Differential Geometry
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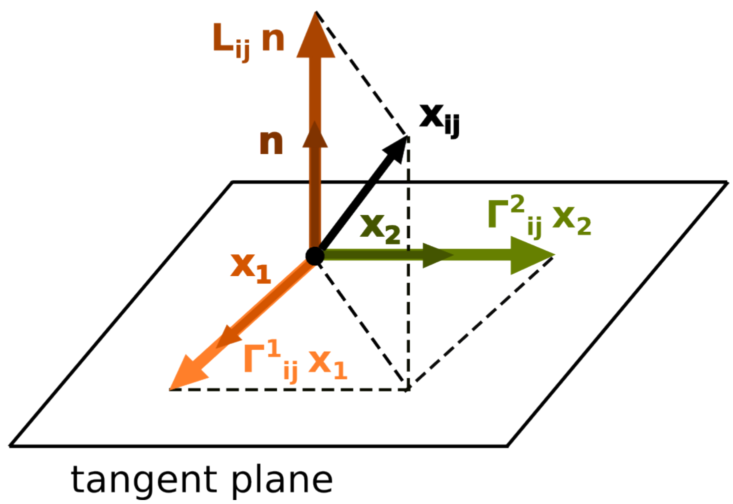
The Second Fundamental Form. Geodesics. The Curvature Tensor. The Fundamental Theorem of Surfaces. Manifolds

The Second Fundamental Form and the Christoffel symbols. Consider a surface $\mathbf{x} = \mathbf{x}(u, v)$. Following the reasoning that $\mathbf{x}_1$ and $\mathbf{x}_2$ denote the derivatives $\frac{\partial \mathbf{x}}{\partial u}$ and $\frac{\partial \mathbf{x}}{\partial v}$ respectively, we denote the second derivatives

$$\frac{\partial^2 \mathbf{x}}{\partial u^2} \text{ by } \mathbf{x}_{11},\ \frac{\partial^2 \mathbf{x}}{\partial v \partial u} \text{ by } \mathbf{x}_{12},\ \frac{\partial^2 \mathbf{x}}{\partial u \partial v} \text{ by } \mathbf{x}_{21}, \text{ and } \frac{\partial^2 \mathbf{x}}{\partial v^2} \text{ by } \mathbf{x}_{22}.$$

The terms $\mathbf{x}_{ij}$, $i, j = 1, 2$ can be represented as a linear combination of tangential and normal component. Each of the vectors $\mathbf{x}_{ij}$ can be represented as a combination of the tangent component (which itself is a combination of vectors $\mathbf{x}_1$ and $\mathbf{x}_2$) and the normal component (which is a multiple of the unit normal vector $\mathbf{n}$). Let Γ^1_{ij} and Γ^2_{ij} denote the coefficients of the tangent component and L_{ij} denote the coefficient with $\mathbf{n}$ of vector $\mathbf{x}_{ij}$. Thus,

$$\boxed{\mathbf{x}_{ij} = \Gamma^1_{ij}\mathbf{x}_1 + \Gamma^2_{ij}\mathbf{x}_2 + L_{ij}\mathbf{n} = \textstyle\sum_k \Gamma^k_{ij}\mathbf{x}_k + L_{ij}\mathbf{n}.}$$



The formula above is called the **Gauss formula**.

The coefficients Γ^k_{ij} where $i, j, k = 1, 2$ are called **Christoffel symbols** and the coefficients L_{ij}, $i, j = 1, 2$ are called **the coefficients of the second fundamental form**.

Einstein notation and tensors. The term "Einstein notation" refers to the certain summation convention that appears often in differential geometry and its many applications. Consider a formula can be written in terms of a sum over an index that appears in subscript of one and superscript of the other variable. For example, $\mathbf{x}_{ij} = \sum_k \Gamma^k_{ij}\mathbf{x}_k + L_{ij}\mathbf{n}$. In cases like this *the summation symbol is omitted*. Thus, the **Gauss formula** for $\mathbf{x}_{ij}$ in Einstein notation is written simply as

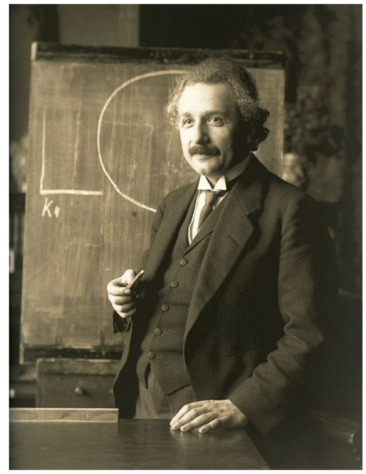
$$\mathbf{x}_{ij} = \Gamma^k_{ij}\mathbf{x}_k + L_{ij}\mathbf{n}.$$

An important benefit of the use of Einstein notation can be seen when considering n-dimensional manifolds – all the formulas we consider for surfaces generalize to formulas for n-dimensional manifolds. For example, the formula $\mathbf{x}_{ij} = \Gamma^k_{ij}\mathbf{x}_k + L_{ij}\mathbf{n}$ remains true except that the indices i, j take integer values ranging from 1 to n not just values 1 and 2.

If we consider the scalar components in certain formulas as arrays of scalar functions, we arrive to the concept of a

$$\boxed{\textbf{tensor}.}$$

For example, a 2×2 matrix with entries g_{ij} is considered to be a tensor of rank 2. This matrix is referred to as the **metric tensor.** The scalar functions Γ^k_{ij} are considered to be the components of a tensor Γ of rank 3 (or type (2,1)). The Einstein notation is crucial for simplification of some complicated tensors.



Another good example of the use of Einstein notation is the matrix multiplication *(students who did not take Linear Algebra can skip this example and the next several paragraphs that relate to matrices).* If A is an $n \times m$ matrix and $\mathbf{v}$ is a $m \times 1$ (column) vector, the product $A\mathbf{v}$ will be a $n \times 1$ column vector. If we denote the elements of A by a^i_j where $i = 1, \ldots, n$, $j = 1, \ldots, m$ and x^j denote the entries of vector $\mathbf{x}$, then the entries of the product $A\mathbf{x}$ are given by the sum $\sum_j a^i_j x^j$ that can be denoted by $a^i_j x^j$ using Einstein notation.

Note also that the entries of a column vector are denoted with indices in superscript and the entries of row vectors with indices in subscript. This convention agrees with the fact that the entries of the column vector $a^i_j x^j$ depend just on the superscript i.

Another useful and frequently considered tensor is the Kronecker delta symbol. Recall that the identity matrix I is a matrix with the ij-th entry 1 if $i = j$ and 0 otherwise. Denote these entries by δ^i_j. Thus,

$$\delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

In this notation, the equation $I\mathbf{x} = \mathbf{x}$ can be written as $\delta^i_j x^j = x^i$.

Recall that the inverse of a matrix A is the matrix A^{-1} with the property that the products AA^{-1} and $A^{-1}A$ are both equal to the identity matrix I. If a_{ij} denote the elements of the matrix A, a^{ij} denote the elements of the inverse matrix A^{ij}, the ij-th element of the product $A^{-1}A$ in Einstein notation is given by $a^{ik}a_{kj}$. Thus $a^{ik}a_{kj} = \delta^i_j$.

In particular, let g^{ij} denote the entries of the inverse matrix of $[g_{ij}]$ whose entries are the coefficients of the first fundamental form. The fact that the matrix and its inverse multiply to the identity gives us the following formulas (all given in Einstein notation).

$$g_{ik}g^{kj} = \delta^j_i \text{ and } g^{ik}g_{kj} = \delta^i_j.$$

The coefficients g^{ij} of the inverse matrix are given by the formulas

$$g^{11} = \frac{g_{22}}{g}, \quad g^{12} = \frac{-g_{12}}{g}, \quad \text{and } g^{22} = \frac{g_{11}}{g}$$

where g is the determinant of the matrix $[g_{ij}]$.

Computing the second fundamental form and the Christoffel symbols. The formula computing the Christoffel symbols can be obtained by multiplying the equation $\mathbf{x}_{ij} = \Gamma^k_{ij}\mathbf{x}_k + L_{ij}\mathbf{n}$ by $\mathbf{x}_l$ where $k = 1, 2$. Since $\mathbf{n} \cdot \mathbf{x}_l = 0$, and $\mathbf{x}_k \cdot \mathbf{x}_l = g_{kl}$, we obtain that

$$\mathbf{x}_{ij} \cdot \mathbf{x}_l = \Gamma^k_{ij} g_{kl}$$

To solve for Γ^k_{ij}, we have to get rid of the terms g_{kl} from the left side. This can be done by using the inverse matrix g^{ls}.

$$(\mathbf{x}_{ij} \cdot \mathbf{x}_l) g^{ls} = \Gamma^k_{ij} g_{kl} g^{ls} = \Gamma^k_{ij} \delta^s_k = \Gamma^s_{ij}$$

Thus, we obtain that the Christoffel symbols can be computed by the formula

$$\boxed{\Gamma^k_{ij} = (\mathbf{x}_{ij} \cdot \mathbf{x}_l) g^{lk}.}$$

To compute the coefficients of the second fundamental form, multiply the equation $\mathbf{x}_{ij} = \Gamma^k_{ij}\mathbf{x}_k + L_{ij}\mathbf{n}$ by $\mathbf{n}$. Since $\mathbf{x}_l \cdot \mathbf{n} = 0$, we have that $\mathbf{x}_{ij} \cdot \mathbf{n} = L_{ij}\mathbf{n} \cdot \mathbf{n} = L_{ij}$. Thus,

$$\boxed{L_{ij} = \mathbf{x}_{ij} \cdot \mathbf{n} = \mathbf{x}_{ij} \cdot \tfrac{\mathbf{x}_1 \times \mathbf{x}_2}{|\mathbf{x}_1 \times \mathbf{x}_2|}.}$$

While the first fundamental form determines the intrinsic geometry of the surface, the second fundamental form reflects the way how the surface embeds in the surrounding space and how it curves relative to that space. Thus, the second fundamental form reflects the **extrinsic geometry of the surface**. The presence of the normal vector $\mathbf{n}$ in the formula for L_{ij} reflects this fact also since $\mathbf{n}$ "sticks out" of the surface. In contrast, we shall see that the Christoffel symbols can be computed using the first fundamental form only which shows that they are completely intrinsic.

Example 1. The Christoffel symbols and the second fundamental form of a plane. Consider the xy-plane $z = 0$ for simplicity. Thus $\mathbf{x}(x, y) = (x, y, 0)$. Recall (or compute again) that $\mathbf{x}_1 = \mathbf{x}_x = (1, 0, 0)$, $\mathbf{x}_2 = \mathbf{x}_y = (0, 1, 0)$, $g_{11} = g_{22} = 1$, and $g_{12} = 0$.

Since the first derivatives $\mathbf{x}_1$ and $\mathbf{x}_2$ are constant, the second derivatives $\mathbf{x}_{11}, \mathbf{x}_{12} = \mathbf{x}_{21}, \mathbf{x}_{22}$ are all zero. As a result, the second fundamental form L_{ij} are zero.

$$L_{ij} = \mathbf{x}_{ij} \cdot \mathbf{n} = (0, 0, 0) \cdot \mathbf{n} = 0$$

Represented as a matrix, the second fundamental form is $\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right]$. This means that the surface does not deviate from its tangent plane at all and this should not be surprising since the tangent plane of a plane is that plane itself.

The Christoffel symbols are also zero.

$$\Gamma^k_{ij} = (\mathbf{x}_{ij} \cdot \mathbf{x}_l) g^{lk} = ((0, 0, 0) \cdot \mathbf{x}_l) g^{lk} = 0.$$

We shall see later that this means that surfaces with this property are such that their shortest-distance curves (geodesics) are really lines.

Example 2. The Christoffel symbols and the second fundamental form of a cylinder. Consider the cylinder $(a\cos t, a\sin t, z)$ with a circular base of radius a. Recall (or compute again) that $\mathbf{x}_1 = (-a\sin t, a\cos t, 0)$, $\mathbf{x}_2 = (0, 0, 1)$ and so $g_{11} = a^2$, $g_{12} = 0$ and $g_{22} = 1$. Thus, $g = a^2$ and so $g^{11} = \frac{g_{22}}{g} = \frac{1}{a^2}$, $g^{12} = \frac{-g_{12}}{g} = 0$ and $g^{22} = \frac{g_{11}}{g} = \frac{a^2}{a^2} = 1$ so the matrix inverse to the first fundamental form is $\left[\begin{array}{cc} \frac{1}{a^2} & 0 \\ 0 & 1 \end{array}\right]$. The unit normal vector is $\mathbf{n} = (\cos t, \sin t, 0)$.

The second derivatives are $\mathbf{x}_{11} = (-a\cos t, -a\sin t,\, 0)$, $\mathbf{x}_{12} = (0,0,0)$, and $\mathbf{x}_{22} = (0,0,0)$. Thus, $L_{11} = \mathbf{x}_{11}\cdot\mathbf{n} = -a\cos^2 t - a\sin^2 t = -a$, $L_{12} = \mathbf{x}_{12}\cdot n = 0$, and $L_{22} = \mathbf{x}_{22}\cdot\mathbf{n} = 0$. So, the second fundamental form is $\begin{bmatrix} -a & 0 \\ 0 & 0 \end{bmatrix}$. When comparing the second fundamental form of the cylinder to that of the plane from the previous example, we can see that they differ only in the first coefficient. Note also that these two matrices have the same determinant – we shall elaborate on this fact later.

Since $\mathbf{x}_{12} = \mathbf{x}_{22} = (0,0,0)$, $\Gamma^k_{12} = \Gamma^k_{21} = \Gamma^k_{22} = 0$. The remaining two Christoffel symbols, Γ^1_{11} and Γ^2_{11} can be computed as follows.

$$\Gamma^1_{11} = \mathbf{x}_{11}\cdot\mathbf{x}_1 g^{11} + \mathbf{x}_{11}\cdot\mathbf{x}_2 g^{21} = 0g^{11} + 0(0) = 0 \text{ and}$$

$$\Gamma^2_{11} = \mathbf{x}_{11}\cdot\mathbf{x}_1 g^{12} + \mathbf{x}_{11}\cdot\mathbf{x}_2 g^{22} = 0(0) + 0g^{22} = 0.$$

Hence, all Christoffel symbols are zero.

Example 3. The Christoffel symbols and the second fundamental form of a sphere. Consider the sphere $\mathbf{x} = (a\cos\theta\cos\phi, a\sin\theta\cos\phi, a\sin\phi)$ of radius a parametrized by geographic coordinates. Recall (or compute again) that $\mathbf{x}_1 = (-a\sin\theta\cos\phi, a\cos\theta\cos\phi, 0)$, $\mathbf{x}_2 = (-a\cos\theta\sin\phi, -a\sin\theta\sin\phi, a\cos\phi)$, $g_{11} = a^2\cos^2\phi$, $g_{12} = 0$, $g_{22} = a^2$, and $\mathbf{n} = (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi)$.

Compute that $\mathbf{x}_{11} = (-a\cos\theta, -a\sin\theta\cos\phi, 0)$, $\mathbf{x}_{12} = \mathbf{x}_{21} = (a\sin\theta\sin\phi, -a\cos\theta\sin\phi, 0)$, and $\mathbf{x}_{22} = (-a\cos\theta\cos\phi, -a\sin\theta\cos\phi, -a\sin\phi)$, so that $L_{11} = -a\cos^2\phi$, $L_{12} = 0$ and $L_{22} = -a$ and the second fundamental form is $\begin{bmatrix} -a\cos^2\phi & 0 \\ 0 & -a \end{bmatrix}$.

Compute then that $g^{11} = \frac{1}{a^2\cos^2\phi}$, $g^{12} = 0$, $g^{22} = \frac{1}{a^2}$, so that the inverse of the first fundamental form is $\begin{bmatrix} \frac{1}{a^2\cos^2\phi} & 0 \\ 0 & \frac{1}{a^2} \end{bmatrix}$. Lastly, compute the dot products of the second and the first derivatives to be $\mathbf{x}_{11}\cdot\mathbf{x}_1 = 0$, $\mathbf{x}_{11}\cdot\mathbf{x}_2 = a^2\sin\phi\cos\phi$, $\mathbf{x}_{12}\cdot\mathbf{x}_1 = \mathbf{x}_{21}\cdot\mathbf{x}_1 = -a^2\sin\phi\cos\phi$, $\mathbf{x}_{12}\cdot\mathbf{x}_2 = \mathbf{x}_{21}\cdot\mathbf{x}_2 = 0$, $\mathbf{x}_{22}\cdot\mathbf{x}_1 = 0$, $\mathbf{x}_{22}\cdot\mathbf{x}_2 = 0$.

Thus, the Christoffel symbols are $\Gamma^1_{11} = \mathbf{x}_{11}\cdot\mathbf{x}_1 g^{11} + \mathbf{x}_{11}\cdot\mathbf{x}_2 g^{21} = 0g^{11} + \mathbf{x}_{11}\cdot\mathbf{x}_2(0) = 0$,

$$\Gamma^2_{11} = \mathbf{x}_{11}\cdot\mathbf{x}_1 g^{12} + \mathbf{x}_{11}\cdot\mathbf{x}_2 g^{22} = 0(0) + a^2\sin\phi\cos\phi\frac{1}{a^2} = \sin\phi\cos\phi,$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \mathbf{x}_{12}\cdot\mathbf{x}_1 g^{11} + \mathbf{x}_{12}\cdot\mathbf{x}_2 g^{21} = -a^2\sin\phi\cos\phi\frac{1}{a^2\cos^2\phi} + 0(0) = \frac{-\sin\phi}{\cos\phi}$$

and $\Gamma^2_{12} = \Gamma^2_{21} = \mathbf{x}_{12}\cdot\mathbf{x}_1 g^{12} + \mathbf{x}_{12}\cdot\mathbf{x}_2 g^{22} = 0+0 = 0$, $\Gamma^1_{22} = \mathbf{x}_{22}\cdot\mathbf{x}_1 g^{11} + \mathbf{x}_{22}\cdot\mathbf{x}_2 g^{21} = 0+0 = 0$, and $\Gamma^2_{22} = \mathbf{x}_{22}\cdot\mathbf{x}_1 g^{12} + \mathbf{x}_{22}\cdot\mathbf{x}_2 g^{22} = 0+0 = 0$. Thus, all but three Christoffel symbols are zero. We shall use the calculation above to compute the geodesics on the sphere.

Example 4. Find the coefficients of the second fundamental form of the surface $z = z(x,y)$.

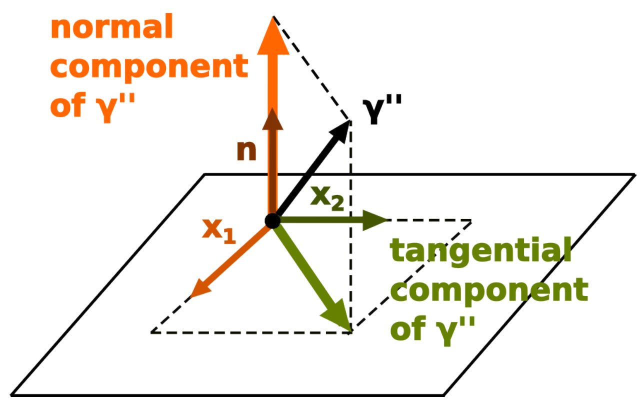
With x, y as parameters, we have that $\mathbf{x}(x,y) = (x, y, z(x,y))$. Let us shorten the notation by using z_1 for z_x, z_2 for z_y, and $z_{11} = z_{xx}$, $z_{12} = z_{21} = z_{xy}$ and $z_{22} = z_{yy}$. We have $\mathbf{x}_1 = (1, 0, z_1)$ and $\mathbf{x}_2 = (0, 1, z_1)$ so that $g_{11} = 1 + z_1^2$, $g_{12} = z_1 z_2$ and $g_{22} = 1 + z_2^2$. Thus, $g = 1 + z_1^2 + z_2^2$. One can also compute that $\mathbf{n} = \frac{1}{\sqrt{g}}(-z_1, -z_2, 1)$ and $\mathbf{x}_{11} = (0,0,z_{11})$, $\mathbf{x}_{12} = (0,0,z_{12})$, $\mathbf{x}_{22} = (0,0,z_{22})$. Thus $L_{ij} = \frac{z_{ij}}{\sqrt{g}}$.

Normal and Geodesic curvature. Geodesics

The tangential and the normal component. Let us consider a curve $\boldsymbol{\gamma} = \boldsymbol{\gamma}(t)$ on a surface $\mathbf{x}$ so that
$$\boldsymbol{\gamma}(t) = \mathbf{x}(u(t), v(t)).$$
The chain rule gives us that
$$\boldsymbol{\gamma}'(t) = u'\mathbf{x}_1 + v'\mathbf{x}_2.$$
Using the notation $\mathbf{x}_{ij}, i, j = 1, 2,$ for the second partial derivatives of $\mathbf{x}$, and differentiating the equation for $\boldsymbol{\gamma}'$ with respect to t again, we obtain that
$$\boldsymbol{\gamma}'' = u''\mathbf{x}_1 + u'(u'\mathbf{x}_{11} + v'\mathbf{x}_{12}) + v''\mathbf{x}_2 + v'(u'\mathbf{x}_{21} + v'\mathbf{x}_{22}) = u''\mathbf{x}_1 + v''\mathbf{x}_2 + u'^2\mathbf{x}_{11} + u'v'\mathbf{x}_{12} + u'v'\mathbf{x}_{21} + v'^2\mathbf{x}_{22}.$$

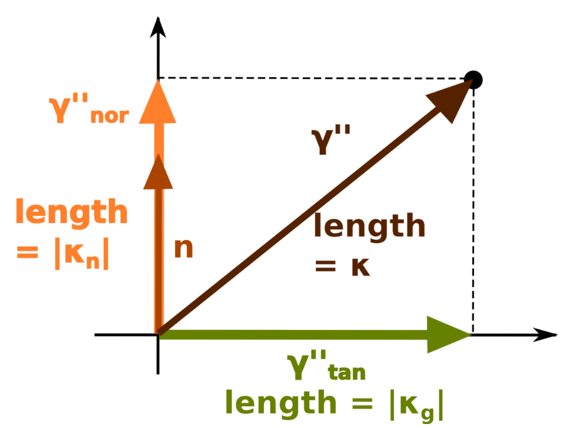
To be able to use Einstein notation, let us
$$\boxed{\text{denote } u \text{ by } u^1 \text{ and } v \text{ by } u^2.}$$
Thus the part $u''\mathbf{x}_1 + v''\mathbf{x}_2$ can be written as $(u^i)''\mathbf{x}_i$ and the part $u'v'\mathbf{x}_{12} + u'v'\mathbf{x}_{21} + v'^2\mathbf{x}_{22}$ as $(u^i)'(u^j)'\mathbf{x}_{ij}$. This gives us the short version of the formula above
$$\boldsymbol{\gamma}'' = (u^i)''\mathbf{x}_i + (u^i)'(u^j)'\mathbf{x}_{ij}.$$

Substituting the Gauss formula $\mathbf{x}_{ij} = \Gamma^k_{ij}\mathbf{x}_k + L_{ij}\mathbf{n}$ in the above formula, we obtain that
$$\boldsymbol{\gamma}'' = (u^i)''\mathbf{x}_i + (u^i)'(u^j)'(\Gamma^k_{ij}\mathbf{x}_k + L_{ij}\mathbf{n}) =$$
$$((u^k)'' + \Gamma^k_{ij}(u^i)'(u^j)')\mathbf{x}_k + (u^i)'(u^j)'L_{ij}\mathbf{n}.$$
The part $((u^k)'' + \Gamma^k_{ij}(u^i)'(u^j)')\mathbf{x}_k$ is the **tangential component**, it is denoted by $\boldsymbol{\gamma}''_{tan}$, and it is in the tangent plane. The part $(u^i)'(u^j)'L_{ij}\mathbf{n}$ is the **normal component** and it is denoted by $\boldsymbol{\gamma}''_{nor}$. The normal component is colinear with $\mathbf{n}$ and, thus, orthogonal to the tangent plane.



Up to a sign, the length of the tangential component $\boldsymbol{\gamma}''_{tan}$ determines the **geodesic curvature** κ_g and the length of the normal component $\boldsymbol{\gamma}''_{nor}$ determines the **normal curvature** κ_n. Thus, $\kappa_g = \pm|\boldsymbol{\gamma}''_{tan}|$, and $\kappa_n = \pm|\boldsymbol{\gamma}''_{nor}|$.

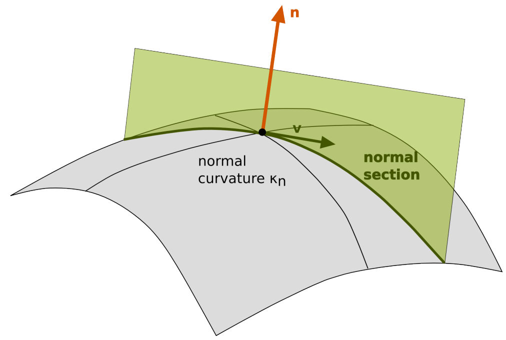
If $\boldsymbol{\gamma}$ is parametrized by the arc length, $\kappa = |\boldsymbol{\gamma}''|$. The figure on the right illustrates that the curvatures κ, κ_g and κ_n are related by the formula
$$\kappa^2 = \kappa_g^2 + \kappa_n^2.$$



Since κ can be interpreted as the total extent of curving of $\boldsymbol{\gamma}$, the formula $\kappa^2 = \kappa_g^2 + \kappa_n^2$ means that the two factors, κ_g and κ_n contribute to this extend of curving: κ_n indicates the extent of curving coming from the curving of a surface and κ_g the extent of the interior curving. Thus, one can think of κ_n as an external and κ_g as an internal curvature. The following two scenarios may be helpful for understanding this.

1. **External curvature of the surface**. If a surface itself is curved relative to the surrounding space in which it embeds, then a curve on this surface will be forced to bend as well. The level of this bending is measured by the normal curvature κ_n.

   For example, the curving of any normal section of a surface comes only from curving of the surface itself. In particular, a horizontal circle on the cylinder $x^2 + y^2 = a^2$ is curved just because the cylinder is a rolled-up plane. If we "un-roll" the cylinder back to a plane, the circle becomes a straight line. The horizontal circle has $\kappa_g = 0$ and $\kappa_n \neq 0$.



Example with $\kappa_n \neq 0, \kappa_g = 0$

2. **Internal curvature of the surface.** Consider a curve "meandering" in a plane. The curvature of this curve comes only from the "meandering", not from any exterior curving of the plane since the plane is flat. This level of bending is measured by the geodesic curvature κ_g. In this case, $\kappa_n = 0$ and $\kappa_g \neq 0$.

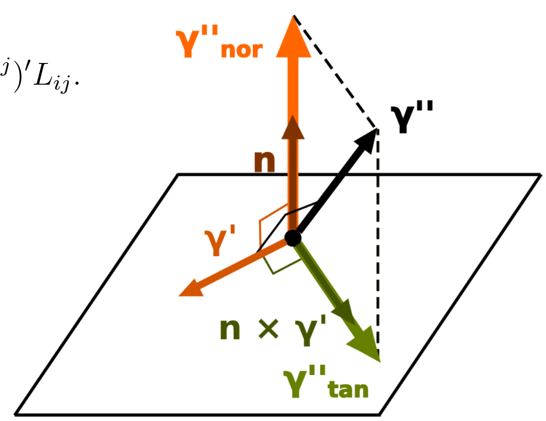
Example with $\kappa_n = 0, \kappa_g \neq 0$

We now examine more closely the computation of the two curvatures. For κ_n, start from the formula

$$\boldsymbol{\gamma}'' = \boldsymbol{\gamma}''_{tan} + \boldsymbol{\gamma}''_{nor} = ((u^k)'' + \Gamma^k_{ij}(u^i)'(u^j)')\mathbf{x}_k + (u^i)'(u^j)'L_{ij}\mathbf{n}$$

and dot it by $\mathbf{n}$. Since $\mathbf{x}_k \cdot \mathbf{n} = 0$ and $\mathbf{n} \cdot \mathbf{n} = 1$, we obtain that $\boldsymbol{\gamma}'' \cdot \mathbf{n} = (u^i)'(u^j)'L_{ij}$. This last expression computes the normal curvature. Thus

$$\kappa_n = \boldsymbol{\gamma}'' \cdot \mathbf{n} = (u^i)'(u^j)'L_{ij}.$$



Next, we show that κ_g can be computed as $(\mathbf{n} \times \boldsymbol{\gamma}') \cdot \boldsymbol{\gamma}''$ if $\boldsymbol{\gamma}$ is *parametrized by the arc length.* To show this, start by noting that $\boldsymbol{\gamma}''_{nor} \cdot \boldsymbol{\gamma}' = 0$ since $\boldsymbol{\gamma}'$ is in the tangent plane and $\boldsymbol{\gamma}''_{nor}$ is perpendicular to it.

Since $\boldsymbol{\gamma}$ is unit-speed, the length of $\boldsymbol{\gamma}'$ is constant (and equal to 1) so $\boldsymbol{\gamma}'' \cdot \boldsymbol{\gamma}' = 0$ (recall the argument that differentiating the relation $\boldsymbol{\gamma}' \cdot \boldsymbol{\gamma}' = 1$ produces $\boldsymbol{\gamma}'' \cdot \boldsymbol{\gamma}' + \boldsymbol{\gamma}' \cdot \boldsymbol{\gamma}'' = 2\boldsymbol{\gamma}'' \cdot \boldsymbol{\gamma}' = 0$). Thus

$$0 = \boldsymbol{\gamma}'' \cdot \boldsymbol{\gamma}' = (\boldsymbol{\gamma}''_{tan} + \boldsymbol{\gamma}''_{nor}) \cdot \boldsymbol{\gamma}' = \boldsymbol{\gamma}''_{tan} \cdot \boldsymbol{\gamma}' + \boldsymbol{\gamma}''_{nor} \cdot \boldsymbol{\gamma}' = \boldsymbol{\gamma}''_{tan} \cdot \boldsymbol{\gamma}' + 0 = \boldsymbol{\gamma}''_{tan} \cdot \boldsymbol{\gamma}'.$$

So, $\boldsymbol{\gamma}''_{tan}$ is orthogonal to $\boldsymbol{\gamma}'$ as well. Since $\boldsymbol{\gamma}''_{tan}$ is orthogonal to both $\boldsymbol{\gamma}'$ and $\mathbf{n}$, it is *colinear with* $\mathbf{n} \times \boldsymbol{\gamma}'$. Hence, $\boldsymbol{\gamma}''_{tan}$ is a multiple of $\mathbf{n} \times \boldsymbol{\gamma}'$. Since the length $|\boldsymbol{\gamma}''_{tan}|$ is $\pm\kappa_g$ and the length

$|\mathbf{n} \times \boldsymbol{\gamma}'| = |\mathbf{n}||\boldsymbol{\gamma}'|\sin(\pm\frac{\pi}{2}) = \pm 1$ (recall that both $\mathbf{n}$ and $\boldsymbol{\gamma}'$ have length 1), we have that

$$\boldsymbol{\gamma}''_{tan} = \kappa_g(\mathbf{n} \times \boldsymbol{\gamma}').$$

Dotting the above identity by $\mathbf{n}\times\boldsymbol{\gamma}'$, we obtain $(\mathbf{n}\times\boldsymbol{\gamma}')\cdot\boldsymbol{\gamma}''_{tan} = \kappa_g$. But since $\mathbf{n}\times\boldsymbol{\gamma}'$ is perpendicular to $\boldsymbol{\gamma}''_{nor}$, the mixed product $(\mathbf{n} \times \boldsymbol{\gamma}')\cdot\boldsymbol{\gamma}''_{tan}$ is equal to $(\mathbf{n}\times\boldsymbol{\gamma}')\cdot\boldsymbol{\gamma}''$. Thus $\kappa_g = (\mathbf{n}\times\boldsymbol{\gamma}')\cdot\boldsymbol{\gamma}''$ or, using the bracket notation

$$\boxed{\kappa_g = (\mathbf{n}\times\boldsymbol{\gamma}')\cdot\boldsymbol{\gamma}'' = [\mathbf{n}, \boldsymbol{\gamma}', \boldsymbol{\gamma}''] = [\mathbf{n}, \mathbf{T}, \mathbf{T}'].}$$

Geodesics. A curve $\boldsymbol{\gamma}$ on a surface is said to be **a geodesic** if $\kappa_g = 0$ at every point of $\boldsymbol{\gamma}$. Thus, a geodesic curves *only* because of the curving of the surface – the extent of internal curving is zero.

We show that the following conditions are equivalent. For some of the equivalences below, we need to assume that γ is *parametrized by the arc length.*

1. $\boldsymbol{\gamma}$ is a geodesic.	2. $[\mathbf{n}, \mathbf{T}, \mathbf{T}'] = 0$.
3. $\boldsymbol{\gamma}''_{tan} = 0$ at every point of $\boldsymbol{\gamma}$.	4. $\boldsymbol{\gamma}'' = \boldsymbol{\gamma}''_{nor}$ at every point of $\boldsymbol{\gamma}$.
5. $(u^k)'' + \Gamma^k_{ij}(u^i)'(u^j)' = 0$ for $k = 1, 2$.	6. $\kappa = \pm\kappa_n$ at every point of $\boldsymbol{\gamma}$.
7. $\mathbf{N}$ is colinear with $\mathbf{n}$ (i.e. $\mathbf{N} = \pm\mathbf{n}$).	

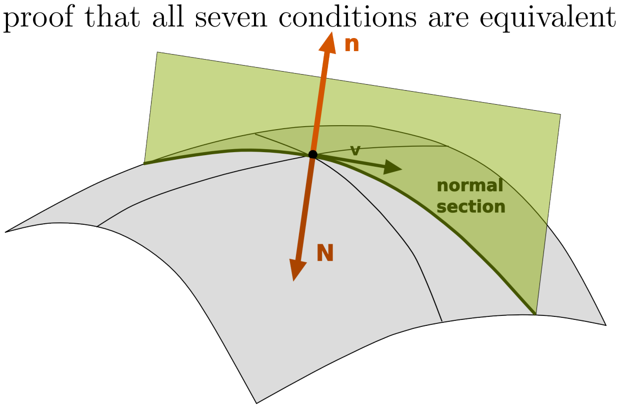
Conditions 1 and 2 are equivalent since $\kappa_g = [\mathbf{n}, \mathbf{T}, \mathbf{T}']$. Conditions 3 and 4 are clearly equivalent. Conditions 3 and 5 are equivalent since $\boldsymbol{\gamma}''_{tan} = ((u^k)'' + \Gamma^k_{ij}(u^i)'(u^j)')\mathbf{x}_k$. Conditions 1 and 3 are equivalent since $\boldsymbol{\gamma}''_{tan} = 0 \Leftrightarrow \kappa_g = |\boldsymbol{\gamma}''_{tan}| = 0$.

To see that conditions 1 and 6 are equivalent, recall the formula $\kappa_n^2 = \kappa_g^2 + \kappa_n^2$. Thus, if $\kappa_g = 0$ then $\kappa^2 = \kappa_n^2 \Rightarrow \kappa = \pm\kappa_n$. Conversely, if $\kappa = \pm\kappa_n$, then $\kappa^2 = \kappa_n^2 \Rightarrow \kappa_g^2 = 0 \Rightarrow \kappa_g = 0$.

Finally, to show that 1 and 7 are equivalent, recall that $\boldsymbol{\gamma}'' = \mathbf{T}' = \kappa\mathbf{N}$ if $\boldsymbol{\gamma}$ is parametrized by the arc length. Assuming that $\boldsymbol{\gamma}$ is a geodesic, we have that $\boldsymbol{\gamma}'' = \boldsymbol{\gamma}''_{nor} = \kappa_n\mathbf{n}$. Thus, $\kappa\mathbf{N} = \kappa_n\mathbf{n}$ and so the vectors $\mathbf{N}$ and $\mathbf{n}$ are colinear, in particular $\mathbf{N} = \pm\mathbf{n}$ since they both have unit length. Conversely, if $\mathbf{N}$ and $\mathbf{n}$ are colinear, then $\boldsymbol{\gamma}''$ (always colinear with $\mathbf{N}$ if unit-speed parametrization is used) is colinear with $\mathbf{n}$ as well. So $\boldsymbol{\gamma}'' = \boldsymbol{\gamma}_{nor}$ and so condition 4 holds. Since we showed that 1 and 4 are equivalent, 1 holds as well. This concludes the proof that all seven conditions are equivalent.

Examples.

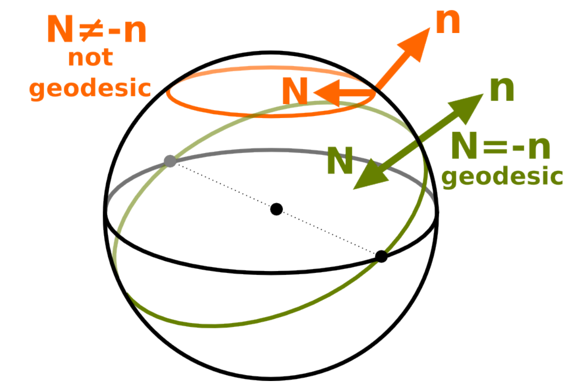
1. If $\boldsymbol{\gamma}$ is the normal section in the direction of a vector $\mathbf{v}$ in the tangent plane (intersection of the surface with a plane orthogonal to the tangent plane), then the normal vector $\mathbf{N}$ has the same direction as the unit normal vector $\mathbf{n}$ and so $\mathbf{N} = \pm\mathbf{n}$ (the sign is positive if the acceleration vector has the same direction as $\mathbf{n}$). So, **every normal section is a geodesic.**



2. A great circle on a sphere is the normal section and so, it is a geodesic. Having two points on a sphere which are not antipodal (i.e. exactly opposite to one another with respect to the center), there is a great circle on which the two points lie. Thus, the "straightest possible" curve on a sphere that connects any two points is a great circle.

Thus, κ_g of a great circle is 0 and its curvature κ comes just from the normal curvature κ_n (equal to $\frac{1}{a}$ if the radius is a).

Any circle on a sphere which is not "great" (i.e. whose center does not coincide with a center of the sphere and the radius is smaller than a) is not a geodesic. Any such "non-great" circle is an example of a curve on a surface whose normal vector $\mathbf{N}$ is not colinear with the normal vector of the sphere $\mathbf{n}$.



Just great circles are geodesics

Computing the geodesics. Consider the two equations $(u^k)'' + \Gamma^k_{ij}(u^i)'(u^j)' = 0$ for $k = 1, 2$. The expressions on the left side correspond to the coefficients of $\boldsymbol{\gamma}''$ with $\mathbf{x}_1$ and $\mathbf{x}_2$. Two equations considered together represent a system of two second order differential equations whose solutions compute geodesics on a surface. So, this system of differential equations is a tool for explicitly obtaining formulas of geodesics on a surface. This system is frequently being solved in everyday life, for example when determining the shortest flight route for an airplane.

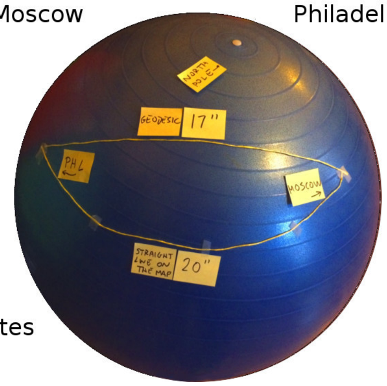
Consider, for example, the air traffic routes from Philadelphia to London, Moscow and Hong Kong represented below. Each city being further from Philadelphia than the previous one, makes the geodesic path appear more curved when represented on a flat plane. Still, all three routes are determined as geodesics – as intersections of great circles on Earth which contain Philadelphia and the destination city.

Philadelphia to London

Philadelphia to Moscow

Philadelphia to Hong Kong

Philadelphia to Moscow via geodesics and non-geodesic routes



Example 1. Geodesics of a plane. For simplicity, let us consider the xy-plane, $z = 0$ again. Thus $\mathbf{x} = (x, y, 0)$ so $u = x$ and $v = y$. We computed the Christoffel symbols before and obtained that they all vanish. Thus, the equations of geodesics are $x'' = 0$ and $y'' = 0$. Integrating both equations with respect to the unit-speed parameter s twice produces the following.

$$x = as + b \text{ and } y = cs + d$$

These equations are parametric equations of a line. Hence, geodesics are straight lines and an arc-length curve $\boldsymbol{\gamma}(s)$ in xy-plane is a geodesic exactly if it is a straight line.

Example 2. Geodesics of a cylinder. Consider the cylinder $x^2 + y^2 = a^2$ parametrized by $\mathbf{x} = (a\cos t, a\sin t, z)$ so that $u = t$ and $v = z$. We computed the Christoffel symbols before and obtained that they are all zero. Thus, the equations of geodesics are given by $t'' = 0$ and $z'' = 0$. These equations have solutions $t = as + b$ and $z = cs + d$ which are parametric equations of a line in tz-plane. This shows that a curve on a cylinder is geodesic if and only if it is a straight line in zt-plane. Thus, a unit-speed curve $\boldsymbol{\gamma}(s)$ on a cylinder is a geodesic exactly if it becomes a straight line if the cylinder is "un-rolled" into a plane.

In particular, both meridians and parallels on the cylinder are geodesics. The meridians are z-curves. They are parametrized by unit-speed since $\mathbf{x}_2 = (0, 0, 1)$ has unit length. Since $t = t_0$ is a constant on a z-curve, $t' = t'' = 0$ so the first equation holds. The second holds since $z' = \frac{dz}{dz} = 1$ and so $z'' = 0$. Hence, both geodesic equations hold.

The parallels (or circles of latitude), are t-curves with $z = z_0$ a constant. They are parametrized by unit-speed for $t = \frac{s}{a}$. Thus, $t' = \frac{1}{a}$ and $t'' = 0$ and $z' = z'' = 0$ so both geodesic equations hold.

Another way to see that the circles of latitude $\boldsymbol{\gamma} = (a\cos t, a\sin t, z_0) = (a\cos\frac{s}{a}, a\sin\frac{s}{a}, z_0)$ are geodesics is to compute the second derivative (colinear with $\mathbf{N}$) and to note that it is a multiple of $\mathbf{n}$ (thus condition 4. holds). The first derivative is $\boldsymbol{\gamma}' = (-\sin\frac{s}{a}, \cos\frac{s}{a}, 0)$ and the second is $\boldsymbol{\gamma}'' = (-\frac{1}{a}\cos\frac{s}{a}, -\frac{1}{a}\sin\frac{s}{a}, 0)$. The second derivative is a multiple of $\mathbf{n} = (\cos\frac{s}{a}, \sin\frac{s}{a}, 0)$ ($\boldsymbol{\gamma}'' = \frac{-1}{a}\mathbf{n}$) and so $\boldsymbol{\gamma}$ is a geodesic.

Example 3. Meridians of a cone are geodesics. Consider the cone obtained by revolving the line $(3t, 4t)$ in rz-plane about the z-axis. Note that the unit-speed parametrization of this line is $(\frac{3}{5}s, \frac{4}{5}s)$ since $\sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{25}{25}} = 1$. Using this parametrization, the cone is given by $\mathbf{x} = \frac{1}{5}(3s\cos\theta, 3s\sin\theta, 4s)$. Since the line has the unit-speed parametrization, the s-curves (the meridians) on the cone have the unit-speed parametrization also.

We show that all the meridians are geodesics. Compute that $\mathbf{x}_1 = \frac{1}{5}(3\cos\theta, 3\sin\theta, 4)$ and $\mathbf{x}_2 = \frac{1}{5}(-3s\sin\theta, 3s\cos\theta, 0)$ so that $g_{11} = \mathbf{x}_1 \cdot \mathbf{x}_1 = 1$ (this also tells you that the meridians have the unit-speed parametrization), $g_{12} = 0$ and $g_{22} = \frac{9}{25}s^2$. So that the first fundamental form is $\begin{bmatrix} 1 & 0 \\ 0 & \frac{9s^2}{25} \end{bmatrix}$. Then compute that $\mathbf{x}_{11} = (0, 0, 0)$, $\mathbf{x}_{12} = \frac{1}{5}(-3\sin\theta, 3\cos\theta, 0)$ and $\mathbf{x}_{22} = \frac{1}{5}(-3s\cos\theta, -3s\sin\theta, 0)$. The inverse matrix of $[g_{ij}]$ is $\begin{bmatrix} 1 & 0 \\ 0 & \frac{25}{9s^2} \end{bmatrix}$. Since $\mathbf{x}_{11} = 0, \Gamma^1_{11} = \Gamma^2_{11} = 0$. Since $\mathbf{x}_{12} \cdot \mathbf{x}_1 = 0$ and $g^{21} = 0$, $\Gamma^1_{12} = \Gamma^1_{21} = 0$. Since $\mathbf{x}_{22} \cdot \mathbf{x}_2 = 0$ and $g^{12} = 0$, $\Gamma^2_{22} = 0$. The remaining symbols are

$$\Gamma^2_{12} = \Gamma^2_{21} = \mathbf{x}_{12} \cdot \mathbf{x}_1 g^{12} + \mathbf{x}_{12} \cdot \mathbf{x}_2 g^{22} = 0 + \frac{9s}{25}\frac{25}{9s^2} = \frac{1}{s} \text{ and}$$

$$\Gamma^1_{22} = \mathbf{x}_{22} \cdot \mathbf{x}_1 g^{11} + \mathbf{x}_{22} \cdot \mathbf{x}_2 g^{21} = \frac{-9s}{25} + 0 = \frac{-9}{25}s.$$

Thus, two equations of geodesics are

$$s'' - \tfrac{9}{25}s(\theta')^2 = 0 \text{ and } \theta'' + \tfrac{2}{s}s'\theta' = 0$$

for a unit-speed curve $\boldsymbol{\gamma}$ on the cone for which s and θ depend on a parameter t.

In particular, on a meridian s is the parameter (think that "t" is s here) and θ is a constant. Thus $\theta' = \theta'' = 0$ and $s' = 1$, $s'' = 0$. So, both geodesic equations are satisfied and this shows that the meridians on the cone are geodesics.

Example 4. Meridians of a sphere and the equator are geodesics. Recall that the unit-speed parametrization of a circle of radius a is $(a\cos\frac{s}{a}, a\sin\frac{s}{a})$. So, to obtain the unit-speed parametrization of the sphere of radius a, one can consider the sphere as the surface of revolution of a unit-speed circle in rz-plane. This produces the parametrization

$$\mathbf{x} = (a\cos\theta\cos\frac{s}{a},\ a\sin\theta\cos\frac{s}{a},\ a\sin\frac{s}{a}).$$

Compute $\mathbf{x}_1 = \mathbf{x}_\theta = (-a\sin\theta\cos\frac{s}{a}, a\cos\theta\cos\frac{s}{a}, 0)$, $\mathbf{x}_2 = \mathbf{x}_s = (-\cos\theta\sin\frac{s}{a}, -\sin\theta\sin\frac{s}{a}, \cos\frac{s}{a})$ and $g_{11} = a^2\cos^2\frac{s}{a}$, $g_{12} = 0$, g_{22} simplifies to 1 (this also tells you that the meridians have the unit-speed parametrization). The first fundamental form is $\begin{bmatrix} a^2\cos^2\frac{s}{a} & 0 \\ 0 & 1 \end{bmatrix}$. $g = a^2\cos^2\frac{s}{a}$ and $\mathbf{n} = \frac{1}{\sqrt{g}}(\mathbf{x}_1 \times \mathbf{x}_2) = (-\cos\theta\cos\frac{s}{a}, -\sin\theta\cos\frac{s}{a}, -\sin\frac{s}{a})$.

Compute that $\mathbf{x}_{11} = (-a\cos\theta\cos\frac{s}{a}, -a\sin\theta\cos\frac{s}{a}, 0)$, $\mathbf{x}_{12} = (\sin\theta\sin\frac{s}{a}, -\cos\theta\sin\frac{s}{a}, 0)$, and $\mathbf{x}_{22} = (-\frac{1}{a}\cos\theta\cos\frac{s}{a}, -\frac{1}{a}\sin\theta\cos\frac{s}{a}, -\frac{1}{a}\sin\frac{s}{a})$. The inverse of the first fundamental form is $\begin{bmatrix} \frac{1}{a^2\cos^2\frac{s}{a}} & 0 \\ 0 & 1 \end{bmatrix}$ and the Christoffel symbols are $\Gamma^2_{11} = a\cos\frac{s}{a}\sin\frac{s}{a}$, $\Gamma^1_{12} = \Gamma^1_{21} = \frac{-1}{a}\tan\frac{s}{a}$, and the rest are 0.

Thus, two equations of geodesics are

$$s'' + a\cos\tfrac{s}{a}\sin\tfrac{s}{a}(\theta')^2 = 0 \text{ and } \theta'' - \tfrac{2}{a}\tan\tfrac{s}{a}s'\theta' = 0$$

On a meridian, s is a parameter and θ is constant. Hence $s' = 1, s'' = 0$ and $\theta' = 0, \theta'' = 0$. Hence both equations of the geodesics are satisfied.

On the equator, $\phi = \frac{s}{a} = 0$ so $s = 0$. Hence the equation of the equator is $(a\cos\theta, a\sin\theta, 0)$. Note that this has the unit-speed parametrization if $\theta = \frac{\overline{\theta}}{a}$. Thus if $\overline{\theta}$ is considered as a parameter, $\theta' = \frac{1}{a}$, $\theta'' = 0$ and $s' = s'' = 0$. Hence the two equations become

$$a\cos\tfrac{s}{a}\sin\tfrac{s}{a}\tfrac{1}{a^2} = 0 \text{ and } 0 = 0$$

Using that $s = 0$, the first equation is $0 = 0$ and so so both equations are satisfied.

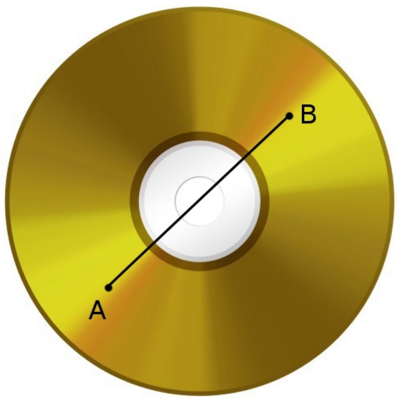
Properties of geodesics. Since a geodesic curves solely because of curving of the surface, a geodesic has the role of a straight line on a surface. Moreover, geodesics have the following properties of straight lines.

1. If a curve $\boldsymbol{\gamma}(s)$ for $a \le s \le b$ is the shortest route on the surface that connects the points $\boldsymbol{\gamma}(a)$ and $\boldsymbol{\gamma}(b)$, then $\boldsymbol{\gamma}$ is a geodesic. One of the project topics focuses on the proof of this claim.

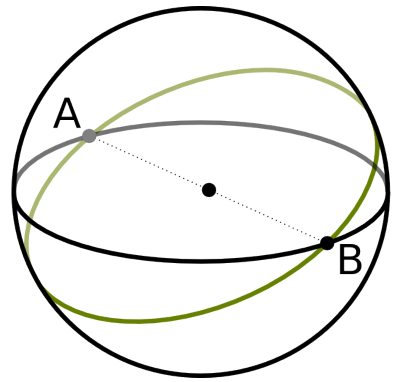
   Note that the converse does not have to hold – if a curve is geodesic, it may not give the shortest route between its two points. For example, a north pole and any other point on a sphere but the south pole, determine two geodesics connecting them, just one of which will be the shortest route.

2. Every point P on a surface and a vector $\mathbf{v}$ in the tangent plane uniquely determine a geodesic $\boldsymbol{\gamma}$ with $\boldsymbol{\gamma}(0) = P$ and $\boldsymbol{\gamma}'(0) = \mathbf{v}$.

   As opposed to the straight lines, a geodesic connecting two points does not have to exist. For example, consider the xy-plane without the origin. Then there is no geodesic connecting (1,0) and (-1,0). Also, there can be infinitely many geodesics connecting two given points on a surface (for example, take north and south poles on a sphere).



Two points do not determine a "line"



There are many "lines" passing two points

The Gaussian Curvature

In this section, we obtain a simple formula computing the Gaussian K. Recall that the formula for the normal curvature is given by $\kappa_n = L_{ij}(u^i)'(u^j)'$. If the curve $\boldsymbol{\gamma}$ is not given by the arc-length parametrization, this formula becomes $\kappa_n = \frac{L_{ij}(u^i)'(u^j)'}{|\boldsymbol{\gamma}'|^2}$. Recall the formula for $|\boldsymbol{\gamma}'|^2$ from earlier section

$$|\boldsymbol{\gamma}'(t)|^2 = g_{11}((u^1)')^2 + 2g_{12}(u^1)'(u^2)' + g_{22}((u^2)')^2 \text{ in Einstein notation } = g_{ij}(u^i)'(u^j)'$$

Thus, the normal curvature can be computed as

$$\kappa_n = \frac{L_{ij}(u^i)'(u^j)'}{g_{kl}(u^k)'(u^l)'}$$

Differentiating this equation with respect to $(u^r)'$ for $r = 1, 2$, and setting derivatives to zero in order to get conditions for extreme values, we can obtain the conditions that $(L_{ij} - \kappa_n g_{ij})(u^j)' = 0$ for $i = 1, 2$. A nonzero vector $((u^1)', (u^2)')$ can be a solution of these equations just if the determinant of the system $|L_{ij} - \kappa_n g_{ij}|$ is zero.

This determinant is equal to $(L_{11} - \kappa_n g_{11})(L_{22} - \kappa_n g_{22}) - (L_{12} - \kappa_n g_{12})^2$. Substituting that determinant of $[g_{ij}]$ is g and denoting the determinant of $[L_{ij}]$ by L, we obtain the following quadratic equation in κ_n

$$g\kappa_n^2 - (L_{11}g_{22} + L_{22}g_{11} - 2L_{12}g_{12})\kappa_n + L = 0$$

The solutions of this quadratic equation are the principal curvatures κ_1 and κ_2. The Gaussian K is equal to the product $\kappa_1\kappa_2$ and from the above quadratic equation this product is equal to the quotient $\frac{L}{g}$ (recall that the product of the solutions x_1 and x_2 of a quadratic equation ax^2+bx+c is equal to $\frac{c}{a}$). Thus,

$$\boxed{K = \frac{L}{g}}$$

that is the Gaussian K is the **quotient of the determinants of the coefficients of the second and the first fundamental forms**.

From the formula $(L_{ij} - \kappa_n g_{ij})(u^j)' = 0$ it follows that if $L_{12} = L_{21} = g_{12} = g_{21} = 0$, then the principal curvatures are given by $\frac{L_{11}}{g_{11}}$ and $\frac{L_{22}}{g_{22}}$ and the principal directions are $\mathbf{x}_1$ and $\mathbf{x}_2$. Conversely, if directions $\mathbf{x}_1$ and $\mathbf{x}_2$ are principal, then $L_{12} = L_{21} = g_{12} = g_{21} = 0$. Using this observation, we can conclude that the principal directions on a surface of revolution are determined by the meridian and the circle of latitude through every point.

Note that from the equation $(L_{ij} - \kappa_n g_{ij})(u^j)' = 0$ also follows that the principal curvatures are the eigenvalues of the operator determined by the first and the second fundamental form that can be expressed as

$$S = g^{-1}\begin{bmatrix} L_{11}g_{22} - L_{12}g_{12} & L_{12}g_{22} - L_{22}g_{12} \\ L_{12}g_{11} - L_{11}g_{12} & L_{22}g_{11} - L_{12}g_{12} \end{bmatrix}.$$

and is called the **shape operator**.

Examples. Gaussian of a plane, a cylinder, a sphere, and the surface $z = z(x, y)$.

1. Recall that the first and the second fundamental forms of the xy-plane are the identity and the zero matrix. So, $g = 1$ and $L = 0$ which readily gives you that $K = \frac{0}{1} = 0$.

2. Recall that the first and the second fundamental forms of the cylinder $x^2+y^2 = a^2$ are $\begin{bmatrix} a^2 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -a & 0 \\ 0 & 0 \end{bmatrix}$. Thus $g = a^2$, $L = 0$ and so $K = \frac{0}{a^2} = 0$. Recall that the fact that a plane and a cylinder have the same Gaussian indicates that they are locally indistinguishable (locally isometric).

3. The first and the second fundamental form of the sphere of radius a centered at the origin are $\begin{bmatrix} a^2\cos^2\phi & 0 \\ 0 & a^2 \end{bmatrix}$ and $\begin{bmatrix} -a\cos^2\phi & 0 \\ 0 & -a \end{bmatrix}$. Thus $g = a^4\cos^2\phi$ and $L = a^2\cos^2\phi$. Hence

$$K = \frac{a^2\cos^2\phi}{a^4\cos^2\phi} = \frac{1}{a^2}.$$

   This agrees with our earlier conclusion on K of a sphere.

4. We have computed the first and the second fundamental form of the surface $z = z(x, y)$ to be $\begin{bmatrix} 1+z_1^2 & z_1z_2 \\ z_1z_2 & 1+z_2^2 \end{bmatrix}$, and $\frac{1}{g}\begin{bmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{bmatrix}$. Thus $g = 1 + z_1^2 + z_2^2$, $L = \frac{z_{11}z_{22} - z_{12}^2}{g}$

$$K = \frac{z_{11}z_{22} - z_{12}^2}{g^2} = \frac{z_{11}z_{22} - z_{12}^2}{(1+z_1^2+z_2^2)^2}.$$

The Curvature Tensor. Theorema Egregium

Recall that Theorema Egregium states that *the Gaussian curvature K can be calculated intrinsically*, that is using the first fundamental form only. The formula $K = \frac{L}{g}$ enables one to prove the "Remarkable Theorem" by proving that the determinant L can be computed intrinsically only in terms of g_{ij} (the determinant $g = g_{11}g_{22} - g_{12}^2$ is already clearly intrinsic). Note that while $L_{ij} = \mathbf{x}_{ij} \cdot \mathbf{n}$ are not intrinsic (note the presence of $\mathbf{n}$ in this formula), the determinant L, surprisingly, is intrinsic. We show that L is intrinsic by showing the following steps.

1. The Christoffel symbols Γ_{ij}^k are intrinsic. Note that this implies that the geodesic curvature is intrinsic also.
2. We introduce the Riemann curvature tensor R_{ijk}^l and represent it via the Christoffel symbols. By step 1, R_{ijk}^l is intrinsic also.
3. We represent L via R_{ijk}^l and g_{ij}.

The Christoffel symbols Γ_{ij}^k can be computed intrinsically. To prove this statement, start by differentiating the equation $g_{ij} = \mathbf{x}_i \cdot \mathbf{x}_j$ with respect to u^k. Get

$$\frac{\partial g_{ij}}{\partial u^k} = \mathbf{x}_{ik} \cdot \mathbf{x}_j + \mathbf{x}_i \cdot \mathbf{x}_{jk}$$

In a similar manner, we obtain

$$\frac{\partial g_{ik}}{\partial u^j} = \mathbf{x}_{ij} \cdot \mathbf{x}_k + \mathbf{x}_i \cdot \mathbf{x}_{kj} \text{ and } \frac{\partial g_{jk}}{\partial u^i} = \mathbf{x}_{ji} \cdot \mathbf{x}_k + \mathbf{x}_j \cdot \mathbf{x}_{ki}$$

Note that the second equation can be obtained from the first by permuting the indices j and k and the third equation can be obtained from the second by permuting the indices i and j. This is called *cyclic permutation of indices.*

At this point, we require the second partial derivatives to be continuous as well. This condition will guarantee that the partial derivatives $\mathbf{x}_{ij}$ and $\mathbf{x}_{ji}$ are equal. In this case, adding the second and third equation and subtracting the first gives us

$$\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k} = \mathbf{x}_{ij} \cdot \mathbf{x}_k + \underline{\mathbf{x}_i \cdot \mathbf{x}_{kj}} + \mathbf{x}_{ji} \cdot \mathbf{x}_k + \underline{\mathbf{x}_j \cdot \mathbf{x}_{ki}} - \underline{\mathbf{x}_{ik} \cdot \mathbf{x}_j} - \underline{\mathbf{x}_i \cdot \mathbf{x}_{jk}} = 2\mathbf{x}_{ij} \cdot \mathbf{x}_k$$

Thus,

$$\boxed{\Gamma_{ij}^k = (\mathbf{x}_{ij} \cdot \mathbf{x}_l)g^{lk} = \frac{1}{2}\left(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l}\right) g^{lk}.}$$

This shows that the Christoffel symbols Γ_{ij}^k can be computed just in terms of the metric coefficients g_{ij} that can be determined by measurements within the surface. Since the geodesics can be computed from the two differential equations which only feature the Christoffel symbols and the Christoffel symbols can be computed only using g_{ij}, we showed the following theorem.

Theorem. The geodesic curvature is intrinsic.

The Riemann curvature tensor. The coefficients of the $\boxed{\textbf{Riemann curvature tensor}}$ (or Riemann-Christoffel curvature tensor) are defined via the Christoffel symbols by

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ik}}{\partial u^j} - \frac{\partial \Gamma^l_{ij}}{\partial u^k} + \Gamma^p_{ik}\Gamma^l_{pj} - \Gamma^p_{ij}\Gamma^l_{pk}$$

The geometric meaning of this tensor cannot really be seen from this formula. Roughly speaking, this tensor measures the extent of deviation of initial vector and the vector resulting when the initial vector is parallel transported around a loop on a surface. For example, when a vector in space is parallel transported around a loop in a plane, it will always return to its original position and the Riemann curvature tensor directly measures the failure of this on a general surface. The extent of this failure is known as the holonomy of the surface.

To represent the determinant L via R^l_{ijk} and g_{ij}, we need a set of formulas known as the Gauss's equations. These equations are proven using Weingarten's equations and as a byproduct of the proof, we obtain a set of equations know as the Codazzi-Mainardi equations.

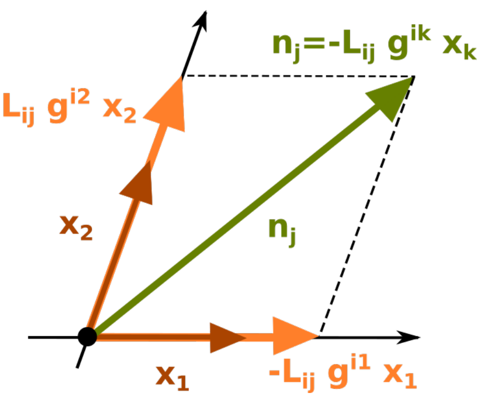
Proposition.

Weingarten's equations $\quad \mathbf{n}_j = -L_{ij} g^{ik} \mathbf{x}_k$

Gauss's equations $\quad R^l_{ijk} = L_{ik} L_{jp} g^{pl} - L_{ij} L_{kp} g^{pl}.$

Codazzi-Mainardi equations $\quad \frac{\partial L_{ij}}{\partial u^k} - \frac{\partial L_{ik}}{\partial u^j} = \Gamma^l_{ik} L_{lj} - \Gamma^l_{ij} L_{lk}.$

Proof. Let us prove Weingarten's equations first. Since $\mathbf{n} \cdot \mathbf{n} = 1$, $\mathbf{n}_j \cdot \mathbf{n} = 0$ and so $\mathbf{n}_j$ is in tangent plane. Thus, it can be represented as a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$. Let a^l_j denote the coefficients of $\mathbf{n}_j$ with $\mathbf{x}_l$. Thus $\mathbf{n}_j = a^l_j \mathbf{x}_l$.



Differentiate the equation $\mathbf{n} \cdot \mathbf{x}_i = 0$ with respect to u^j and obtain $\mathbf{n}_j \cdot \mathbf{x}_i + \mathbf{n} \cdot \mathbf{x}_{ij} = 0$. Recall that $L_{ij} = \mathbf{n} \cdot \mathbf{x}_{ij}$.

Thus, $0 = \mathbf{n}_j \cdot \mathbf{x}_i + L_{ij} = a^l_j \mathbf{x}_l \cdot \mathbf{x}_i + L_{ij} = a^l_j g_{li} + L_{ij}$ and so $a^l_j g_{li} = -L_{ij}$. To solve for a^l_j, multiply both sides by g^{ik} and recall that $g_{li} g^{ik} = \delta^k_l$. Thus we have $-L_{ij} g^{ik} = a^l_j g_{li} g^{ik} = a^l_j \delta^k_l = a^k_j$. This gives us

$$\mathbf{n}_j = a^k_j \mathbf{x}_k = -L_{ij} g^{ik} \mathbf{x}_k.$$

To prove the remaining two sets of equations, let us start by Gauss formulas for the second derivatives

$$\mathbf{x}_{ij} = \Gamma^l_{ij} \mathbf{x}_l + L_{ij} \mathbf{n}$$

Differentiate with respect to u^k and obtain

$$\begin{array}{rcl}
\mathbf{x}_{ijk} & = & \frac{\partial \Gamma^l_{ij}}{\partial u^k}\mathbf{x}_l + \Gamma^l_{ij}\mathbf{x}_{lk} + \frac{\partial L_{ij}}{\partial u^k}\mathbf{n} + L_{ij}\mathbf{n}_k \\
& = & \frac{\partial \Gamma^l_{ij}}{\partial u^k}\mathbf{x}_l + \Gamma^l_{ij}(\Gamma^p_{lk}\mathbf{x}_p + L_{lk}\mathbf{n}) + \frac{\partial L_{ij}}{\partial u^k}\mathbf{n} - L_{ij}L_{pk}g^{pl}\mathbf{x}_l \text{ (sub Gauss and Wein. eqs)} \\
& = & \frac{\partial \Gamma^l_{ij}}{\partial u^k}\mathbf{x}_l + \Gamma^l_{ij}\Gamma^p_{lk}\mathbf{x}_p - L_{ij}L_{pk}g^{pl}\mathbf{x}_l \ + \ \Gamma^l_{ij}L_{lk}\mathbf{n} + \frac{\partial L_{ij}}{\partial u^k}\mathbf{n} \text{ (regroup the terms)} \\
& = & \frac{\partial \Gamma^l_{ij}}{\partial u^k}\mathbf{x}_l + \Gamma^p_{ij}\Gamma^l_{pk}\mathbf{x}_l - L_{ij}L_{pk}g^{pl}\mathbf{x}_l \ + \ \Gamma^l_{ij}L_{lk}\mathbf{n} + \frac{\partial L_{ij}}{\partial u^k}\mathbf{n} \text{ (make tangent comp via } \mathbf{x}_l) \\
& = & \left(\frac{\partial \Gamma^l_{ij}}{\partial u^k} + \Gamma^p_{ij}\Gamma^l_{pk} - L_{ij}L_{pk}g^{pl}\right)\mathbf{x}_l \ + \ \left(\Gamma^l_{ij}L_{lk} + \frac{\partial L_{ij}}{\partial u^k}\right)\mathbf{n} \text{ (factor } x_l \text{ and } \mathbf{n})
\end{array}$$

Interchanging j and k we obtain

$$\mathbf{x}_{ikj} = \left(\frac{\partial \Gamma^l_{ik}}{\partial u^j} + \Gamma^p_{ik}\Gamma^l_{pj} - L_{ik}L_{pj}g^{pl}\right)\mathbf{x}_l \ + \ \left(\Gamma^l_{ik}L_{lj} + \frac{\partial L_{ik}}{\partial u^j}\right)\mathbf{n}$$

Since $\mathbf{x}_{ijk} = \mathbf{x}_{ikj}$, both the tangent and the normal components of $\mathbf{x}_{ijk} - \mathbf{x}_{ikj}$ are zero. The coefficient of the tangent component is

$$\frac{\partial \Gamma^l_{ij}}{\partial u^k} + \Gamma^p_{ij}\Gamma^l_{pk} - L_{ij}L_{pk}g^{pl} - \frac{\partial \Gamma^l_{ik}}{\partial u^j} - \Gamma^p_{ik}\Gamma^l_{pj} + L_{ik}L_{pj}g^{pl} = L_{ik}L_{pj}g^{pl} - L_{ij}L_{pk}g^{pl} - R^l_{ijk} = 0.$$

This proves the Gauss's equations.

The coefficient of the normal component is

$$\Gamma^l_{ij}L_{lk} + \frac{\partial L_{ij}}{\partial u^k} - \Gamma^l_{ik}L_{lj} - \frac{\partial L_{ik}}{\partial u^j} = 0$$

proving Codazzi-Mainardi equations. QED.

We can now prove Theorema Egregium.

Theorema Egregium. The Gaussian K is dependent solely on the coefficient of the first fundamental form by

$$\boxed{K = \frac{g_{1i}R^i_{212}}{g}.}$$

Proof. Multiplying Gauss's equation $R^l_{ijk} = L_{ik}L_{jp}g^{pl} - L_{ij}L_{kp}g^{pl}$ by g_{lm}, we obtain $R^l_{ijk}g_{lm} = L_{ik}L_{jp}g^{pl}g_{lm} - L_{ij}L_{kp}g^{pl}g_{lm} = (L_{ik}L_{jp} - L_{ij}L_{kp})\delta^p_m = L_{ik}L_{jm} - L_{ij}L_{km}$. Taking $i = k = 2$, and $j = m = 1$, we obtain $L = L_{22}L_{11} - L_{21}L_{21} = R^l_{212}g_{l1}$.

From here we have that $K = \frac{L}{g} = \frac{L_{11}L_{22} - L_{12}L_{21}}{g} = \frac{R^l_{212}g_{l1}}{g}$. QED.

Once we obtain the relation $R^l_{ijk}g_{lm} = L_{ik}L_{jm} - L_{ij}L_{km}$ in the above proof, we could also take $i = k = 1$, and $j = m = 2$, and obtain $R^l_{121}g_{l2} = L_{11}L_{22} - L_{12}L_{12} = L$. Thus shows that

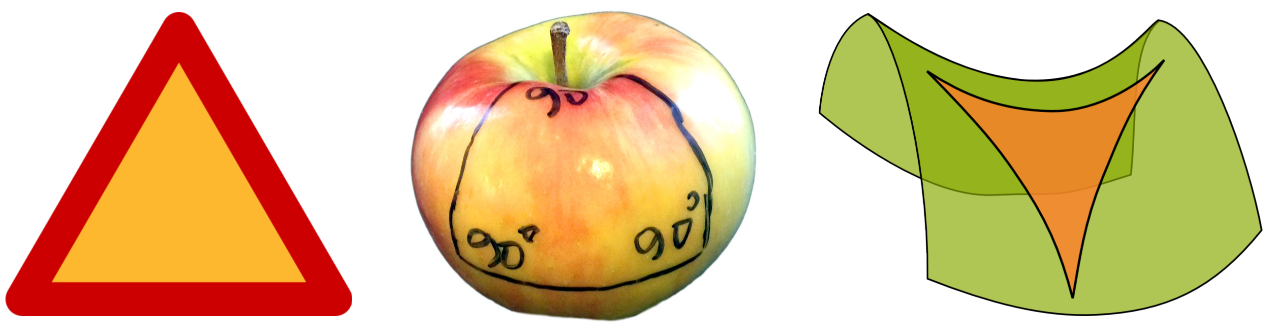
$$R^i_{212}g_{i1} = R^i_{121}g_{i2}$$

and so K can be computed both as

$$\boxed{K = \frac{g_{2i}R^i_{121}}{g}} \text{ and as } \boxed{K = \frac{g_{1i}R^i_{212}}{g}.}$$

Total curvature. The surface integral of the Gaussian curvature over some region of a surface is called the **total curvature**. The total curvature directly corresponds to the deviation of the sum of the angles of a geodesic triangle from 180 degrees. In particular,

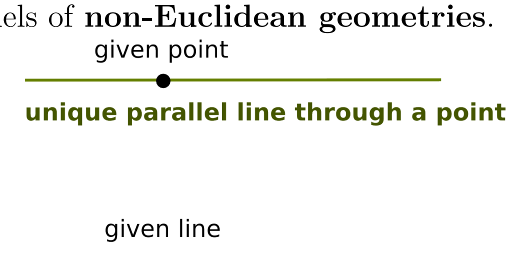
- On a surface of total curvature zero, (such as a plane for example), the sum of the angles of a triangle is precisely 180 degrees.
- On a surface of positive curvature, the sum of angles of a triangle exceeds 180 degrees. For example, consider a triangle formed by the equator and two meridians on a sphere. Any meridian intersects the equator by 90 degrees. However, if the angle between the two meridians is $\theta > 0$, then the sum of the angles in the triangle is $180 + \theta$ degrees. In the figure on the right, the angles add to 270 degrees.
- On a surface of negative curvature, the sum of the angles of a triangle is less than 180 degrees.



$K = 0 \Rightarrow \sum$ angles $= 180$, $\quad K > 0 \Rightarrow \sum$ angles > 180, $\quad K < 0 \Rightarrow \sum$ angles < 180

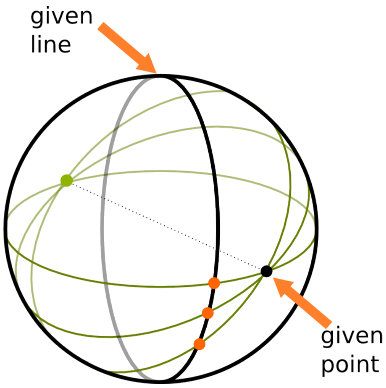
The total curvature impact also the number of lines passing a given point, parallel to a given line. Surfaces for which this number is not equal to one are models of **non-Euclidean geometries**.

Recall that the parallel postulate in **Euclidean geometry** is stating that in a plane there is exactly one line passing a given point that does not intersect a given line, i.e. there is **exactly one line** parallel to a given line passing a given point.

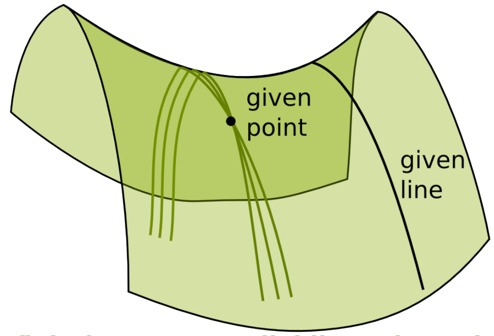


In **elliptic geometry** the parallel postulate is replaced by the statement that there is **no line** through a given point parallel to a given line. In other words, all lines intersect.

In **hyperbolic geometry** the parallel postulate is replaced by the statement that there are at least two distinct lines through a given point that do not intersect a given line. As a consequence, there are **infinitely many lines** parallel to a given line passing a given point.

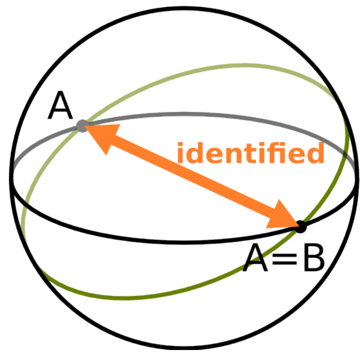


no parallel lines through a point



infinitely many parallel lines through a point

We present the projective plane which is a model of elliptic geometry and Poincaré half plane and disc which are models of hyperbolic geometry.



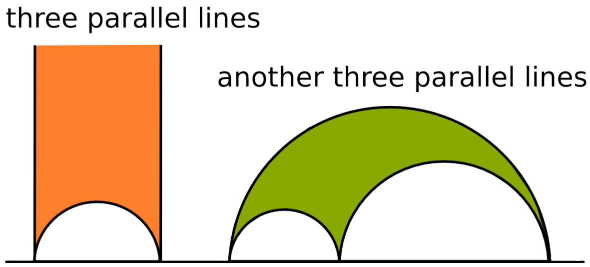
The projective plane $\mathbb{R}P^2$ is defined as the image of the map that identifies antipodal points of the sphere S^2. More generally, n-projective plane $\mathbb{R}P^n$ is defined as the image of the map that identifies antipodal points of the n-sphere S^n.

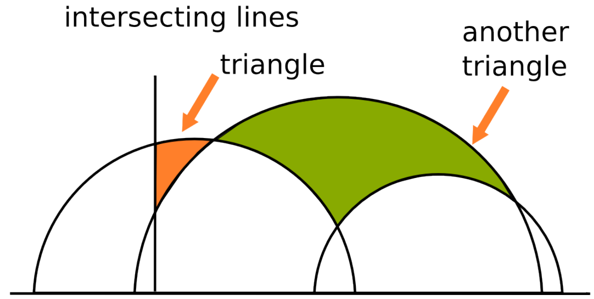
While there are lines which do not intersect (i.e. parallel lines) in a regular plane, every two "lines" (great circles on the sphere with antipodal points identified) in the projective plane intersect in one and only one point. This is because every pair of great circles intersect in exactly two points antipodal to each other. After the identification, the two antipodal points become a single point and hence every two "lines" of the projective plane intersect in a single point. The standard metric on the sphere gives rise to the metric on the projective plane. In this metric, the curvature K is positive.

The projective plane can also be represented as the set of lines in $\mathbb{R}^3$ passing the origin. The distance between two such elements of the projective plane is the angle between the two lines in $\mathbb{R}^3$. The "lines" in the projective planes are the planes in $\mathbb{R}^3$ that pass the origin. Every two such "lines" intersect at a point (since every two planes in $\mathbb{R}^3$ that contain the origin intersect in a line passing the origin).

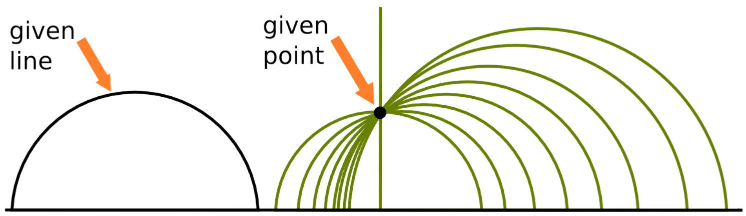
All lines intersect

Poincaré half-plane. Consider the upper half $y > 0$ of the plane $\mathbb{R}^2$ with metric given by the first fundamental form $\begin{bmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{bmatrix}$. In this metric, the geodesic (i.e. the "lines") are circles with centers on x-axis and half-lines that are perpendicular to x-axis.



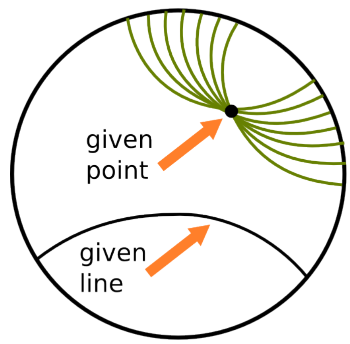


Given one such line and a point, there is more than one line passing the point that does not intersect the given line. In the given metric, the Gaussian curvature is negative.



infinitely many parallel lines through a point

Poincaré disc. Consider the disc $x^2+y^2<1$ in $\mathbb{R}^2$ with metric given by the first fundamental form $\begin{bmatrix} \frac{1}{(1-x^2-y^2)^2} & 0 \\ 0 & \frac{1}{(1-x^2-y^2)^2} \end{bmatrix}$. In this metric, the geodesic (i.e. the "lines") are diameters of the disc and the circular arcs that intersect the boundary orthogonally. Given one such "line" and a point in the disc, there is more than one line passing the point that does not intersect the given line. In the given metric, K is negative.



infinitely many parallel lines through a point

The Fundamental Theorem of Surfaces

The Fundamental Theorem of Surfaces states that a surface is uniquely determined by **the coefficients of the first and the second fundamental form.** More specifically, if g_{ij} and L_{ij} are symmetric functions (i.e. $g_{ij}=g_{ji}$ and $L_{ij}=L_{ji}$) such that $g_{11}>0$ and $g>0$, and such that both Gauss's and Codazzi-Mainardi equations hold, there is a coordinate patch $\mathbf{x}$ such that g_{ij} and L_{ij} are coefficients of the first and the second fundamental form respectively. The patch $\mathbf{x}$ is unique up to a rigid motion (i.e. rotations and translations in space).

Recall that the Gauss's and Codazzi-Mainardi equations relate the coefficients R^l_{ijk}, Γ^k_{ij} and g_{ij} with L_{ij}. Since R^l_{ijk} can be expressed via Γ^k_{ij} and Γ^k_{ij} can be expressed via g_{ij}, the Gauss's and Codazzi-Mainardi equations can be viewed as equations connecting the coefficients of the first fundamental form g_{ij} with the coefficients of the second fundamental form L_{ij}.

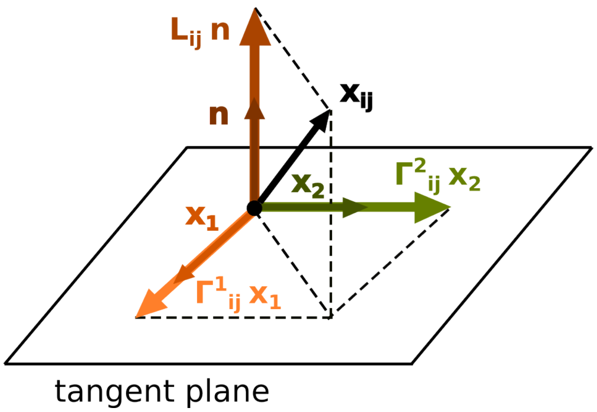
The idea of the proof of the Fundamental Theorem of Surfaces is similar to the proof of the Fundamental Theorem of Curves. Namely, note that the vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ in tangent plane are independent by assumption that the patch is proper. Moreover, the vector $\mathbf{n}$ is independent of $\mathbf{x}_1$

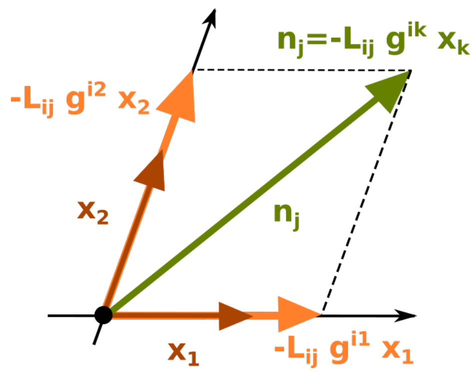
and $\mathbf{x}_2$ since it is not in the tangent plane. Thus the three vectors $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{n}$ represent a basis, or a "moving frame", of the surface analogous to the moving frame $\mathbf{T}, \mathbf{N}$ and $\mathbf{B}$ of a curve.

Gauss formula and Weingarten's equations represent (partial) differential equations relating the derivatives of $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{n}$ in terms of the three vectors themselves

Gauss formula $\quad \mathbf{x}_{ij} = \Gamma^k_{ij}\mathbf{x}_k + L_{ij}\mathbf{n}.$

Weingarten's equations $\quad \mathbf{n}_j = -L_{ij}g^{ik}\mathbf{x}_k$

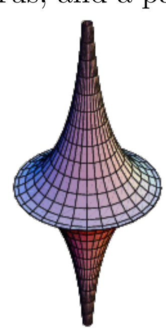




As opposed to a system of ordinary differential equations, there is no theorem that guarantees an existence and uniqueness of a solution of a system of partial differential equations. However, in case of the equations for $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{n}$, the existence and uniqueness of solution follows from the fact that both Gauss's and Codazzi-Mainardi equations hold. Thus, the apparatus $\mathbf{x}_1, \mathbf{x}_2$, $\mathbf{n}$, g_{ij}, L_{ij} describes a surface.

Example. Surfaces of revolution with constant Gaussian. It can be shown that a surface of revolution obtained by revolving a unit-speed curve $(r(s), z(s))$ about z-axis, has the Gaussian curvature K equal to $\frac{-r''}{r}$ (try to fill in the blanks here. You can use that the curve has unit speed, so that $r'^2 + z'^2 = 1$). If K is constant, this yields a differential equation $r'' + Kr = 0$ which can be solved for r. If r is obtained, z can be obtained from the condition $z'^2 + r'^2 = 1$ as $z = \pm\int_0^s \sqrt{1 - r'^2}ds$. Consider the three cases as below.

- $K = a^2 > 0$. In this case, the equation is $r'' + a^2 r = 0$. Its characteristic equation has two complex zeros $\pm ai$ so that the general solution is $r(s) = c_1 \cos as + c_2 \sin as$. Using some trigonometric identities, this solution can be represented as $r(s) = C_1 \cos(as + C_2)$. Combining this with the z equation produces a sphere, the outer part of a torus, and a part of one of these two classes of surfaces.



- $K = -a^2 < 0$. In this case, the equation is $r'' - a^2 r = 0$. Its characteristic equation has two real zeros $\pm a$ and the general solution is $r(s) = c_1 e^{as} + c_2 e^{-as} = C_1 \cosh as + C_2 \sinh as$. A pseudo-sphere (see the figure on the right), the inner part of a torus and a part of one of these two classes of surfaces.

- $K = 0$. In this case, $r'' = 0$ and so $r(s) = c_1 s + c_2$. The $z(s)$ equation is also a line. If the line $(r(s), z(s))$ is horizontal, the surface is the xy-plane or a part of it. If $(r(s), z(s))$ is vertical, the surface is a cylinder or a part of it. In general, the surface is a cone or a part of it.

Manifolds

The concept of an n-dimensional manifold generalizes that of a (2-dimensional) surface. In fact, all the definitions we introduced for surfaces (starting from the definition of a surface, and including definitions of the first and the second fundamental forms, the tangent and normal vectors, Gaussian and the other curvatures) can be generalized from two dimensions to n-dimensions.

Intuitively, a surface is an object in the 3-dimensional space $\mathbb{R}^3$ which locally looks like the 2-dimensional space $\mathbb{R}^2$. Generalizing this idea to n-dimensions, we arrive to the concept of an **n-dimensional manifold** or an **n-manifold** for short. Intuitively, an n-manifold locally looks like the space $\mathbb{R}^n$. We can also generalize the concept of a coordinate patch so that the inverse of a coordinate patch of an n-manifold is a mapping of a region on the manifold to $\mathbb{R}^n$. The coordinate patches are required to be continuous with continuous derivatives, one-to-one, to overlap smoothly on the intersection of their domains, and such that the concept of the tangent plane at any point is well-defined. The coordinate patches provide **local coordinates** on the n-manifold.

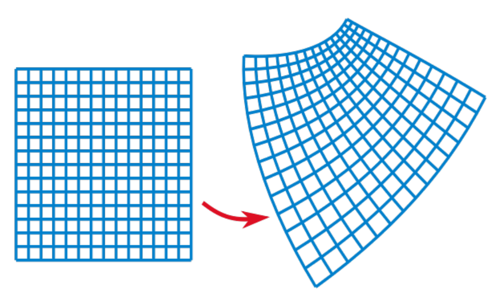
We note some advantages of considering manifolds instead of only surfaces.

1. Surfaces are 2-manifolds so this more general study of n-manifolds agrees with that of surfaces for $n = 2$.

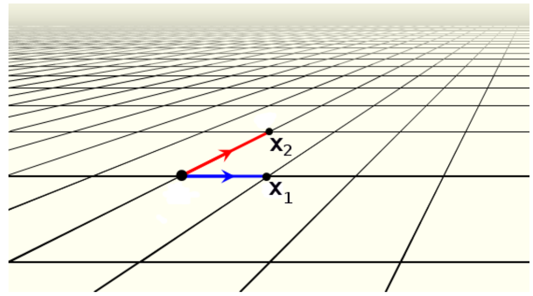
2. Formulas for surfaces we have considered involve the indices ranging from 1 to 2. All these formulas remain true for n-dimensional manifolds if we let the indices range from 1 to n. In particular, the proof of Theorema Egregium generalizes to an n-manifold.

3. The study of n-manifolds can be carried out without assuming the embedding into the space $\mathbb{R}^{n+1}$. Thus, one can study surfaces without considering an embedding of it in the 3-dimensional space $\mathbb{R}^3$.

   Although the n-manifolds for $n > 2$ may not be embedded in the physical, three-dimensional space, the theory of n-manifolds is used in high energy physics, quantum mechanics and relativity theory and, as such, is relevant. The Einstein space-time manifold, for example, has dimension four.



Coordinate Patches. Recall that a proper coordinate patch of a surface is given by parametric equations $\mathbf{x} = (x(u, v), y(u, v), z(u, v))$ such that x, y, z are one-to-one continuous functions with continuous inverses, continuous derivatives and such that $\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \neq \mathbf{0}$. Also, one requires that two such patches overlap smoothly: patches $\mathbf{x}$ and $\bar{\mathbf{x}}$ overlap smoothly provided that the composite functions $\mathbf{x}^{-1} \circ \bar{\mathbf{x}}$ and $\bar{\mathbf{x}}^{-1} \circ \mathbf{x}$ are one-to-one and onto continuous functions with continuous derivatives on the intersection of the domain of the patches.



We define coordinate patches on an n-manifold analogously: a coordinate patch of a nonempty set of points M is a one-to-one continuous mapping from an open region D in $\mathbb{R}^n$ into M given by $\mathbf{x}(u^1, u^2, \ldots, u^n)$ such that the image of a small enough part of D is indistinguishable from a small enough piece of $\mathbb{R}^n$. The n-tuple $(u^1, u^2, \ldots, u^n)$ represents the **local coordinates** on M. Two coordinate patches $\mathbf{x}$ and $\bar{\mathbf{x}}$ **overlap smoothly** if the composite functions $\mathbf{x}^{-1} \circ \bar{\mathbf{x}}$ and $\bar{\mathbf{x}}^{-1} \circ \mathbf{x}$ are one-to-one and onto continuous functions with continuous derivatives up to order at least three on the intersection of the domains. The condition that the partial derivatives up to order three are continuous guarantees the validity of the formulas involving the second fundamental form and equations from the previous section. If derivatives of any order are continuous, such patch is said to be **smooth**.

With these requirements, M is an **n-manifold** if there is a collection of coordinate patches such that: (1) The coordinate patches cover every point of M and they overlap smoothly. (2) Every two different points on M can be covered by two different patches. (3) The collection of patches is maximal with respect to the conditions (1) and (2). A coordinate patch of an n-manifold is also called a **chart** and a collection of coordinate patches is called an **atlas**.

Examples.

1. **Euclidean space** $\mathbb{R}^n$. This space consists of all points of the form $(x_1, x_2, \ldots, x_n)$. Considering $x_1, \ldots, x_n$ as n parameters of a coordinate patch makes $\mathbb{R}^n$ into an n-manifold. In case $n = 2$ this is the xy-plane consisting of points (x, y). In case $n = 3$, this is the three-dimensional space consisting of points (x, y, z).

   The **dot (or inner) product** of two elements of $\mathbb{R}^n$ can be defined analogously to the dot product in the case $n = 3$ as follows.

$$(x_1, x_2, \ldots, x_n) \cdot (y_1, y_2, \ldots, y_n) = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n.$$

   The dot product enables one to define the concept of an angle α between two vectors $\vec{x}$ and $\vec{y}$ using the same formula as in the 3-dimensional case: $\cos\alpha = \frac{\vec{x}\cdot\vec{y}}{|\vec{x}||\vec{y}|}$ where the length of $\vec{x} = (x_1, x_2, \ldots, x_n)$ is defined by

$$|\vec{x}| = \sqrt{\vec{x}\cdot\vec{x}} = \sqrt{x_1^2 + x_2^2 + \ldots x_n^2}.$$

   Thus, two vectors are **perpendicular** exactly when their dot product is zero. One of your project topics explores the generalization of the cross product for n-dimensional vectors.

2. **Hypersurfaces**. Let f be a function with continuous derivatives that maps $\mathbb{R}^{n+1}$ into $\mathbb{R}$. The set of all vectors $\mathbf{x} = (x_1, \ldots, x_{n+1})$ in $\mathbb{R}^{n+1}$ such that $f(\mathbf{x}) = 0$ defines an n-manifold usually referred to as hypersurface. You can think of the equation $f(\mathbf{x}) = 0$ as the relation which relates the variables. If you can solve for x_{n+1}, for example and obtain a relation how x_{n+1} depends on the previous n-variables which can be then considered as parameters. The fact that there is n parameters makes this manifold an n-manifold.

For example, the n-**plane** can be defined as the set of vectors $\mathbf{x} = (x_1, \ldots, x_{n+1})$ in $\mathbb{R}^{n+1}$ such that

$$a_1x_1 + a_2x_2 + \ldots a_{n+1}x_{n+1} = d$$

for some constant vector $\mathbf{a} = (a_1, a_2, \ldots, a_{n+1})$ and a constant d. Thus, the equation of an n-plane containing a point $\mathbf{b}$ and which is perpendicular to $\mathbf{a}$ is given by $\mathbf{a} \cdot (\mathbf{x} - \mathbf{b}) = 0$ (in this case the constant d is equal to $\mathbf{a} \cdot \mathbf{b}$). Note that the vector $\mathbf{x} - \mathbf{b}$ lies in the n-plane. The equation $\mathbf{a} \cdot (\mathbf{x} - \mathbf{b}) = 0$ means that the vector $\mathbf{a}$ is perpendicular to the n-plane.

In the case when $n = 2$, $\mathbf{a} = (a, b, c)$, and $\mathbf{b} = (x_0, y_0, z_0)$, this produces the familiar plane equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

3. The n-**sphere** S^n is another example of a hypersurface. It can be defined as the set of vectors $\mathbf{x} = (x_1, \ldots, x_{n+1})$ in $\mathbb{R}^{n+1}$ such that

$$x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = a^2$$

i.e. that $|\mathbf{x}|^2 = a^2$. So, this is a hypersurface defined by the relation $|\mathbf{x}|^2 = a^2$ or $\mathbf{x} \cdot \mathbf{x} = a^2$.

In the case when $n = 2$, this produces the familiar equation

$$x^2 + y^2 + z^2 = a^2.$$

4. The 2-torus T^2 in three-dimensional space can be defined as the set of coordinates (x, y, u, v) where (x, y) is on one circle S^1 and (u, v) on the other S^1. This is written as $T^2 = S^1 \times S^1$.

If the circles are $x^2 + y^2 = a^2$ and $u^2 + v^2 = b^2$, we obtain another parametrization of the torus from practice problem 3 of the previous handout. Generalizing this to more dimensions, the n-**torus** T^n is defined as the set of vectors $\mathbf{x} = (x_1, \ldots, x_{2n})$ in $\mathbb{R}^{2n}$ such that $x_1^2 + x_2^2 = 1, x_3^2 + x_4^2 = 1, \ldots, x_{2n-1}^2 + x_{2n}^2 = 1$.

Partial Derivatives. Note that the domain of a coordinate patch is in the space $\mathbb{R}^n$ so the concept of partial derivatives is well-defined there. When working with manifolds, we may want to be able to differentiate on the range of the coordinate patch as well. This can be done by considering derivative of a real-valued functions.

Let f be function that maps a neighborhood U of a point P on n-manifold M into a subset of $\mathbb{R}$. The function f is smooth if the composition $f \circ \mathbf{x}$ is smooth where $\mathbf{x}$ is a coordinate patch that contains P (thus meets U). Note that $f \circ \mathbf{x}$ is a function that maps domain D of $\mathbf{x}$ into $\mathbb{R}$. Thus, we can define the derivative of f with respect to coordinate u^i as $\frac{\partial f}{\partial u^i} = \frac{\partial (f \circ \mathbf{x})}{\partial u^i} \circ (\mathbf{x}^{-1})$ and use it to define the partial derivative operator at point P as

$$\frac{\partial}{\partial u^i}(P)(f) = \frac{\partial f}{\partial u^i}(P).$$

Directional Derivative and Tangent Vectors. We can define tangent vectors using partial derivatives. In the case of surfaces, tangent vectors are defined as velocity vectors of curves on surfaces. However, the definition of velocity vector is not available if we want to avoid referring to a specific embedding in $\mathbb{R}^3$. We can still define tangent vectors using an alternate route - via directional derivative. To understand the idea, consider a vector $\mathbf{v}$ in $\mathbb{R}^3$ given by (v^1, v^2, v^3). This defines a **directional derivative** operator by

$$D_{\mathbf{v}} = \mathbf{v} \cdot \nabla = v^1 \frac{\partial}{\partial x} + v^2 \frac{\partial}{\partial y} + v^3 \frac{\partial}{\partial z}.$$

This operator is defined on the set of all real-valued functions f by $D_{\mathbf{v}} f = \mathbf{v} \cdot \nabla f = v^1 \frac{\partial f}{\partial x} + v^2 \frac{\partial f}{\partial y} + v^3 \frac{\partial f}{\partial z}$. Thus, any linear combination of the partial derivatives can be considered as a directional derivative.

The set of all tangent vectors corresponds exactly to the set of all directional derivatives. For every curve $\boldsymbol{\gamma}(t)$ on the surface $\mathbf{x}(u^1, u^2)$, the velocity vector $\boldsymbol{\gamma}'(t)(f) = \frac{d}{dt}(f \circ \boldsymbol{\gamma})$ can be seen as an operator $\boldsymbol{\gamma}'(t)(f) = \frac{\partial (f \circ \mathbf{x})}{\partial u^i}(\mathbf{x}^{-1} \circ \boldsymbol{\gamma}(t)) \frac{du^i}{dt} = \frac{\partial f}{\partial u^i} \boldsymbol{\gamma}(t) \frac{du^i}{dt}$. Thus, $\boldsymbol{\gamma}(t)' = \frac{du^i}{dt} \frac{\partial}{\partial u^i}(\boldsymbol{\gamma}(t))$ is the directional derivative $D_{((u^1)', (u^2)')}$.

Having defined partial derivatives on an n-manifold M allows us to define the tangent vectors at a point P as the set of all linear combinations of the partial derivatives $\frac{\partial}{\partial u^i}(P)$. Thus, $\mathbf{v}$ is a **tangent vector** if $\mathbf{v}$ is a linear combination of partial derivatives $\frac{\partial}{\partial u^i}(P)$ i.e. $\mathbf{v}$ is of the form $v^i \frac{\partial}{\partial u^i}(P)$.

The set of all vectors tangent to M at a point P is called the tangent space at P and is denoted by $T_P M$. This space represents the generalization of tangent plane of 2-manifolds. After showing that the partial derivatives are linearly independent, the vectors $\frac{\partial}{\partial u^i}(P)$ can be viewed as the basis of the tangent space $T_P M$.

Hilbert space, inner product, and metric. The coefficients of the first fundamental form are defined using the dot product of the basis vectors $\mathbf{x}_i = \frac{\partial}{\partial u^i}$. If an n-manifold with a patch $\mathbf{x}$ is embedded in $\mathbb{R}^{n+1}$, the dot product of the partial derivatives $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ generalizes precisely and

$$g_{ij} = \mathbf{x}_i \cdot \mathbf{x}_j \quad \text{for } \; i, j = 1, 2, \ldots, n.$$

If a manifold is to be considered without any embedding in any Euclidean space, one has to be more more flexible about the interpretation of a "vector" and the meaning of the inner product in order to generalize the definition of g_{ij} to arbitrary manifold. This leads us spaces known as **Hilbert spaces**.

A Hilbert space is a vector space with real coefficients meaning that we can refer to its elements as vectors, that we can add two vectors, that we can multiply a vector with a real number, and that the axioms of a vector space from Linear Algebra hold. In addition, this space is equipped with an **inner product** $\cdot$ which produces a real number $\mathbf{v} \cdot \mathbf{w}$ given two vectors $\mathbf{v}$ and $\mathbf{w}$, and which satisfies the following properties of the dot product in $\mathbb{R}^n$.

David Hilbert

Below, $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are vectors in the Hilbert space and a and b are real numbers.

1. The inner product is linear: $(a\,\mathbf{u} + b\,\mathbf{v}) \cdot \mathbf{w} = a\,\mathbf{u} \cdot \mathbf{w} + b\,\mathbf{v} \cdot \mathbf{w}$.

2. The inner product is symmetric: $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$.

3. The inner product is positive definite: $\mathbf{v} \cdot \mathbf{v} \geq 0$ and $\mathbf{v} \cdot \mathbf{v} = 0$ if and only if $\mathbf{v}$ is the zero vector.

The inner product enables one to measures the **length** of a vector $\mathbf{v}$ by

$$|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}.$$

Thus, one can define a metric using the inner product. Lastly, one requires that a Hilbert space is is **complete** with respect to this metric. This means that if the metric recognizes a sequence of vectors $\mathbf{v}_n$ as convergent, then such sequence has a limit in the space. More precisely, if the distance $|\mathbf{v}_m - \mathbf{v}_n|$ of $\mathbf{v}_m$ and $\mathbf{v}_n$ is small enough for large enough m and n, then there is a vector $\mathbf{v}$ in the space such that the sequence $\mathbf{v}_n$ converges to $\mathbf{v}$.

One can also consider the coefficients to be complex instead of real numbers. In this case, the definition is the same except that the requirement that the inner product is symmetric becomes $\mathbf{v} \cdot \mathbf{w} = \overline{\mathbf{w} \cdot \mathbf{v}}$ where $\overline{a + ib}$ is the complex-conjugate $a - ib$.

The space $\mathbb{R}^n$ with the dot product is an example of a Hilbert space. For a more exotic example, consider a "vector" to be a function $f(t)$ defined on interval $[a, b]$ and such that $\int_a^b f(t)^2 dt$ is defined (such a function is said to be **square-integrable** on $[a, b]$) and consider the inner product of two square-integrable functions f and g to be

$$f \cdot g = \int_a^b f(t)g(t)dt.$$

You can check that this product satisfies the three properties defining an inner product above.

Riemannian manifold and Theorema Egregium. Going back to manifolds, if the tangent space $T_P M$ at every point P of an n-manifold M is equipped with an inner product, we say that M is a **Riemannian manifold** and the inner product is called a **Riemannian metric**. In this case, the coefficients of the first fundamental form at any point can be defined as follows.

$$g_{ij} = \frac{\partial}{\partial u^i} \cdot \frac{\partial}{\partial u^j}$$

If g^{ij} denotes the matrix inverse to the matrix g_{ij}, then we can define **Christoffel Symbols** using the same formula which holds for surfaces

$$\Gamma^k_{ij} = \frac{1}{2}\left(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l}\right) g^{lk}.$$

With the inner product around, one can talk about the length between vectors and, hence, define the concepts of a unit-speed curve on a Riemannian n-manifold. With Christoffel symbols around also, we can also generalize the concept of a **geodesic** as follows. A curve $\boldsymbol{\gamma}$ on manifold M is geodesic if in each coordinate system defined along $\boldsymbol{\gamma}$ the equation $(u^k)'' + \Gamma^k_{ij}(u^i)'(u^j)' = 0$ holds for $k = 1, 2, \ldots n$. With this definition, every point P and every tangent vector $\mathbf{v}$ uniquely determine a geodesic $\boldsymbol{\gamma}$ with $\boldsymbol{\gamma}(0) = P$ and $\boldsymbol{\gamma}'(0) = \mathbf{v}$. In addition, a curve with the shortest possible length between two points is necessarily a geodesic connecting these points.

The Riemann curvature tensor can be defined via Christoffel symbols, using the same formula which holds for surfaces

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ik}}{\partial u^j} - \frac{\partial \Gamma^l_{ij}}{\partial u^k} + \Gamma^p_{ik}\Gamma^l_{pj} - \Gamma^p_{ij}\Gamma^l_{pk}$$

and the **sectional curvature** K, generalization of the Gaussian, at every point P of a manifold M can be defined as

$$K = \frac{g_{1i} R^i_{212}}{g}.$$

Since it can be expressed using the first fundamental form only, it can be computed completely intrinsically – without any reference to an embedding of the manifold into any external space. It is possible to define other types of curvatures: Ricci curvature and scalar curvature. These concepts are used in physics, especially in relativity theory.

Besides generalized Theorema Egregium, there are others fascinating results in differential geometry. We mention some of them below.

1. If the coefficients of the Riemann curvature tensor are equal to zero, then the n-manifold is locally isometric to $\mathbb{R}^n$. This generalizes the statement that if a surface has zero Gaussian, then it is locally isometric to a plane.

2. If a connected and complete Riemannian manifold of even dimension has constant sectional curvature $\frac{1}{a^2}$, then it is either a $2n$-sphere of radius a or a projective space.

3. Since the sectional curvature corresponds to the Gaussian for 2-manifolds, the following generalizes the description of the surfaces of revolution with constant Gaussian curvature. A complete, connected and simply connected (every closed curve can be collapsed to a point) Riemannian manifold of constant sectional curvature c is

   - the sphere S^n of radius $\frac{1}{\sqrt{c}}$ if $c > 0$,
   - the space $\mathbb{R}^n$ if $c = 0$, and
   - a hyperbolic space if $c < 0$. The hyperbolic space H^n is the set of vectors in $\mathbb{R}^n$ of length smaller than 1 with the metric coefficients g_{ij} given at point $\mathbf{v}$ by $g_{ij}(\mathbf{v}) = \frac{4\delta_{ij}}{-c(1-|\mathbf{v}|^2)^2}$.

4. Poincaré conjecture (recently proven, see Wikipedia for more details): Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere. Informally phrased, this means that if a 3-manifold "behaves" like a sphere in certain sense, then such manifold is topologically indistinguishable from a 3-sphere.