Differential Geometry
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Project Topics and Report Instructions

Th first draft is due on **Friday Nov 19, 2021.**

Project Overview. Students should be able to research a topic or learn about a method on their own, not just via an extensive classroom instruction. The purpose of this project is to practice this skill. You can choose a topic from the **sample topics list below**. You can also pick a topic that is *not* listed below, but should consult with me to double check it is relevant and extensive enough.

Important Dates. The first draft of your project is due on Fri. Nov 19, 2021. You will receive feedback on your draft from me and then the final version is going to be due on Fri. Dec 3, 2021.

Report Instructions. First you need to decide on the topic and find the material that you may need to cover your topic, then write a project report. The purpose of the report is to practice your skills of effectively and convincingly writing about your methods, results and conclusions obtained.

The report should consist of three parts: the introduction, the main part and the conclusion. Throughout the report, make sure that your sentences are clear and your spelling, grammar and punctuation correct. Avoid long sentences and do not use complicated words when you can communicate something using simple and clear phrases. The technical level of your report should be such that your class peers will be able to understand and follow your arguments.

In the **introduction**, you should state the main goal of the project. You can start with the problem statement and the background of the problem. You should clearly state the project objective. Then you can also give a brief summary of the methods used and conclusions obtained.

In the **main part**, you should list your results or prove the statement you need to demonstrate. You should explain the mathematical methods used, prove the main statement using well supported mathematically correct claims, or explain how the mathematical methods apply to other disciplines. All mathematical data manipulation, calculations, algebraic or numerical work should be in the main part. Make sure that each step is clear and justified.

At the end of your report, you should summarize your **conclusions** and reflect on their relevance.

A possibility. In the case that you end up researching a topic especially thoroughly and beyond the scope of the published material you have been using, you could consider presenting your findings during the USciences Annual Research Day (usually in early April) or even to consider preparing your findings for a publication. You can consult with me about this.

List of Sample Project Topics

Curves

1. General helix and Lancret's theorem (Millman-Parker, page 32).
2. Prove the formulas for Frenet-Serret apparatus for a curve with arbitrary parametrization (Millman-Parker, page 47).
3. Global Theory of Curves. Four Vertex Theorem. Knots (Millman-Parker, selected parts of chapter 5).

Surfaces

4. Show that the surface area does not depend on the parametrization of a surface (Millman-Parker, page 96).

5. Show that principal curvatures are the eigenvalues of the second fundamental form. From this fact, it can be deduced that if $\mathbf{v}_1$ is an eigenvector of the second fundamental form and if $\mathbf{v}_2$ is perpendicular to $\mathbf{v}_1$, then $\mathbf{v}_2$ is an eigenvector as well. Thus, the principal directions are perpendicular (Millman-Parker page 128).

6. Geodesics. Prove the formula computing the geodesic curvature ("Surfaces 3" handout, second half of page 6 and the first part of 7).

7. Geodesics. Prove that a curve minimizing the distance between the two points is a geodesic (Millman-Parker, pages 113 and 114).

8. Prove the Weingarten's, Gauss's, and Codazzi-Mainardi equations. ("Surfaces 3" handout, pages 14 and 15).

9. Parallel transport (Millman-Parker, section 4.6). Gauss curvature via parallel transport (Faber, page 52).

10. Classification of surfaces of revolution with constant Gaussian curvature (Millman-Parker, section 4.11).

11. Global Theory of Surfaces. Euler characteristic. Gauss-Bonnet theorem (relates the Euler characteristic with the Gaussian curvature) (Millman-Parker, selected parts of chapter 6).

Applications and Generalizations.

12. Einstein notation and tensors. Covariant and contravariant tensors (find the source materials on your own).

13. Three Kepler's laws of planetary motion (Stewart, section 14.4) and their generalizations. Volume of hypersphere (Stewart p. 1052) and n-dimensional manifolds.

14. Exterior product: the cross product in n-dimensional space (liavas.net/files/Tim_paper.pdf).

15. Differential Geometry in Theory of Special and General Relativity (Faber, selected parts of chapter 3).

16. The metric of the Big Bang: Friedmann-Lemaître-Robertson-Walker metric (find the source materials on your own).

If you are using the Millman-Parker book, note that $\langle \mathbf{a}, \mathbf{b} \rangle$ **is used to denote the dot product** $\mathbf{a} \cdot \mathbf{b}$. Also, for one or two topics, it may be relevant to note that if $\mathbf{a} = a^i \mathbf{x}_i$ and $\mathbf{b} = b^i \mathbf{x}_i$ are two vectors in the tangent plane of a surface $\mathbf{x}$, then $\mathrm{I}(\mathbf{a}, \mathbf{b})$ denotes $\sum_{ij} g_{ij} a^i b^j$ and $\mathrm{II}(\mathbf{a}, \mathbf{b})$ denotes $\sum_{ij} L_{ij} a^i b^j$. Here I and II stand for the first and the second fundamental forms respectively.