines the optimal ration of cars in the two cities.

Thus, if the total number of cars is 7000, $O+T = 7000 \longrightarrow \frac{3}{4}T+T = 7000 \longrightarrow \frac{7}{4}T = 7000 \longrightarrow T = 4000$ and $O = \frac{3}{4}4000 = 3000$. More generally, if m is the total number of cars, from $O+T = m$ we obtain that the limiting values of O and T would be $\frac{3}{7}m$ and $\frac{4}{7}m$ respectively.

Thus, in either one of the following two initial conditions scenarios, the long term trend is to have 3000 cars in Orlando and 4000 cars in Tampa.

Differential equations model. Alternatively, you can model this situation using differential equations.

$$\frac{dO}{dt} = -0.4O + 0.3T \qquad \text{and} \qquad \frac{dT}{dt} = 0.4O - 0.3T$$

Setting the equations to 0 yields the equilibrium condition $O = \frac{3}{4}T$. Together with initial condition $O + T = 7000$, we arrive to the same values $O = 3000$ and $T = 4000$. The graphs of the two solutions corresponding to two scenarios are on Figure 10. They can be obtained by **f = inline('[-0.4*y(1)+ 0.3*y(2); 0.4*y(1)- 0.3*y(2)]', 't', 'y');** followed by **[t, y]=ode45(f, [0,20], [2000;5000]); plot(t,y)** and **[t, y]=ode45(f, [0,20], [0;7000]); plot(t,y)**.

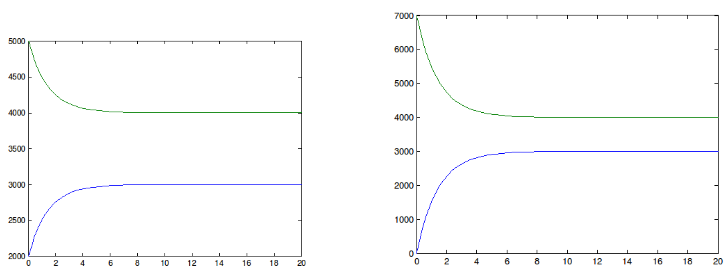

Figure 10: Two Scenarios in Orlando-Tampa Example

The exact solution of the system of differential equations can be found using Matlab. They turn out to be $O = 3c_1 + c_2e^{-.7t}$ and $T = 4c_1 - c_2e^{-.7t}$. When $t \to \infty$, $O \to 3c_1$ and $T \to 4c_1$. Since $O(0) = 3c_1$ and $T(0) = 4c_1$, in case that $7000 = 3c_1 + 4c_1$ we have that $O \to 3000$ and $T \to 4000$ when $t \to \infty$.

Note also that dividing the two equations would yield

$$\frac{dO}{dT} = \frac{\frac{dO}{dt}}{\frac{dT}{dt}} = \frac{-0.4O + 0.3T}{0.4O - 0.3T} = \frac{-(0.4O - 0.3T)}{0.4O - 0.3T} = -1$$

This gives us the solutions $O = -T + c$. Thus the trajectories in the phase plane are parallel line segments with slope -1 (see Figure 11). In general, if m is the total number of cars, we obtain $c = m$. The direction of the trajectories is towards the point $(\frac{3}{7}m, \frac{4}{7}m)$ (that is the intersection of the line $O = -T+m$ with the line $O = \frac{3}{4}T$). For any given value of initial conditions $(T(0), O(0))$ the graph of the particular solution is a line segment on $O = -T + m$ ending at the intersection of $O = -T + m$ and $O = \frac{3}{4}.T$

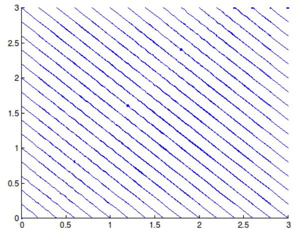

Figure 11: Trajectories in Orlando-Tampa Example

2. Let R_n, D_n and I_n stands for the number of Republican, Democrat and Independent voters respectively in the n-th election. We can model the situation above by the following system

$$\begin{array}{rcllll} R_{n+1} & = & 0.75R_n & +0.1D_n & +0.1I_n \\ D_{n+1} & = & 0.20R_n & +0.8D_n & +0.3I_n \\ I_{n+1} & = & 0.05R_n & +0.1D_n & +0.6I_n \end{array}$$

Let R, D and I denote the limiting values. The equilibrium values are solutions of the system $R = 0.75R + 0.1D + 0.1I$, $D = 0.20R + 0.8D + 0.3I$, $I = 0.05R + 0.1D + 0.6I$. The solutions are $R = \frac{10}{6}I$ and $D = \frac{19}{6}I$. Note that if we assume that $R + D + I = 100\%$ (we assume that the number of voters remains the same for many elections), we have that $(\frac{10+19+6}{6})I = \frac{35}{6}I = 1$ and so $I = \frac{6}{35} = 17.14\%$ $R = \frac{10}{35} = 28.57\%$ and $D = \frac{19}{35} = 54.29\%$.

Alternatively, you can consider the system of difference equations $\Delta R_n = -0.25R_n + 0.1D_n + 0.1I_n$, $\Delta D_n = 0.20R_n - 0.2D_n + 0.3I_n$, $\Delta I_n = 0.05R_n + 0.1D_n - 0.4I_n$.

3. (a) Use **f = inline('[3*y(1) -y(2); 4*y(1) -2*y(2)]','t','y');** followed by **ode45(f, [0,20], [1;2])** to obtain the first graph on Figure 12. (b) Using the same M-file as in examples from this section, we obtain the second graph on Figure 12. We conclude that both x and y increase to ∞ for $t \to \infty$. Thus, (0,0) is unstable.

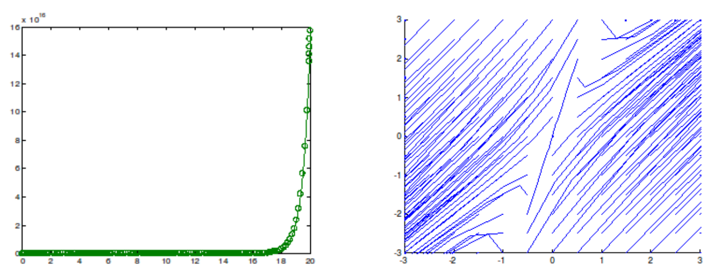

Figure 12: Saddle Point

4. System: $a' = -0.65a + 0.10b$ and $b' = 0.65a - 0.10b$. Setting the equations to zero yields the equilibrium condition $b = \frac{13}{2}a$. Assume that there is no new pollution added to either lake (thus $a+b$ is constant). If we denote the total amount of pollutant with $a+b = c$, this last equation represents the solutions (this can be obtained by dividing the two equations, getting that $\frac{db}{da} = -1$ and obtaining that $b = -a + c$ from there). Thus the trajectories in the phase plane are parallel line segments with slope -1 (see Figure 4). The direction of the

trajectories is towards the point $(\frac{2}{15}c, \frac{13}{15}c)$ (that is the intersection of the line $b = -a + c$ with the line $b = \frac{13}{2}a$). Thus about 13% of the pollutant will end up in the first and 87% in the second lake.

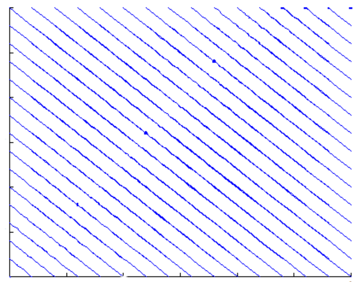

Figure 13: Two lakes

For any given value of initial conditions $(a(0), b(0))$ the graph of the particular solution is a line segment on $b = -a + c$ ending at the point $(\frac{2}{15}c, \frac{13}{15}c)$.

5. When dividing the equations, one obtains a separable differential equation $\frac{dy}{dx} = \frac{b}{a} \Rightarrow$ $a dy = b dx \Rightarrow ay = bx + c$. $+ c \Rightarrow c = ay_0 - bx_0$. Thus, the solution is $y = \frac{b}{a}x$

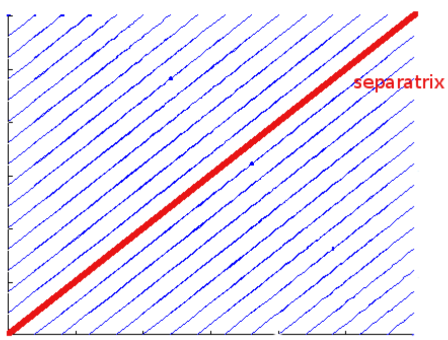

Figure 14: Guerrilla war phase plane

Winning criterion. The side Y wins if the y-intercept is positive i.e. if $y_0 - \frac{b}{a}x_0 > 0 \longrightarrow \frac{y_0}{x_0} > \frac{b}{a}$. The side X wins if the x intercept is positive (i.e. if y-intercept is negative) and that is if $\frac{y_0}{x_0} < \frac{b}{a}$.

The battle is a draw if the initial conditions lie on the separatrix i.e. if $ay_0 - bx_0 = 0 \longrightarrow \frac{y_0}{x_0} = \frac{b}{a}$. The equation of the separatrix is $y = \frac{b}{a}x$.

Size of the winning army. In case that X wins, the size of X at the end can be found as the x-intercept. Solving $0 = \frac{b}{a}x + y_0 - \frac{b}{a}x_0$ for x, we have $x = x_0 - \frac{a}{b}y_0$.

In case that Y wins, the size of Y at the end can be found as y-intercept which is $y = y_0 - \frac{b}{a}x_0$