Differential Geometry
Lia Vas

Measuring lengths and angles – the first fundamental form

A curve on a surface. If a surface $\mathbf{x}$ is parametrized with variables u and v, a curve $\boldsymbol{\gamma}$ is on the surface $\mathbf{x}$ if a parametrization of $\boldsymbol{\gamma}$ with a parameter t can be obtained from equations of $\mathbf{x}$ when u and v become functions of t

$$\boldsymbol{\gamma}(t) = \mathbf{x}(u(t), v(t)).$$

Examples.

1. A curve given by the equations $\boldsymbol{\gamma}(t) = (x(t), y(t), 0)$ is a curve in the plane $z = 0$. Recall that this plane can be parametrized by $(x, y, 0)$. Thus, in this example, "u" is x and "v" is y.

2. The helix $\boldsymbol{\gamma}(t) = (a\cos t, a\sin t, bt)$ is on the cylinder $x^2 + y^2 = a^2$ since the x and y coordinates satisfy the equation of the cylinder. Recall that this cylinder has parametric equations $(a\cos t, a\sin t, h)$, so the equations $(a\cos t, a\sin t, bt)$ can be perceived as a "special case" of the cylinder equations when one parameter is deleted using the relation $h = bt$. In this example, "u" is t and "v" is h, and, on the helix, $t = t$, and $h = bt$.

3. Recall the practice problems 4 and 5 from "Curves". In these problems, we showed that the curve $\boldsymbol{\gamma} = (\frac{5}{13}\cos s, \frac{8}{13} - \sin s, \frac{-12}{13}\cos s)$ is in the plane $z = \frac{-12}{5}x$.

4. A **spherical curve** is a curve that lies on a sphere.

   Note that the equation of a sphere $(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2$ has the vector form $((x, y, z) - \mathbf{c}) \cdot ((x, y, z) - \mathbf{c}) = r^2$ where $\mathbf{c} = (x_0, y_0, z_0)$ is the center, r the radius, and (x, y, z) any point on the sphere. Hence, a curve $\boldsymbol{\gamma}(t) = (x(t), y(t), z(t))$ is on this sphere if and only if $(\boldsymbol{\gamma} - \mathbf{c}) \cdot (\boldsymbol{\gamma} - \mathbf{c}) = r^2$. Differentiating this equation, one obtains the condition that $\boldsymbol{\gamma}' \cdot (\boldsymbol{\gamma} - \mathbf{c}) = 0.$
   Conversely, integrating this last condition (see the proof of Claim 1 in "Curves"), we obtain that $(\boldsymbol{\gamma} - \mathbf{c}) \cdot (\boldsymbol{\gamma} - \mathbf{c})$ is constant. Denoting this constant by r^2 we obtain an equation of the sphere of radius r centered at $\mathbf{c}$.

   For example, $\boldsymbol{\gamma} = (4\cos 2t, 4\sin 2t, 4\sin t)$ is spherical for $\mathbf{v} = (-1, 0, 0)$ since one checks that $\boldsymbol{\gamma}' \cdot (\boldsymbol{\gamma} - \mathbf{c}) = 0$. The distance from any point on $\boldsymbol{\gamma}$ to $(-1, 0, 0)$ is the radius of that sphere. So, if we calculate the distance of $(4, 0, 0)$ with $t = 0$, for example, to $(-1, 0, 1)$, we obtain the radius $r = \sqrt{(4+1)^2 + 0 + 0} = 5$. Thus, $\boldsymbol{\gamma}$ is on the sphere $(x+1)^2 + y^2 + z^2 = 25$.

The u-curves and the v-curves. Two important special cases of curves on a surface are the following.

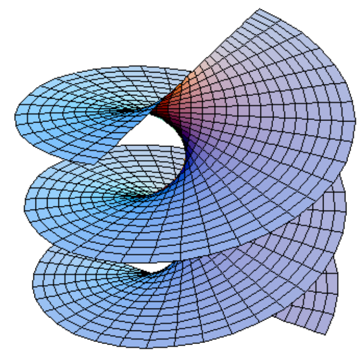
- Taking v to be a constant v_0, one obtains the curve $\boldsymbol{\gamma}_1(u) = \mathbf{x}(u, v_0)$ (so "t" is u here). This curve is called a **u-curve.** The velocity vector $\frac{\partial \mathbf{x}}{\partial u}$ is in the tangent plane.

- Taking u to be a constant u_0, one obtains the curve $\boldsymbol{\gamma}_2(v) = \mathbf{x}(u_0, v)$ (so "t" is v here). This curve is called a **v-curve.** The velocity vectors $\frac{\partial \mathbf{x}}{\partial v}$ is in the tangent plane.

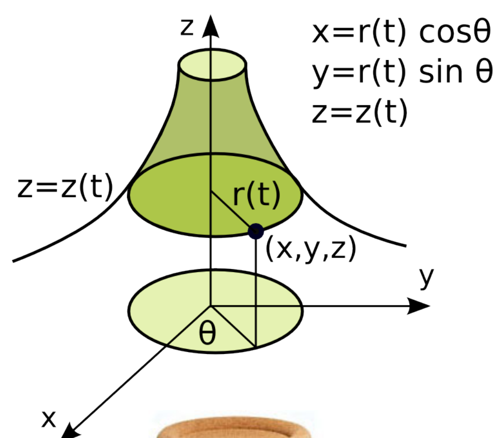
Surfaces are often represented by by graphing a mesh determined by u and v curves. Because of this, we can think that the mesh of u and v curves forms a **coordinate system** on the surface.

Examples.

1. On the sphere $\mathbf{x} = (a\cos\theta\cos\phi, a\sin\theta\cos\phi, a\sin\phi)$, the ϕ-curves are circles of constant longitude **meridians** and the θ-curves are circles of constant latitude, **parallels**.



2. **Helicoid**. The surface parametrized by $(r\cos\theta, r\sin\theta, a\theta)$ has θ-curves helices spiraling about cylinder of radius r_0 and r-curves lines $(r\cos\theta_0, r\sin\theta_0, a\theta_0)$ passing through z-axis in a plane parallel to xy-plane. The surface resembles the spiral ramps like those found in garages.



3. **Surfaces of revolution**. A surface of revolution of curve $\alpha = (r(t), z(t))$ about z-axis can be given by $(r(t)\cos\theta, r(t)\sin\theta, z(t))$. The θ-curves are circles of radii $r(t_0)$ in horizontal planes passing $z(t_0)$. They are also called the **circles of latitude** or the **parallels** by analogy with parallels on a sphere. The t-curves have the same shape as the curve α except that they are positioned in vertical planes at longitude θ_0. They are called the **meridians**.

4. A surface is said to be a **ruled surface** if it is generated by moving a line along some direction. Such a surface can also be described by the property that through every point there is a line completely contained in the surface. This line is called a **ruling**.

   If $\boldsymbol{\alpha}(t)$ is a curve that cuts across all the rulings and $\boldsymbol{\beta}(t)$ is the direction of ruling, you can think that the surface is obtained by moving vector $\boldsymbol{\beta}$ along the curve $\boldsymbol{\alpha}$.

Such a surface has the equation

$$\mathbf{x}(t,s) = \boldsymbol{\alpha}(t) + s\boldsymbol{\beta}(t)$$

The s-curves are ruling lines $\boldsymbol{\alpha}(t_0) + s\boldsymbol{\beta}(t_0)$. In case that $\boldsymbol{\beta}$ is a constant vector, the t-curves represent curve $\boldsymbol{\alpha}$ translated in space.

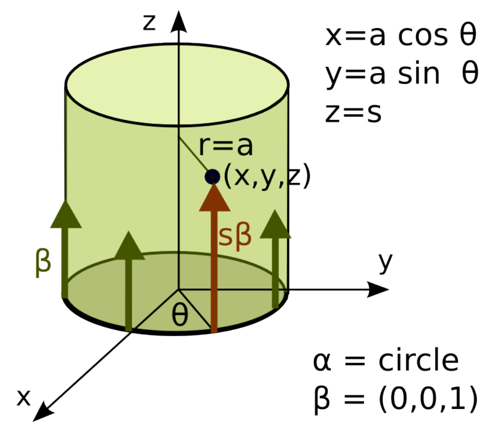
Examples of ruled surfaces.

1. A plane can be considered to be a ruled surface letting $\boldsymbol{\alpha}(t)$ be a line and $\boldsymbol{\beta}(t)$ be a constant vector.

2. A **cone** is a ruled surface with $\boldsymbol{\alpha}$ constant, say point P. The point P is called the **vertex** of the cone. In case that $\boldsymbol{\beta}$ makes a constant angle with fixed line through P (called the **axis**) of the cone), we obtain the **right circular cone**. For example, the cone $z = \sqrt{x^2+y^2}$ can be parametrized by taking $\boldsymbol{\alpha} = (0,0,0)$ and $\boldsymbol{\beta} = (\cos t, \sin t, 1)$ and getting $(s\cos t, s\sin t, s)$.

3. A **cylindrical surface** is defined as a ruled surface with $\boldsymbol{\beta}$ constant vector. If $\boldsymbol{\alpha}$ is a circle, the cylindrical surface is said to be **circular cylinder**. If $\boldsymbol{\beta}$ is a vector perpendicular to the plane of circle $\boldsymbol{\alpha}$ the circular cylinder is said to be **right**.

   For example, the cylinder $x^2 + y^2 = a^2$ can be considered as a ruled surfaces with $\boldsymbol{\alpha}$ being the circle in xy-plane and $\boldsymbol{\beta} = (0,0,1)$.

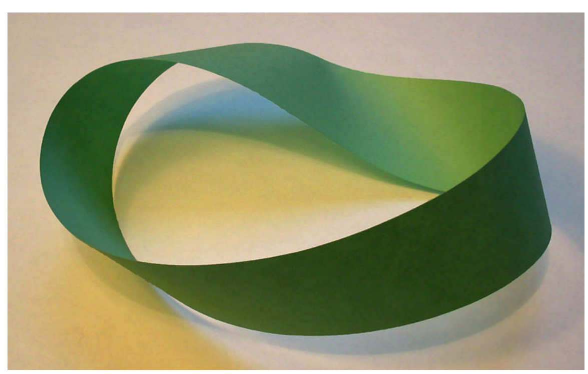


4. The **helicoid** $(r\cos\theta, r\sin\theta, a\theta)$ can be considered to be a ruled surface by taking $\boldsymbol{\alpha} = (0,0,a\theta)$ and $\boldsymbol{\beta} = (\cos\theta, \sin\theta, 0)$.

5. Another example of a ruled surface is a **Möbius strip** (or Möbius band). A model can be created by taking a paper strip and giving it a half-twist (180°-twists), and then joining the ends of the strip together to form a loop.

The Möbius strip has several curious properties: it is a surface with **only one side and only one boundary**. To convince yourself of these facts, create your own Möbius strip and play with it (or go to Wikipedia and study the images there).

Another interesting property is that if you cut a Möbius strip along the center line, you will get one long strip with two full twists in it, not two separate strips. The resulting strip will have two sides and two boundaries. So, cutting created the second boundary.



Möbius strip

Continuing this construction you can deduce that a strip with an odd-number of half-twists will have only one surface and one boundary while a strip with an even-number of half-twists will have two surfaces and two boundaries.

For more curious properties and alternative construction of Möbius strip, see Wikipedia.

A Möbius strip can be obtained as a ruled surface by considering $\boldsymbol{\alpha}$ to be a unit-circle in xy-plane $(\cos t, \sin t, 0)$. Through each point of $\boldsymbol{\alpha}$ pass a line segment of unit length with midpoint $\boldsymbol{\alpha}(t)$ in direction of $\boldsymbol{\beta}(t) = \sin \frac{t}{2}\boldsymbol{\alpha}(t) + \cos \frac{t}{2}(0, 0, 1)$. The ruled surface $\mathbf{x}(t, s) = \boldsymbol{\alpha}(t) + s\boldsymbol{\beta}(t)$ is a Möbius strip.

There are many applications of Möbius strip in science, technology and everyday life. For example, Möbius strips have been used as conveyor belts (that last longer because the entire surface area of the belt gets the same amount of wear), fabric computer printer and typewriter ribbons. Medals often have a neck ribbon configured as a Möbius strip that allows the ribbon to fit comfortably around the neck while the medal lies flat on the chest. Examples of Möbius strip can be encountered: in physics as compact resonators and as superconductors with high transition temperature; in chemistry as molecular knots with special characteristics (e.g. chirality); in music theory as dyads and other areas.

The First Fundamental Form

The first fundamental form describes the way of measuring the distances on a surface. An apparatus that enables one to measure the distances is called **metric**. This is why the first fundamental form is often referred to as the metric form.

Since the basis of the tangent plane $\frac{\partial \mathbf{x}}{\partial u}$ and $\frac{\partial \mathbf{x}}{\partial v}$ will play a major role in the definition of the metric form, we use the usual abbreviation and denote them by $\mathbf{x}_1$ and $\mathbf{x}_2$.

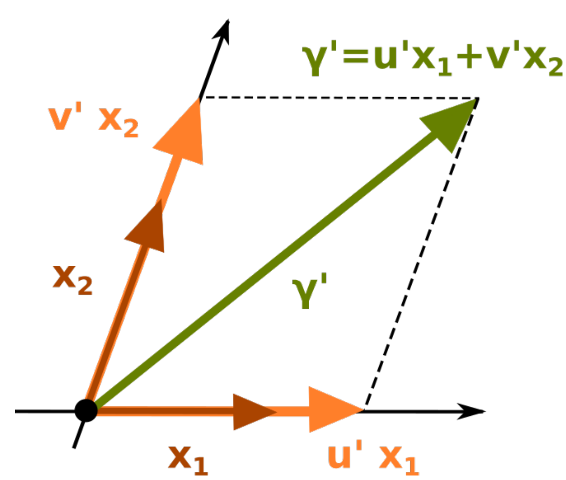
The condition $\mathbf{x}_1 \times \mathbf{x}_2 \neq \mathbf{0}$ guarantees that the tangent plane is not collapsed into a line or a point i.e. that it is a two-dimensional plane. It also implies that the vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ can be taken to be a **basis of the tangent plane.**

In particular, this means that the velocity vector of every curve on the surface can be represented via $\mathbf{x}_1$ and $\mathbf{x}_2$. Since the arc length of the curve can be found by integrating the length of the velocity vector, such **length will be computed by an integral involving $\mathbf{x}_1$ and $\mathbf{x}_2$**. This leads to the definition of the first fundamental form.

Let us start by considering the arc length of a curve $\boldsymbol{\gamma}(t) = \mathbf{x}(u(t), v(t))$ on a surface $\mathbf{x}$. The velocity vector $\boldsymbol{\gamma}'(t)$ is given by the chain rule $\frac{\partial \mathbf{x}}{\partial u}\frac{du}{dt} + \frac{\partial \mathbf{x}}{\partial v}\frac{dv}{dt}$ which, using our new abbreviation, can be written as

$$\boldsymbol{\gamma}'(t) = \mathbf{x}_1 \frac{du}{dt} + \mathbf{x}_2 \frac{dv}{dt} = u'\mathbf{x}_1 + v'\mathbf{x}_2.$$

Thus, the velocity vector is a linear combination of the basis vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ with coefficients u' and v'.



The length on the curve is given by $L = \int_a^b |\boldsymbol{\gamma}'(t)|dt$. The square of length $|\boldsymbol{\gamma}'(t)|^2$ is equal to the dot product $\boldsymbol{\gamma}'(t) \cdot \boldsymbol{\gamma}'(t) = (u'\mathbf{x}_1 + v'\mathbf{x}_2) \cdot (u'\mathbf{x}_1 + v'\mathbf{x}_2)$, thus

$$|\boldsymbol{\gamma}'(t)|^2 = (u')^2\, \mathbf{x}_1 \cdot \mathbf{x}_1 + 2u'v'\, \mathbf{x}_1 \cdot \mathbf{x}_2 + (v')^2\, \mathbf{x}_2 \cdot \mathbf{x}_2.$$

Thus, the three dot products featured in this formula completely determine the arc length of any curve on the surface. To further abbreviate the notation, the dot products are denotes as follows

$$g_{11} = \mathbf{x}_1 \cdot \mathbf{x}_1, \quad g_{12} = \mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_2 \cdot \mathbf{x}_1 = g_{21}, \quad g_{22} = \mathbf{x}_2 \cdot \mathbf{x}_2$$

and are called **the coefficients of the first fundamental form.**

The traditional notation $g_{11} = E, g_{12} = F$, and $g_{22} = G$ comes from Gauss. The more modern notation $g_{11}, g_{12} = g_{21}$, and g_{22} is convenient for representing the relevant dot products as a matrix $[g_{ij}] = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$.

Using this notation, $|\boldsymbol{\gamma}'(t)|^2 = g_{11}(u')^2 + 2g_{12}u'v' + g_{22}(v')^2$ so that the length of the curve $\boldsymbol{\gamma}$ on the surface $\mathbf{x}$ is given by

$$L = \int_a^b |\boldsymbol{\gamma}'(t)|dt = \int_a^b \left(g_{11}(u')^2 + 2g_{12}u'v' + g_{22}(v')^2\right)^{1/2} dt = \int_a^b \left(g_{11}\, du^2 + 2g_{12}\, du\, dv + g_{22}\, dv^2\right)^{1/2}$$

The expression under the root, $g_{11}\, du^2 + 2g_{12}\, du\, dv + g_{22}\, dv^2$ is called the **first fundamental form**.

Example 1. The first fundamental form of a plane. Consider the xy-plane $z = 0$ for simplicity. Thus $\mathbf{x}(x, y) = (x, y, 0)$ and you can consider that "u" is x and "v" is y here. Then $\mathbf{x}_1 = \frac{\partial \mathbf{x}}{\partial x} = (1, 0, 0)$ and $\mathbf{x}_2 = \frac{\partial \mathbf{x}}{\partial y} = (0, 1, 0)$ and these are the basis vectors of the tangent plane. Note that they span exactly the plane $z = 0$ which shouldn't be surprising since the tangent plane of a plane is that same plane.

The coefficients of the first fundamental form are $g_{11} = \mathbf{x}_1 \cdot \mathbf{x}_1 = 1, g_{12} = \mathbf{x}_1 \cdot \mathbf{x}_2 = 0$, and $g_{22} = \mathbf{x}_2 \cdot \mathbf{x}_2 = 1$, or, represented in a matrix, they are $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Note that this is the identity matrix reflecting the fact that the metric on xy-plane is the usual, standard metric. If $\boldsymbol{\gamma}$ is a parametric curve $(x(t), y(t))$ in xy-plane, the general arc-length formula above becomes the familiar Calculus 2 arc-length formula $L = \int_a^b \sqrt{(x')^2 + (y')^2}dt$.

Example 2. The first fundamental form of a cylinder. Consider the cylinder $(a\cos t, a\sin t, z)$ with a circular base of radius a. Thus you can consider that "u" is t and "v" is z. Then $\mathbf{x}_1 = \frac{\partial \mathbf{x}}{\partial t} = (-a\sin t, a\cos t, 0)$ and $\mathbf{x}_2 = \frac{\partial \mathbf{x}}{\partial z} = (0, 0, 1)$. $g_{11} = \mathbf{x}_1 \cdot \mathbf{x}_1 = a^2, g_{22} = \mathbf{x}_2 \cdot \mathbf{x}_2 = 1$, and $g_{12} = \mathbf{x}_1 \cdot \mathbf{x}_2 = 0$, or, represented in a matrix, they are $\begin{bmatrix} a^2 & 0 \\ 0 & 1 \end{bmatrix}$. This matrix is "almost" the identity matrix from the previous example (in fact if $a = 1$ the lengths on the cylinder are exactly the same as in a plane). The close proximity of these two fundamental forms is due to the fact that a cylinder is obtained only by rolling up a plane without any deformations. So, one can measure the distances on a cylinder by "unrolling" it and then measuring the distances in so-obtained plane.

Example 3. The first fundamental form of a sphere. Consider the sphere of radius a centered at the origin. Using the geographical coordinates

$$\mathbf{x} = (a\cos\theta\cos\phi, a\sin\theta\cos\phi, a\sin\phi).$$

If you consider that "u" is θ and "v" is ϕ, then $\mathbf{x}_1 = (-a\sin\theta\cos\phi, a\cos\theta\cos\phi, 0)$ and $\mathbf{x}_2 = (-a\cos\theta\sin\phi,\ -a\sin\theta\sin\phi,\ a\cos\phi)$. So, $g_{11} = a^2\cos^2\phi$, $g_{12} = 0$, $g_{22} = a^2$. Represented by a matrix, the fist fundamental form is $\begin{bmatrix} a^2\cos^2\phi & 0 \\ 0 & a^2 \end{bmatrix}$. The first fundamental form is not constant so the geometry of the sphere is more complex than the geometry of a plane and a cylinder.

Example 4. The first fundamental form of the surface $z = z(x, y)$. Let us consider this surface as $\mathbf{x} = (x, y, z(x, y))$ parametrized by x and y. The derivatives are $\mathbf{x}_1 = (1, 0, z_x)$ and $\mathbf{x}_2 = (0, 1, z_y)$ where z_x and z_y denote the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. The coefficients of the first fundamental form are $g_{11} = 1 + z_x^2$, $g_{12} = z_x z_y$ and $g_{22} = 1 + z_y^2$.

Measuring angles. We have seen how the first fundamental form enables us to compute lengths. Let us now consider measuring angles on a surface. Recall that the angle α between two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ can be computed from the formula for the dot product

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = |\mathbf{v}_1||\mathbf{v}_2|\cos\alpha \quad \Rightarrow \quad \cos\alpha = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{|\mathbf{v}_1||\mathbf{v}_2|} = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\sqrt{\mathbf{v}_1 \cdot \mathbf{v}_1}\sqrt{\mathbf{v}_2 \cdot \mathbf{v}_2}}$$

Note that the right side of the last equation is described completely in terms of the dot products of vectors. So, if $\mathbf{v}_1$ and $\mathbf{v}_2$ are two vectors in the plane tangent to the surface $\mathbf{x} = \mathbf{x}(u, v)$ at a point, representing the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ via the basis vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ and expressing the dot product and the lengths via the coefficients g_{ij} we can obtain a formula computing the angle α in terms of the coefficients g_{ij}.

The **angle between two curves** is defined to be the angle between their tangent (or velocity) vectors. So, the angle between two curves on a surface can be defined as the angle between the two corresponding tangent vectors in the tangent plane and can be expressed in terms of the coefficients g_{ij}. The next example shows that these formulas are particularly simple in the case of u and v-curves.

Example 1. Find the formula computing the angle between a u-curve and a v-curve on a surface $\mathbf{x} = \mathbf{x}(u, v)$.

Solution. Recall that an u-curve is obtained by fixing v and considering u as the parameter of the curve. Thus the velocity vector of this curve is obtained by finding derivative of $\mathbf{x}$ with respect to u while considering v as a constant. This is exactly $\frac{\partial \mathbf{x}}{\partial u} = \mathbf{x}_1$. Equivalently, the velocity vector of a v-curve is $\frac{\partial \mathbf{x}}{\partial v} = \mathbf{x}_2$.

The angle α between a u-curve and a v-curve is the angle between their velocity vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ hence

$$\cos\alpha = \frac{\mathbf{x}_1 \cdot \mathbf{x}_2}{\sqrt{\mathbf{x}_1 \cdot \mathbf{x}_1}\sqrt{\mathbf{x}_2 \cdot \mathbf{x}_2}} = \frac{g_{12}}{\sqrt{g_{11}}\sqrt{g_{22}}}$$

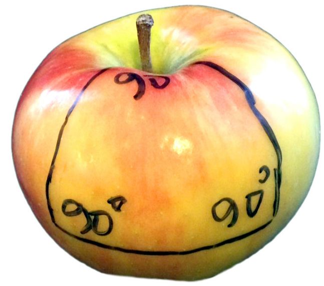
Example 2. The meridians and parallels of a sphere are perpendicular.

Solution. Consider the sphere $\mathbf{x} = (a\cos\theta\cos\phi, a\sin\theta\cos\phi, a\sin\phi)$ and recall that we computed the coefficients of the first fundamental form to be $g_{11} = a^2\cos^2\phi$, $g_{12} = 0$, $g_{22} = a^2$. Since $g_{12} = 0$, $\cos\alpha = 0$ which implies that $\alpha = \pm\frac{\pi}{2}$. In either case, the θ-curves (parallels) and ϕ-curves (meridians) are perpendicular.

Example 3. Compute the angle between meridians with $\theta = 0$ and $\theta = \frac{\pi}{2}$ at the point of their intersection in the north pole.

Solution. The meridian with $\theta = 0$ is $\boldsymbol{\gamma}_1 = (a\cos\phi, 0, a\sin\phi)$. The meridian with $\theta = \frac{\pi}{2}$ is $\boldsymbol{\gamma}_2 = (0, a\cos\phi, a\sin\phi)$, So $\boldsymbol{\gamma}_1' = (-a\sin\phi, 0, a\cos\phi)$, $\boldsymbol{\gamma}_2' = (0, -a\sin\phi, a\cos\phi)$, and $\boldsymbol{\gamma}_1' \cdot \boldsymbol{\gamma}_2' = a^2\cos^2\phi$.

The curves $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ intersect in the north pole where $\phi = \frac{\pi}{2}$. So, at the north pole $\boldsymbol{\gamma}_1' \cdot \boldsymbol{\gamma}_2' = a^2\cos^2\frac{\pi}{2} = 0$. So, $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ are perpendicular.

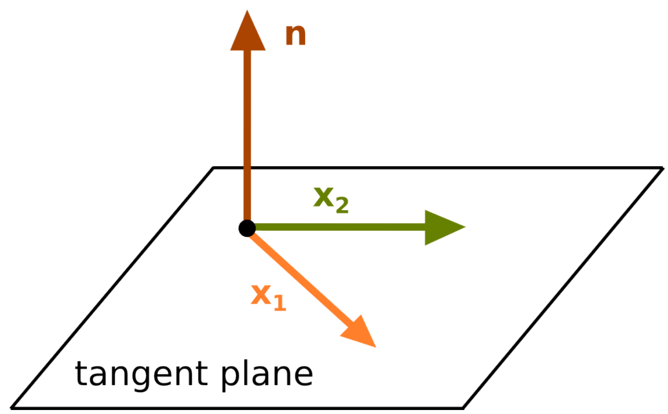


This example also shows that all three angles in the "triangle" formed by the equator and the two meridians above are 90 degrees. So the three angles add up to 270 degrees.

A certain quantity can be **measured intrinsically** if it can be computed using the coefficients of the first fundamental form only which enable measuring distances and angles without using any references to exterior space or the particular embedding. In particular, if certain quantity can be expressed solely in terms of the coefficients of the first fundamental form, it is an **intrinsic quantity**. Thus, to show Theorema Egregium, it is sufficient to show that the Gaussian curvature K can be computed solely using the coefficients of the first fundamental form.

The unit normal vector. The nonzero vector $\mathbf{x}_1 \times \mathbf{x}_2$ is perpendicular to the tangent plane. Thus, the unit normal vector of the tangent plane is given by

$$\boxed{\mathbf{n} = \frac{\mathbf{x}_1 \times \mathbf{x}_2}{|\mathbf{x}_1 \times \mathbf{x}_2|}}$$



This vector should not be confused with the normal vector $\mathbf{N}$ of a curve on a surface. In fact, the vectors $\mathbf{n}$ and $\mathbf{N}$ may have different direction.

For example, let $\boldsymbol{\gamma}$ be a circle obtained by intersection of a sphere and a plane that does not contain the center of the sphere. In this case, the radius of $\boldsymbol{\gamma}$ is less that the radius of the sphere and the center of $\boldsymbol{\gamma}$ is different than the center of the sphere. If P is a point on $\boldsymbol{\gamma}$ then the direction of $\mathbf{n}$ is determined by the line connecting P and the center of the sphere and the direction of $\mathbf{N}$ is determined by the line connecting P and the center of $\boldsymbol{\gamma}$.

Using Lagrange identity $|\mathbf{x}_1 \times \mathbf{x}_2|^2 = (\mathbf{x}_1 \cdot \mathbf{x}_1)(\mathbf{x}_2 \cdot \mathbf{x}_2) - (\mathbf{x}_1 \cdot \mathbf{x}_2)^2$, we have that $|\mathbf{x}_1 \times \mathbf{x}_2|^2 = g_{11}g_{22} - g_{12}^2$, the determinant of the matrix $[g_{ij}]$. The determinant $g_{11}g_{22} - g_{12}^2$ is usually denoted by g. Thus,

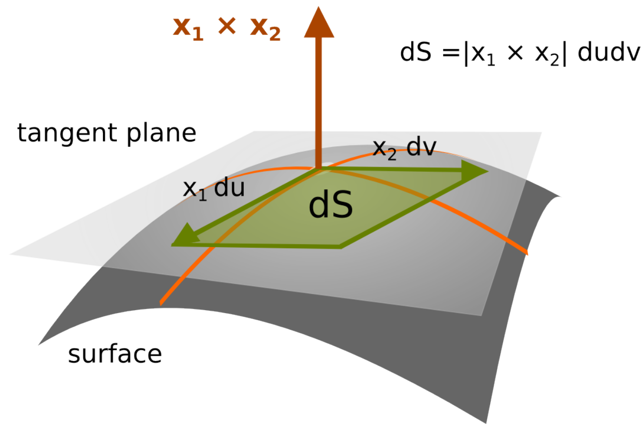
$$|\mathbf{x}_1 \times \mathbf{x}_2|^2 = g \quad \text{and} \quad \mathbf{n} = \frac{\mathbf{x}_1 \times \mathbf{x}_2}{\sqrt{g}}.$$

Measuring surface areas. Besides enabling us to compute lengths and angles on a surface, the first fundamental form also enables us to compute the surface areas. Note that the total area of region D on the surface can be computed by adding up all the areas of "rectangular" regions (i.e. approximately parallelogram shaped pieces on the surface) determined by the intersections of the u and v-curves.

Since the length of the cross product $|\mathbf{x}_1 \times \mathbf{x}_2|$ determines the area of parallelogram determined by $\mathbf{x}_1$ and $\mathbf{x}_2$, the area of one such "rectangular" region is given by

$$dS = |\mathbf{x}_1 \times \mathbf{x}_2|\, du\, dv.$$

This produces the familiar Calculus 3 formula $\int\int_S |\mathbf{x}_1 \times \mathbf{x}_2|\, du\, dv$ that calculates **the total surface area of a parametric surface over region S.**



Using that $|\mathbf{x}_1 \times \mathbf{x}_2| = \sqrt{g}$, we obtain the formula below for the surface area of the surface $\mathbf{x}(u, v)$.

$$\boxed{\text{Surface area } = \int\int_S dS = \int\int_S \sqrt{g}\, du\, dv}$$

Example. Demonstrate the Calculus 3 formula $\int\int_S \sqrt{1 + z_x^2 + z_y^2}\, dx\, dy$ computing the surface area of a surface given by $z = z(x, y)$ over the region S using the general formula for the surface area above.

Solution. Consider x, y as the two parameters of the surface $\mathbf{x} = (x, y, z(x, y))$. We have found that $\mathbf{x}_1 = (1, 0, z_x)$, $\mathbf{x}_2 = (0, 1, z_y)$ and $g_{11} = 1 + z_x^2$, $g_{12} = z_x z_y$ and $g_{22} = 1 + z_y^2$ (see the example above). Thus,

$$g = (1 + z_x^2)(1 + z_y^2) - z_x^2 z_y^2 = 1 + z_x^2 + z_y^2 + z_x^2 z_y^2 - z_x^2 z_y^2 = 1 + z_x^2 + z_y^2$$

Hence the formula $\int\int_S \sqrt{g}\, dx\, dy = \int\int_S \sqrt{1 + z_x^2 + z_y^2}\, dx\, dy$ computes the surface area.

Practice Problems.

1. Compute the unit normal vector for the sphere $\mathbf{x} = (a\cos\theta\cos\phi, a\sin\theta\cos\phi, a\sin\phi)$.

2. Find the surface area of the part of the cylinder $\mathbf{x} = (a\cos\theta, a\sin\theta, h)$ with $0 \leq z \leq c$.

3. Consider the torus obtained by revolving a circle $(x - a)^2 + z^2 = b^2$ in xz-plane along the circle $x^2 + y^2 = a^2$ in xy-plane. Since the first circle can be parametrized by $x = a + b\cos\phi$, $z = b\sin\phi$ and a surface of revolution of a curve $x = f(u), z = g(u)$ in xz-plane about z-axis is given by the parametric equations $\mathbf{x} = (f(u)\cos\theta, f(u)\sin\theta, g(u))$, the torus can be parametrized as

$$\mathbf{x} = ((a + b\cos\phi)\cos\theta, (a + b\cos\phi)\sin\theta, b\sin\phi).$$

   Compute the coefficients of the first fundamental form and find the unit normal vector of the torus.

4. Find the area of the part of the paraboloid $z = a^2 - x^2 - y^2, a > 0$, that lies above the xy-plane.

5. Find the area of the part of the cone $z = a\sqrt{x^2 + y^2}$ below the plane $z = b$ where $a, b > 0$.

6. Consider the cone from the previous problem as a ruled surface $\mathbf{x}(s,t) = \boldsymbol{\alpha}(t) + s\boldsymbol{\beta}(t)$ obtained by ruling the vector $\boldsymbol{\beta} = (\cos t, \sin t, a)$ based at the vertex $\boldsymbol{\alpha} = (0,0,0)$.

   (a) Using this parametrization of the cone, compute the area of the part of the cone below the plane $z = b$.

   (b) Replace the given $\boldsymbol{\beta}$ by $\boldsymbol{\beta} = (a\cos t, a\sin t, a^2)$. Note that this still parametrizes the same cone but that the bounds for s change when you compute the area of the part of the cone below the plane $z = b$. Compute the area in this parametrization.

Solutions.

1. For the sphere, $\mathbf{x}_1 = (-a\sin\theta\cos\phi, a\cos\theta\cos\phi, 0)$, $\mathbf{x}_2 = (-a\cos\theta\sin\phi,\ -a\sin\theta\sin\phi,\ a\cos\phi)$ so $\mathbf{n} = \frac{1}{a^2\cos\phi}(a^2\cos\theta\cos^2\phi,\ a^2\sin\theta\cos^2\phi,\ a^2\sin\phi\cos\phi) = (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi)$. Note that this is exactly $\frac{1}{a}\mathbf{x}$. Hence $\mathbf{n} = \frac{1}{a}\mathbf{x}$ and so $\mathbf{n}$ and $\mathbf{x}$ are colinear.

2. In one of the previous examples, we computed the coefficients of the first fundamental form of the cylinder $\mathbf{x} = (a\cos\theta, a\sin\theta, h)$ to be $g_{11} = a^2$, $g_{12} = 0$ and $g_{22} = 1$. Hence $g = a^2$.

   Since $\sqrt{g} = a$, the formula $S = \int\int_S a\, d\theta\, dh$ computes the surface area. On the region with $0 \le z \le c$, we have that $0 \le \theta \le 2\pi$ and $0 \le h \le c$. Hence,
$$S = \int_0^{2\pi}\int_0^c a\, d\theta\, dh = a\int_0^{2\pi} d\theta \int_0^c dh = a(2\pi)c = 2ac\pi.$$

3. $\mathbf{x} = ((a+b\cos\phi)\cos\theta, (a+b\cos\phi)\sin\theta, b\sin\phi) \Rightarrow \mathbf{x}_1 = (-(a+b\cos\phi)\sin\theta, (a+b\cos\phi)\cos\theta, 0)$ and $\mathbf{x}_2 = (-b\sin\phi\cos\theta,\ -b\sin\phi\sin\theta,\ b\cos\phi)$. So, $g_{11} = (a+b\cos\phi)^2$, $g_{12} = 0$, $g_{22} = b^2$, $g = b^2(a+b\cos\phi)^2$, and $\mathbf{n} = (\cos\theta\cos\phi,\ \sin\theta\cos\phi,\ \sin\phi)$.

4. Considering the paraboloid as a surface of revolution produces the parametrization
$$\mathbf{x} = (r\cos t, r\sin t, a^2 - r^2).$$
   Then $x_1 = (\cos t, \sin t, -2r)$, $\mathbf{x}_2 = (-r\sin t, r\cos t, 0)$, $g_{11} = \cos^2 t + \sin^2 t + 4r^2 = 1 + 4r^2$, $g_{12} = 0$, and $g_{22} = r^2\sin^2 t + r^2\cos^2 t = r^2$ so that $g = r^2(1+4r^2)$. The paraboloid intersects the xy-plane $z = 0$ at the circle with radius $a^2 - r^2 = 0 \Rightarrow r^2 = a^2 \Rightarrow r = a$. Hence the bounds are $0 \le r \le a$ and $0 \le t \le 2\pi$. The area is
$$S = \int_0^{2\pi}\int_0^a r\sqrt{1+4r^2}\, dr\, dt = 2\pi\ \frac{1}{12}(1+4r^2)^{3/2}\bigg|_0^a = \frac{\pi}{6}\left((1+4a^2)^{3/2} - 1\right)$$

5. Considering the cone as a surface of revolution produces the parametrization
$$\mathbf{x} = (r\cos t, r\sin t, ar).$$
   Then $x_1 = (\cos t, \sin t, a)$, $\mathbf{x}_2 = (-r\sin t, r\cos t, 0)$, $g_{11} = \cos^2 t + \sin^2 t + a^2 = 1 + a^2$, $g_{12} = 0$, and $g_{22} = r^2\sin^2 t + r^2\cos^2 t = r^2$ so that $g = r^2(1+a^2)$. The cone intersects the plane $z = b$ at the circle with radius $ar = b \Rightarrow r = \frac{b}{a}$. Hence the bounds are $0 \le r \le \frac{b}{a}$ and $0 \le t \le 2\pi$. The area is
$$S = \int_0^{2\pi}\int_0^{b/a} r\sqrt{1+a^2}\, dr\, dt = \sqrt{1+a^2}\int_0^{2\pi} dt \int_0^{b/a} r dr = \sqrt{1+a^2}\, 2\pi\, \frac{b^2}{2a^2} = \frac{b^2\pi\sqrt{1+a^2}}{a^2}$$

6. If $\boldsymbol{\alpha} = (0,0,0)$ and $\boldsymbol{\beta} = (\cos t, \sin t, a)$, then $\mathbf{x} = \boldsymbol{\alpha} + s\boldsymbol{\beta} = (0,0,0) + (s\cos t, s\sin t, as) = (s\cos t, s\sin t, as)$.

   (a) Note that the parametrization $\mathbf{x} = (s\cos t, s\sin t, as)$is the same parametrization as in the previous problem if we let $r = s$. Hence, the coefficients of the second fundamental form as well as the area are the same as in the previous problem and the area is $\frac{b^2\pi\sqrt{1+a^2}}{a^2}$.

   (b) With $\boldsymbol{\beta} = (a\cos t, a\sin t, a^2)$,

   $$\mathbf{x} = \boldsymbol{\alpha} + s\boldsymbol{\beta} = (0,0,0) + (as\cos t, as\sin t, a^2 s) = (as\cos t, as\sin t, a^2 s).$$

   Then $x_1 = (a\cos t, a\sin t, a^2), \mathbf{x}_2 = (-as\sin t, as\cos t, 0), g_{11} = a^2\cos^2 t + a^2\sin^2 t + a^4 = a^2 + a^4 = a^2(1+a^2), g_{12} = 0$, and $g_{22} = a^2 s^2 \sin^2 t + a^2 s^2 \cos^2 t = a^2 s^2$ so that $g = a^4 s^2 (1+a^2)$. The cone intersects the plane $z = b$ at the circle with radius $a^2 s = b \Rightarrow r = \frac{b}{a^2}$. Hence the bounds are $0 \le s \le \frac{b}{a^2}$ and $0 \le t \le 2\pi$. The area is

   $$S = \int_0^{2\pi} \int_0^{b/a^2} a^2 s\sqrt{1+a^2}\, ds\, dt = a^2\sqrt{1+a^2}\ 2\pi\ \frac{b^2}{2a^4} = \frac{b^2\pi\sqrt{1+a^2}}{a^2}$$

   which is the same as with the other parametrization. The previous two problems illustrate that the surface area is independent of specific parametrization.