Math Methods I
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Calculus of Complex functions. Laurent Series and Residue Theorem

Review of complex numbers. A **complex number** is any expression of the form $x+iy$ where x and y are real numbers, called the **real** part and the **imaginary** part of $x+iy$, and i is $\sqrt{-1}$. Thus, $i^2 = -1$.

Other powers of i can be determined using the relation $i^2 = -1$. For example, $i^3 = i^2 i = -i$ and $i^{10} = (i^2)^5 = (-1)^5 = -1$.

Two complex numbers can be added by adding similar terms and they can be multiplied by foiling.

The complex number $x - iy$ is said to be **complex conjugate** of the number $x + iy$. To find the quotient of two complex numbers, one multiplies both the numerator and the denominator by the complex conjugate of the denominator. In this way, the answer is again in $x + iy$ form.

Example 1. Find the sum, product and quotient of $2 + i$ and $3 - 4i$.

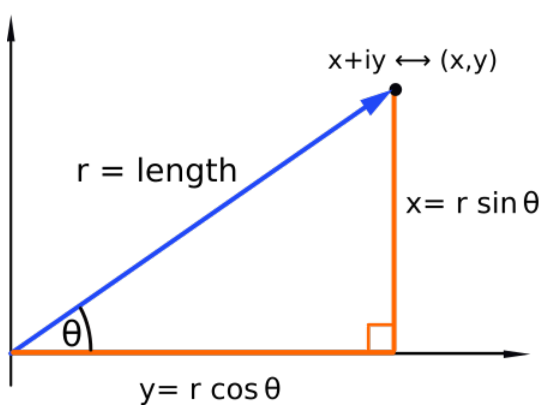
Solutions. The sum is $2 + i + 3 - 4i = 2 + 3 + i - 4i = 5 - 3i$. The product is $(2 + i)(3 - 4i) = 6 - 8i + 3i - 4i^2 = 6 - 8i + 3i + 4 = 10 - 5i$. The complex number $3 + 4i$ is the conjugate of $3 - 4i$, so the quotient is

$$\frac{2+i}{3-4i} = \frac{2+i}{3-4i}\,\frac{3+4i}{3+4i} = \frac{(2+i)(3+4i)}{(3-4i)(3+4i)} = \frac{6+8i+3i-4}{9+12i-12i+16} = \frac{2+11i}{25} = \frac{2}{25} + i\frac{11}{25}$$

Trigonometric Representations. Let us recall the polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. Using this representation, we have that

$$z = x + iy = r\cos\theta + ir\sin\theta.$$

The value r is the distance from the point (x, y) in the plane to the origin. The value r is called the **modulus** or absolute value of z. It is frequently denoted by $|z|$. The angle θ is the angle between the radius vector of (x, y) and the positive part of x-axis. The angle θ is usually called the **argument** or **phase** of z.



Euler's formula. The Euler's formula is stating that

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Euler's formula was proved (in an obscured form) for the first time by Roger Cotes in 1714, then rediscovered and popularized by Euler in 1748.

If $\theta = \pi$, we have that $e^{i\pi} = \cos\pi + i\sin\pi = -1 + 0 = -1$ so that the equation

$$e^{i\pi} = -1$$

holds. Many found this equation fascinating since it contains transcendental real numbers e and π, the imaginary number i on the left hand side and the integer -1 on the right. It states that the combination $e^{i\pi}$ of these transcendental and imaginary numbers is equal to – an integer, no less.

Euler proved the formula $e^{i\theta} = \cos\theta + i\sin\theta$ using power series expansions of exponential, sine and cosine functions (and this proof can be subject of your project). This formula allows the following simplification

$$z = x + iy = r\cos\theta + ir\sin\theta = re^{\theta i}.$$

Euler proved the formula $e^{i\theta} = \cos\theta + i\sin\theta$ using power series expansions of exponential, sine and cosine functions (if you took Calculus 3, you may have encountered this proof). This formula allows the following simplification

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Using the trigonometric representation, the formulas for multiplication and division of two complex numbers become easier than when the Cartesian form of complex numbers is used. If $z_1 = r_1e^{\theta_1 i}$ and $z_2 = r_2e^{\theta_2 i}$, then

$$z_1z_2 = r_1e^{\theta_1 i}r_2e^{\theta_2 i} = r_1r_2e^{i(\theta_1+\theta_2)}.$$

Euler's formula also produces an easy formula for the n-th power of a complex number $z = re^{\theta i}$,

$$z^n = \left(re^{\theta i}\right)^n = r^ne^{n\theta i}.$$

Complex functions. A complex valued function of complex variable is a function $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ where u, v are real functions of two real variables x, y. For example $f(z) = z^2 = (x+iy)^2 = x^2 + 2xyi - y^2$ is one such function. Its real part is $u = x^2 - y^2$ and its imaginary part is $v = 2xy$.

The Euler's formula can be used to define various complex functions that allow the same manipulations as their real-valued counterparts. For example, the exponential function e^z is defined as follows.

$$e^z = e^{x+iy} = e^xe^{iy} = e^x(\cos y + i\sin y)$$

Thus, e^z represents complex function with real part $e^x\cos y$ and the imaginary part $e^x\sin y$.

The exponential function with arbitrary (real) base is defined via the exponential function as

$$a^z = e^{z\ln a}$$

The power function can also be defined via the exponential function. In this course we will work just with integer powers of z. So, let n be an integer.

$$z^n = (re^{i\theta})^n = r^ne^{in\theta}$$

Note that the usual formulas for exponentiation work with this definition.

The complex-valued basic trigonometric functions are also defined via exponential function as follows

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) \qquad \cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

The other trigonometric function can be defined via these two, for example $\tan z = \frac{\sin z}{\cos z}$.

Example 2. Verify the following identities.

(a) $(e^z)^n = e^{nz}$ for n positive integer (b) $\sin^2 z + \cos^2 z = 1$.

Solutions. (a) $(e^z)^n = (e^x e^{iy})^n$. Using the formula for z^n, we have that this is equal to $e^{nx}e^{iny} = e^{n(x+iy)} = e^{nz}$.

(b) $\sin^2 z + \cos^2 z = \frac{-1}{4}(e^{iz} - e^{-iz})^2 + \frac{1}{4}(e^{iz} + e^{-iz})^2$. Square the terms in parenthesis to get $\frac{-1}{4}(e^{2iz} - 2 + e^{-2iz}) + \frac{1}{4}(e^{2iz} + 2 + e^{-2iz}) = \frac{1}{4}(-e^{2iz} + 2 - e^{-2iz} + e^{2iz} + 2 + e^{-2iz}) = \frac{1}{4}(4) = 1$.

Derivatives of Complex Functions

Consider $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ to be a complex valued function of complex variable. If $\frac{df}{dz}$ is a continuous function on the domain of f, then f is said to be differentiable. If f is differentiable at all points of its domain, we say that f is **analytic**. If f is analytic at all but the finitely many points of the domain, then these finitely many points at which f' may be discontinuous are called **singularities**.

If f is differentiable, all the usual rules of differentiation for real valued functions apply to the derivative $\frac{df}{dz}$. This seemingly obvious fact holds thanks to the convenient way how complex functions are defined. So, in practice, you can differentiate a complex function just like you would a real one.

Example 3. Find derivatives of the following functions.

(1) $f(z) = (z^2+5)^4$ (2) $f(z) = e^{zi-2}$ (3) $f(z) = \sin(4z^3+5)$.

Solutions. (1) $f'(z) = 4(z^2+5)^3(2z) = 8z(z^2+5)^3$, (2) $f'(z) = ie^{zi-2}$, (3) $f'(z) = 12z^2\cos(4z^3+5)$.

The differentiation may be trickier if a function cannot be represented in terms of $z = x+iy$. For example $f(x+iy) = 3x + 5yi$. For such function, the set of equations called **Cauchy-Riemann equations** are useful criterion of being analytic.

Namely, if f is an analytic function, then it can be shown that $\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$ so

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

These two equations are called **Cauchy-Riemann equations**. The converse holds as well, if u and v are continuous and the Cauchy-Riemann equations hold, then $f(z)$ is analytic because its derivative $\frac{df}{dz}$ can be found as $\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$.

Example 4. Check if the following functions are analytic or not.

(a) $f(x+iy) = 3y - 3xi$, (b) $f(x+iy) = 3y - 5xi$.

Solution. (a) Here $u = 3y$ and $v = -3x$. The first Cauchy-Riemann equation holds since both $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ are equal to 0. The second equation holds also since $\frac{\partial u}{\partial y} = 3$ and $\frac{\partial v}{\partial x} = -3$ and so $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

(b) Here $u = 3y$ and $v = -5x$. The first Cauchy-Riemann equation holds since both $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ are equal to 0. However, the second equation fails since $\frac{\partial u}{\partial y} = 3$ and $\frac{\partial v}{\partial x} = -5$. So it fails since $3 = \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x} = 5$. Thus, this function is not analytic.

If f is an analytic function and u and v have continuous second partial derivatives, then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{-\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2} \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Similarly,

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{-\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{-\partial v}{\partial y}\right) = -\frac{\partial^2 v}{\partial y^2} \Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

The equations

$$u_{xx} + u_{yy} = 0 \quad \text{and} \quad v_{xx} + v_{yy} = 0$$

are called the **Laplace equations**. So, if f is analytic, both the real and the imaginary part satisfy the Laplace equations. However, the converse does not have to hold: there are non-analytic functions with u and v which satisfy the Laplace equations. For example, function f from part (b) of Example 4 has $u_{xx} + u_{yy} = 0 + 0 = 0$ and $v_{xx} + v_{yy} = 0 + 0 = 0$ but, as we have seen, it is not analytic.

Still, the Laplace equations provide useful information in the following sense: if a given real part u (or imaginary part v) fails the Laplace equation, we know that an analytic function with u as its real part (or v as its imaginary part) does not exist. Part (a) of the next example demonstrate this.

If a given real part satisfies the Laplace equation, the imaginary part of an analytic function with real part u can be found using *both* Cauchy-Riemann equations:

1. Since $u_x = v_y$, integrating u_x with respect to y determines v up to the terms which only have x in them. If $g(x)$ is the sum of all such terms, then $v = \int u_x dy + g(x)$.

2. Since $u_y = -v_x$, differentiate v from the previous step and set it equal to u_y. This gives you an equation in $g'(x)$. Integrating with respect to x determines g up to an integration constant and since we know $g(x)$, we know entire $v = \int u_x dy + g(x)$ up to an integration constant.

Part (b) of the next example demonstrates this process.

Analogous process can be used if v is given and we are looking for u such that $u + iv$ is analytic as practice problem 3 illustrates.

Example 5. Check if analytic functions with real part equal to the given functions u exist. If so, find all analytic functions that have real parts equal to u.

(a) $u = xe^{3y}$ (b) $u = x^2 + 3x - y^2 + 5y$.

Solution. (a) To see if such an analytic function exists, check if it satisfies Laplace equation. $\frac{\partial u}{\partial x} = e^{3y} \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0$ and $\frac{\partial u}{\partial y} = 3xe^{3y} \Rightarrow \frac{\partial^2 u}{\partial y^2} = 9xe^{3y}$. Since $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 - 9xe^{3y} \neq 0$, no such analytic function exists.

(b) Note that the given u satisfies Laplace equation: $\frac{\partial u}{\partial x} = 2x+3 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 2$ and $\frac{\partial u}{\partial y} = -2y+5 \Rightarrow \frac{\partial^2 u}{\partial y^2} = -2$. Since $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2-2 = 0$.

Since $2x+3 = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ you can find v as $v = \int (2x+3)dy = 2xy+3y+g(x)$. Determine the function g from the second Cauchy-Riemann equation $2y-5 = \frac{-\partial u}{\partial y} = \frac{\partial v}{\partial x} = 2y + g'(x) \Rightarrow g'(x) = -5 \Rightarrow g(x) = -5x+c$. Thus, $v = 2xy+3y-5x+c$. In this case $f(z) = x^2+3x-y^2+5y+i(2xy+3y-5x+c)$. This function turns out to be equal to $x^2+2ixy+y^2+3(x+iy)+5(-ix+y)+ic = (x+iy)^2 + 3(x+iy)-5i(x+iy)+ic = z^2+3z-5iz+ic$.

Integrals of Complex Functions

Let C be a curve in the xy-plane. In order evaluate the integral $\int_C f(z)dz$ of the complex function $f(z)$ over C, parametrize the curve C as $x = x(t)$ and $y = y(t)$ and obtain the bounds for t, $a \leq t \leq b$.

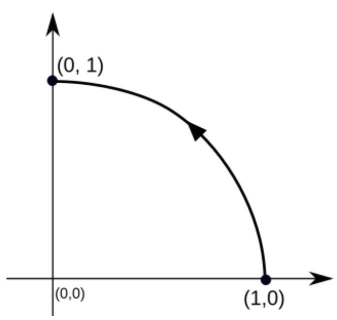
Then represent z as $x(t)+y(t)i$ and note that then $dz = dx + dyi = x'(t)dt + y'(t)dti = z'(t)dt$. Thus,

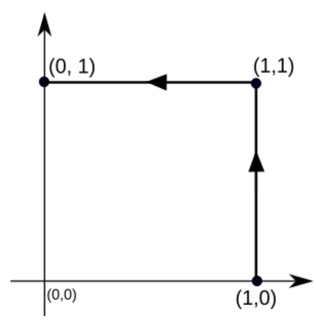
$$\int_L f(z)dz = \int_a^b f(z(t))z'(t)dt$$

Example 6. Evaluate the following integrals.

1. $\int z^2 dz$ over quarter of the unit circle in the first quadrant traversed counterclockwise (the first figure below).

2. $\int$ Re(z) dz over the two line segments on the second figure below. Recall that Re(z) denotes the real part of $z = x+iy$. So, Re(z) $= x$.





Solutions.

1. We can parametrize the unit circle as $x = \cos t$, $y = \sin t$ so $z = \cos t + i\sin t$ and $dz = -\sin t + i\cos t$. Since we consider only the part of the circle in the first quadrant, the bounds are $0 \leq t \leq \frac{\pi}{2}$.

   One could find z^2dz by foiling the terms and then integrate it using some trigonometric identities but, in this case, it is easier to use the exponential representation e^{it} for $z = \cos t + i \sin t$ since then squaring, finding dz and multiplying z^2 and dz is straightforward. Indeed, $z = e^{it} \Rightarrow z^2 = e^{2it}$ and $dz = e^{it}tdt$ and so

   $\int z^2dz = \int_0^{\pi/2} e^{2it}e^{it}idt = i\int_0^{\pi/2} e^{3it}dt = \frac{i}{3i}e^{3it}|_0^{\pi/2} = \frac{1}{3}(e^{\frac{3\pi}{2}i} - e^{0i}) = \frac{1}{3}(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} - 1) = \frac{1}{3}(-i-1) = \frac{-1}{3}(1+i)$.

2. The first line segment can be parametrized by $x = 1$, $y = y$ and the bounds for y are $0 \leq y \leq 1$. In this case, z is $z = 1 + iy$ so Rez=1 and $dz = 0dx + idy = idy$. The integral over this first part is $\int_0^1 idy = iy|_0^1 = i$.

   The second line segment can be parametrized by $x = x$, $y = 1$ and the x-values are decreasing from 1 to 0. Then $z = x + i$ and Re $z = x$, $dz = dx + 0i = dx$. The integral over this part is $\int_1^0 xdx = \frac{x^2}{2}|_1^0 = -\frac{1}{2}$. So, the final answer is $i - \frac{1}{2}$.

Cauchy's Theorem. Let C be a closed piecewise smooth curve and f an analytic function defined on an simply connected domain that contains the interior of the curve C. Without going into the precise definition of "simply connected", we just say that the domain of f is simply connected if it consists of one piece without any holes. For example, a circle is simply connected while an annulus is not. To find more about this concept, you can explore wikipedia.org.

With those assumptions,

$$\oint_C f(z)dz = 0.$$

This statement is known as **Cauchy's Theorem.**

To prove this statement, recall Green's theorem from Calculus 3. If C is a positive oriented, piecewise smooth, closed curve and P and Q have continuous derivatives, then the line integral $\oint_C Pdx + Qdy$ can be reduced to a double integral over the interior D of C.

$$\oint_C Pdx + Qdy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Thus, if $f(z) = u + iv$, and if D denotes the interior of C, then

$$\oint_C f(z)dz = \oint_C (u + iv)(dx + idy) = \oint_C (udx - vdy) + i \oint_C (vdx + udy)$$

Using Green's theorem, we obtain that this last expression is equal to the following.

$$\int\int_D \left(\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i \int\int_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy = 0.$$

The last equality uses the fact that f is analytic so that the Cauchy-Riemann equations hold.

Example 7. Evaluate

$$\int (z^3 + 5z - \sin z)dz$$

over a unit circle in xy-plane.

Solutions. A unit circle is a closed curve and $f(z) = z^3 + 5z - \sin z$ is an analytic function (derivative $3z^2 + 5 - \cos z$ is continuous) defined for every z. Thus, this curve and function f satisfy the assumptions of Cauchy's Theorem. Thus, the integral is equal to zero.

Integrals of non-analytic functions and Laurent Series

In the section on integrals of complex functions, we addressed integration of a complex function over a curve which is *not necessarily closed*. Cauchy's Theorem addresses integration of *analytic* functions over *closed* curves. Next, we address integration of *any* function, not necessarily analytic, over a *closed* curve. As it turns out, the process of such integration can be significantly shortened by determining a certain coefficient of the power series expansion known as the Laurent series of the function. A formula known as the Cauchy's differentiation formula relates the integral and the power series expansion of a complex function.

Cauchy's Differentiation Formula. If f is an analytic function defined on a simple connected region that contains the interior of the closed piecewise smooth curve C, (note that these are the same assumptions as in Cauchy's Theorem) and a is a point in the interior of C, then

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz$$

where $n = 0, 1, \ldots$. This formula is known as **Cauchy's differentiation formula**. Its proof is one of the project topics you can choose from.

Note that for $n = 0$ this gives the formula known as the **Cauchy's integral formula**.

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz \quad \Rightarrow \quad \oint_C \frac{f(z)}{z-a}\, dz = 2\pi i f(a)$$

This formula alone can be useful for integrals of non-analytic functions over closed curves as the following example shows.

Example 8. Evaluate

$$\oint_C \frac{1}{z} dz$$

over the square C with sides (1,0), (0,1), (-1,0), (0,-1).

Solution. Although C is closed, Cauchy's Theorem cannot be applied here since $\frac{1}{z}$ is not analytic at $z = 0$. However, if we consider that $f(z) = 1$ and $a = 0$, then the Cauchy's integral formula applies and it states that the integral is equal to $2\pi i f(0) = 2\pi i(1) = 2\pi i$. Note that evaluating this integral directly would be much more time consuming since you would have to evaluate four integrals, each using different parametrization of a side of the square.

Laurent Series. Recall that a real function $f(x)$ that with continuous derivatives of any order near a point $x = a$ has the power series expansion

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \qquad \text{where} \qquad a_n = \frac{f^{(n)}(a)}{n!}.$$

This formula produces the following power series expansions of some frequently used functions.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Recall also that the first and the last two series converge for any value of x and that the second series converge on interval $(-1, 1)$ which is exactly the interval centered at 0 of "radius" 1. The radius value is the distance from 0 to the singularity 1 of this function.

Because of the way how rational, exponential, and trigonometric functions of complex variable are defined, the process of finding their derivatives produces the same formulas when a real variable x is considered instead of z. For example, the derivative of $\sin x$ is $\cos x$ and the derivative of $\sin z$ is $\cos z$. As a result, the derivatives have the same value at $a = 0$ when we consider f as a function of x as when we consider it as a function of z and, thus, functions composed of rational, exponential, and trigonometric parts have the expansions given by the same formulas as in the real case.

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \qquad \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

The first and the last two series converge for any value of z and the second series converge on the disc centered at zero of radius 1. This radius is, again, the distance from 0 to the singularity 1 of this function.

These elementary function expansions can be used to obtain the expansions of functions composed of the basic four functions as the next example illustrates.

Example 9. Find the power series expansions of the following functions at 0 and determine the radius of convergence.

(a) $f(z) = e^{iz}$, (b) $f(z) = \frac{1}{1-z^2}$.

Solutions. (a) $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \Rightarrow f(z) = e^{iz} = \sum_{n=0}^{\infty} \frac{i^n z^n}{n!}$. This function has no singularities so this series converges for all values of z (i.e. the radius of convergence is infinite).

(b) $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \Rightarrow \frac{1}{1-z^2} = \sum_{n=0}^{\infty} z^{2n}$. 1 and -1 are singularities of this function, thus this series converges within the unit circle, that is, the radius of convergence is 1.

Note that all of the series above do not contain any negative powers of z (or $z - a$ in general). However, examples like $\frac{1}{z}$ indicate that one needs to consider power series expansions which may contain negative powers of z as well. Because of this, one considers *both* constant multiples of $(z-a)^m$ where m is a negative integer *and* the constant multiples of $(z-a)^n$ where n is a non-negative integer. and expand a function $f(z)$ as follows.

$$f(z) = \ldots + a_{-3}(z-a)^{-3} + a_{-2}(z-a)^{-2} + a_{-1}(z-a)^{-1} + a_0 + a_1(z-a) + a_2(z-a)^2 + a_3(z-a)^3 + \ldots$$

Such sum can be written shortly as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

and it is called the *Laurent series* expansion of $f(z)$ at a. It is used for complex functions if the Taylor series expansion cannot be applied. It converges inside of a circle centered at a of radius equal to the distance from a to the nearest singularity of $f(z)$.

Example 10. Find the Laurent series expansion of $\frac{1}{z}$ at 0.

Solution. Since $\frac{1}{z} = z^{-1}$ is already an integer power of z, $\frac{1}{z}$ is the expansion at 0.

$$\ldots + 0(z)^{-3} + 0(z)^{-2} + 1(z)^{-1} + 0 + 0(z) + 0(z)^2 + 0(z)^3 + \ldots = z^{-1} = \frac{1}{z}$$

Since the Taylor series expansion has coefficients a_n given by the formula $a_n = \frac{f^{(n)}(a)}{n!}$, the Cauchy's differentiation formula provides a way to generalize this to negative values of n as well.

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz \quad \Rightarrow \quad a_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz$$

When n is taking negative integer values, then

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz.$$

Thus, the formulas computing positive and negative indexed coefficients *agree.*

The value a in the above formula may potentially be an isolated singularity of $f(z)$ and C is any closed, piecewise smooth curve such that $z = a$ is in its interior and no other singularities of f are in the interior.

The coefficient a_{-1} with $(z-a)^{-1}$ in the series expansion has a special significance and is called the **residue** of $f(z)$ at $z = a$. Its significance can be seen by considering the formula for computing a_n in the case when $n = -1$.

$$a_{-1} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{-1+1}}\, dz = \frac{1}{2\pi i} \oint_C f(z)\, dz \quad \Rightarrow \quad \oint_C f(z)\, dz = 2\pi i\, a_{-1}$$

Thus, knowing a_{-1} can significantly shorten the procedure of evaluating integrals of complex functions. Because of this, we discuss the procedure of determining the residue a_{-1} in the following three cases.

Three types of singularities. Consider the Laurent series expansion of $f(z)$ at $z = a$ and the following three cases.

1. All the coefficients a_{-n} with negative subscripts are *zero.* In this case a is a said to be a **removable singularity**. In this case, just as when f is analytic,

   $$a_{-1} = 0.$$

   For example, consider $f(z) = \frac{\sin z}{z}$. Since $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ we have that the Laurent series expansion of $f(z)$ is $\frac{\sin z}{z} = \frac{1}{z}\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots$ Note that there are no terms with negative powers of z. Thus 0 is a removable singularity of this function.

2. Just finitely many coefficients with negative subscripts are not zero. In this case a is said to be a **pole**. If $a_{-k} \neq 0$ and $a_{-k-1} = a_{-k-2} = \ldots = 0$, then a is a **pole of order** k. Note that a is a pole of order k if $(z-a)^k f(z)$ is an analytic function without singularity or with removable singularity at a. In this case, a_{-1} can be computed by the following formula.

   $$a_{-1} = \frac{1}{(k-1)!} \lim_{z\to a} \frac{d^{k-1}}{dz^{k-1}} \left((z-a)^k f(z) \right).$$

   One of your project topics focuses on the proof of this formula. You can also find a_{-1} by finding the series expansion of $f(z)$ and noting the coefficient with $\frac{1}{z-a}$.

By the formula above, if a is a **pole of the first order**, then

$$a_{-1} = \lim_{z \to a} (z-a) f(z).$$

For example, $f(z) = \frac{1}{z}$ has a pole of the first order 1 at $z=0$ and $a_{-1} = 1$. Similarly, $f(z) = \frac{1}{z^n}$ has a pole of the n-th order at $z = 0$. If $n \neq 1$, $a_{-1} = 0$.

3. Infinitely many coefficients with negative subscripts are zero. In this case a is said to be a **essential singularity**. In this case, the above formula with the limit does not apply so the only way to find a_{-1} is to find the series expansion of $f(z)$ and note the coefficient with $\frac{1}{z-a}$.

   For example, the function

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n! z^n} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3! z^3} + \ldots + \frac{1}{n! z^n} + \ldots$$

   has an essential singularity at $z=0$. The residue a_{-1} is 1.

Practice Problems.

1. Find the derivative of the following functions (a) $f(z) = z e^{5iz^2}$, (b) $f(z) = \frac{1}{(z^2+4)^3}$.

2. Determine if the following functions are analytic.

   (a) $f(z) = z e^{5iz^2}$, (b) $f(x+iy) = x^2 y + ixy^2$, (c) $f(x+iy) = x^2 - y^2 + 2xyi$.

3. Find an analytic function $f(z)$, if such function exists, so that the imaginary part of $f(z)$ is equal to

   (a) $v = 3x^2 + 4y - 3y^2$, (b) $v = e^{xy}$ (c) $v = 3e^x \cos y$.

4. Evaluate $\int z^4 dz$

   (a) over the upper-half of the unit circle traversed counterclockwise;
   (b) over the unit circle traversed counterclockwise.

5. Let $f(z) = z^3 - 2z + e^{z-2}$ and let C be the circle of radius 3 in xy-plane. Evaluate

   (a) $\displaystyle\int_C f(z)\, dz$ (b) $\displaystyle\int_C \frac{f(z)}{z-2}\, dz$

6. Find the power series expansions of the following functions at the given z-value. If the given z-value is an isolated singularity, determine the type of singularity and find the residue.

   (a) $f(z) = \frac{z}{1-z^2}$, $z=0$, (b) $f(z) = \frac{e^{z-1}}{(z-1)^2}$ $z=1$, (c) $f(z) = \frac{1-\cos z}{z^2}$ $z=0$.

7. Classify all the singularities and find their residues for the following functions.

   (a) $f(z) = \frac{1}{(z-2)^2(4+z)}$ (b) $f(z) = \frac{e^z}{(z-1)^5}$, (c) $f(z) = z \cos \frac{1}{z}$.

Solutions.

1. (a) $\frac{d}{dz}\left(ze^{5iz^2}\right) = e^{5iz^2} + 10iz^2e^{5iz^2}$. (b) $\frac{d}{dz}\left((z^2+4)^{-3}\right) = -3(z^2+4)^{-4}(2z) = \frac{-6z}{(z^2+4)^4}$.

2. (a) By the previous problem, the function $f(z) = ze^{5iz^2}$ has derivative $f'(z) = e^{5iz^2} + 10iz^2e^{5iz^2}$ which is a continuous function. Thus, f is analytic.

   (b) For $f(x+iy) = x^2y + ixy^2$, $u = x^2y$ and $v = xy^2$. Check if the Cauchy-Riemann equations hold. $\frac{\partial u}{\partial x} = 2xy = \frac{\partial v}{\partial y}$ so the first equation holds. $\frac{\partial u}{\partial y} = x^2$ and $\frac{\partial v}{\partial x} = y^2$ so the second equation fails. Thus, f is not analytic.

   (c) For $f(x+iy) = x^2 - y^2 + 2xyi$, $u = x^2 - y^2$ and $v = 2xy$. Check if the Cauchy-Riemann equations hold. $\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -2y = \frac{\partial v}{\partial x}$ so both equations hold. Thus, f is analytic.

3. (a) Check first if Laplace equation is satisfied. $v = 3x^2 + 4y - 3y^2 \Rightarrow \frac{\partial v}{\partial x} = 6x \Rightarrow \frac{\partial^2 v}{\partial x^2} = 6$ and $\frac{\partial v}{\partial y} = 4 - 6y \Rightarrow \frac{\partial^2 v}{\partial y^2} = -6$. Thus, $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 6 - 6 = 0$.

   Then find f using Cauchy-Riemann equations. $\frac{-\partial v}{\partial x} = -6x = \frac{\partial u}{\partial y} \Rightarrow u = \int -6x dy = -6xy + g(x)$. Using the second equation, we have that $4 - 6y = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = -6y + g'(x) \Rightarrow g'(x) = 4 \Rightarrow g(x) = 4x + c$. Thus $u = -6xy + 4x + c$. Hence $f(z) = -6xy + 4x + c + i(3x^2 + 4y - 3y^2) = 3(x - iy)^2 + 4(x + iy) + c = 3\overline{z}^2 + 4z + c$.

   (b) Check Laplace equation: $v = e^{xy} \Rightarrow \frac{\partial v}{\partial x} = ye^{xy} \Rightarrow \frac{\partial^2 v}{\partial x^2} = y^2e^{xy}$ and $\frac{\partial v}{\partial y} = xe^{xy} \Rightarrow \frac{\partial^2 v}{\partial y^2} = x^2exy$. Thus $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = (y^2 + x^2)e^{xy} \neq 0$. So, no analytic function f with the imaginary part equal to v exists.

   (c) Check first that Laplace equation is satisfied. Then use the Cauchy-Riemann equations. $v = 3e^x \cos y \Rightarrow \frac{-\partial v}{\partial x} = -3e^x \cos y = \frac{\partial u}{\partial y} \Rightarrow u = \int -3e^x \cos y dy = -3e^x \sin y + g(x)$. Using the second equation, we have that $-3e^x \sin y = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = -3e^x \sin y + g'(x) \Rightarrow g'(x) = 0 \Rightarrow g(x) = c$. Thus $u = -3e^x \sin y + c$. Hence $f(z) = -3e^x \sin y + c + 3ie^x \cos y = 3ie^x(\cos y + i \sin y) + c = 3ie^xe^{iy} + c = 3ie^z + c$.

4. (a) Parametrization of the unit circle is $x = \cos t$, $y = \sin t$, so $z = \cos t + i \sin t = e^{it}$. Thus, $z^4 = e^{4it}$ and $dz = e^{it}idt$. The bounds for t are $0 \leq t \leq \pi$. The integral is $\int z^4 dz = \int_0^\pi e^{4it}e^{it}idt = i\int_0^\pi e^{5it}dt = \frac{i}{5i}e^{5it}|_0^\pi = \frac{1}{5}(e^{5\pi i} - e^{0i}) = \frac{1}{5}(\cos 5\pi + i \sin 5\pi - 1) = \frac{1}{5}(-2) = \frac{-2}{5}$.

   (b) The function $f(z) = z^4$ is analytic because it has derivative $4z^3$ which is a continuous function. Thus, the integral is zero since the integral of an analytic function over a close curve is zero (Cauchy's Theorem).

5. $f(z) = z^3 - 2z + e^{z-2}$ is analytic (derivative $3z^2 - 2 + e^{z-2}$ is continuous) so the integral in part (a) is zero by Cauchy's Theorem. By Cauchy's integral formula, the integral in part (b) is equal to $2\pi i f(2) = 2\pi i(8 - 4 + 1) = 10\pi i$. Alternatively, you can take $g(z) = \frac{f(z)}{z-2}$ and note that g has one singularity, 2, it is a pole of the first order, and it is in the interior of C. The residue of g at 2 is $\lim_{z\to 2}(z-2)\frac{f(z)}{z-2} = \lim_{z\to 2} f(z) = f(2) = 5$. Hence $\int_C g(z)dz = 2\pi i(5) = 10\pi i$.

6. (a) $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \Rightarrow \frac{1}{1-z^2} = \sum_{n=0}^{\infty} z^{2n} \Rightarrow f(z) = \frac{z}{1-z^2} = \sum_{n=0}^{\infty} z^{2n+1}$. The only singularities of $f(z)$ are ± 1. So, the radius of the convergence is the distance from 0 to any of these two

points and so it is 1. The function is analytic at $z=0$ (also note that there are no terms with negative exponents in the series expansion).

(b) $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \Rightarrow e^{z-1} = \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \Rightarrow$

$$f(z) = \frac{e^{z-1}}{(z-1)^2} = \frac{1}{(z-1)^2}\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!} = \sum_{n=0}^{\infty}\frac{(z-1)^{n-2}}{n!} = \frac{1}{(z-1)^2}+\frac{1}{z-1}+\frac{1}{2!}+\frac{z-1}{3!}+\ldots.$$

The only singularity is $z=1$ and, from the power series expansion, we can see that it is a pole of the order 2. The coefficient with the term with $\frac{1}{z-1}$ is 1 so the residue is 1. The series converges at all points except $z=1$.

(c) $f(z) = \frac{1-\cos z}{z^2} = \frac{1}{z^2} - \frac{1}{z^2}\cos z = \frac{1}{z^2} - \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!} = \frac{1}{z^2} - \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n-2}}{(2n)!} = \frac{1}{z^2} - \frac{1}{z^2} + \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \ldots = \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \ldots$ Thus, there are no terms with negative exponents. So, the only singularity $z=0$ is a removable singularity.

7. (a) $f(z) = \frac{1}{(z-2)^2(4+z)}$ has two singularities: 2 is a pole of order 2 and -4 is a pole of order 1. The residue at 2 can be computed as

$$\lim_{z\to 2}\frac{d}{dz}\left((z-2)^2 f(z)\right) = \lim_{z\to 2}\frac{d}{dz}\left(\frac{1}{4+z}\right) = \lim_{z\to 2}\frac{-1}{(4+z)^2} = \frac{-1}{36}.$$

The residue at -4 is $\lim_{z\to -4}(z+4)f(z) = \lim_{z\to -4}\frac{1}{(z-2)^2} = \frac{1}{36}$.

(b) $f(z) = \frac{e^z}{(z-1)^5}$. The only singularity is 1 and it is a pole of order 5. The residue at 1 is $\frac{1}{4!}\lim_{z\to 1}\frac{d^4}{dz^4}\left((z-1)^5 f(z)\right) = \frac{1}{24}\lim_{z\to 1}\frac{d^4}{dz^4}\left(e^z\right) = \frac{1}{24}\lim_{z\to 1} e^z = \frac{e^1}{24} = \frac{e}{24}$.

(c) The only singularity is 0 and it is an essential singularity. Note that the formula with the limit does not apply to this case since 0 is not a pole. We can find the residue by looking at the power series expansion $f(z) = z\cos\frac{1}{z} = z\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!z^{2n}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!z^{2n-1}} = z - \frac{1}{2!z} + \frac{1}{4!z^3} - \ldots$. So, the residue is $\frac{-1}{2}$.

The Residue Theorem

Let $f(z)$ be a complex function which is analytic everywhere except possibly at $z=a$ and C is a closed, piecewise smooth curve such that $z=a$ is at its interior. Recall that the previous section shows that the coefficient a_{-1} in the Laurent series expansion of $f(z)$ can be used for evaluation the integral of f over C.

$$\oint_C f(z)\,dz = 2\pi i a_{-1} = 2\pi i(\text{ coefficient of the term with}\frac{1}{z-a}).$$

Note that if f is analytic at $z=a$, then the residue $a_{-1}=0$ since the Laurent series has no terms with negative powers. In this case, the last formula boils down to Cauchy's Theorem $\oint_C f(z)\,dz = 0$.

Assume now that f is an analytic function with isolated singularities $z_1, z_2, \ldots z_n$ and that C is a closed, piecewise smooth, positive oriented curve whose interior contains the singularities $z_1, z_2, \ldots z_n$. If $R_1, R_2, \ldots R_n$ are the residues at $z_1, z_2, \ldots z_n$, then

$$\oint_C f(z)\,dz = 2\pi i(R_1 + R_2 + \ldots + R_n).$$

This statement is known as **Residue Theorem**.

Example 1. Let $f(z) = \frac{1}{z^2(z^2+1)}$.
(a) Find the residues of all isolated singularities of f.
(b) Evaluate $\oint_C f(z)dz$ where C is the circle of radius 2 centered at 0.

Solution. (a) Note that $z^2 + 1 = (z-i)(z+i)$. Thus, f has three singularities $0, i$ and $-i$. 0 is a pole of order 2 and $\pm i$ are poles of the first order.
The residue at 0 can be computed as $\lim_{z\to 0} \frac{d}{dz}\left(z^2 f(z)\right) = \lim_{z\to 0} \frac{d}{dz}\left(\frac{1}{z^2+1}\right) = \lim_{z\to 0} \frac{-2z}{(z^2+1)^2} = 0$.
The residue at i can be computed as $\lim_{z\to i}(z-i)f(z) = \lim_{z\to i} \frac{1}{z^2(z+i)} = \frac{1}{-1(i+i)} = \frac{-1}{2i} = \frac{i}{2}$.
The residue at $-i$ is $\lim_{z\to -i}(z+i)f(z) = \lim_{z\to i} \frac{1}{z^2(z-i)} = \frac{1}{-1(-i-i)} = \frac{1}{2i} = \frac{-i}{2}$.
(b) Using the Residue Theorem, the integral is equal to $2\pi i(0 + \frac{i}{2} - \frac{i}{2}) = 0$.

The following example illustrate how some real value integrals can be evaluated using Residue Theorem.

Example 2. Let C be the circle of radius 5 centered at the origin. Evaluate

$$\oint_C \frac{e^z}{z^2-4}dz$$

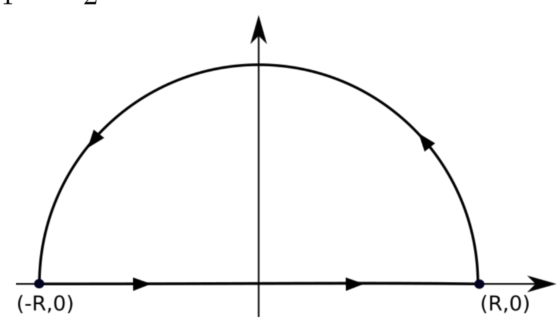
Solution. The function $f(z) = \frac{e^z}{z^2-4} = \frac{e^z}{(z-2)(z+2)}$ has two singularities 2 and -2 and each is a pole of the first order. Both are contained in the interior of C. Thus, the integral is equal to the product of $2\pi i$ and the sum of the two residues.
The residue at 2 is $\lim_{z\to 2}(z-2)\frac{e^z}{(z-2)(z+2)} = \lim_{z\to 2} \frac{e^z}{(z+2)} = \frac{e^2}{4}$.
The residue at -2 is $\lim_{z\to -2}(z+2)\frac{e^z}{(z-2)(z+2)} = \lim_{z\to -2} \frac{e^z}{(z-2)} = \frac{e^{-2}}{-4}$.
Thus,

$$\oint_C \frac{e^z}{(z-2)(z+2)}dz = 2\pi i(\frac{e^2}{4} + \frac{e^{-2}}{-4}) = \frac{\pi i}{2}(e^2 - e^{-2}).$$

Example 3. Let a and R be constants such that $R > a$. Evaluate

$$\int_C \frac{1}{(z^2+a^2)^2}dz$$

where C is the contour given on the figure on the right.



Extra credit level. Using the above integral, evaluate the integral

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2}dx.$$

Note that it would be quite difficult to evaluate this integral directly. Use your answer to evaluate also $\int_0^\infty \frac{1}{(x^2+a^2)^2}dx$.

Solution. The function $f(z) = \frac{1}{(z^2+a^2)^2} = \frac{1}{(z-ai)^2(z+ai)^2}$ has two poles, $\pm ai$, of the second order. The given contour includes only ai so we do not need to consider the residue at $-ai$ Thus, the integral is equal to the product of $2\pi i$ and the residue at ai. The residue at ai is $\lim_{z\to ai}\frac{d}{dz}\left((z-ai)^2 f(z)\right) = \lim_{z\to ai}\frac{d}{dz}\left(\frac{1}{(z+ai)^2}\right) = \lim_{z\to ai}\frac{-2}{(z+ai)^3} = \frac{-2}{(2ai)^3} = \frac{-1}{4a^3(-i)} = \frac{1}{4a^3 i} = \frac{-i}{4a^3}$.

Thus, $\int_{C_R}\frac{1}{(z^2+a^2)^2}dz = 2\pi i\frac{-i}{4a^3} = \frac{\pi}{2a^3}$.

Solution to the extra credit part. On the bottom, line segment part of C can be parametrized as $z = x+0i = x$ and $dz = dx$ so the integral of $f(z)$ over this part is $\int_{-R}^{R}\frac{1}{(x^2+a^2)^2}dx$. If we let $R\to\infty$ we will obtain the integral we need to evaluate.

On the upper, semi-circular part of C, we have that $z = Re^{it}$ for $t: 0\to\pi$ and $dz = Rie^{it}dt$. The integral over this part is $\int_0^{\pi}\frac{Rie^{it}}{(R^2e^{2it}+a^2)^2}dt$. We shall show that the integral of the modulus of f converges to 0 for $R\to\infty$. Thus, the integral of f converges to 0 as well.

Consider the modulus of this function is $\frac{R}{|R^2e^{2it}+a^2|^2}$. The denominator $|R^2e^{2it}+a^2|^2$ is larger than $|R^2e^{2it}|^2 = R^4$. So, the whole function is smaller than $\frac{1}{R^3}$. So, when $R\to\infty$, this function converges to 0. Thus, the integral is equal to 0.

When we let $R\to\infty$, the integral on the left converges to the sum over the lower part that converges to $\int_{-\infty}^{\infty}\frac{1}{(x^2+a^2)^2}dx$ and the integral of the top part that converges to 0. Thus, we obtain that this integral is equal to $\frac{\pi}{2a^3}$.

$$\int_{-\infty}^{\infty}\frac{1}{(x^2+a^2)^2}dx = \frac{\pi}{2a^3}$$

We can use $\int_{-\infty}^{\infty}\frac{1}{(x^2+a^2)^2}dx$ to evaluate $\int_0^{\infty}\frac{1}{(x^2+a^2)^2}dx$ as well. Note that the function under the integral is even, thus $\int_{-\infty}^{\infty}\frac{1}{(x^2+a^2)^2}dx = \int_{-\infty}^{0}\frac{1}{(x^2+a^2)^2}dx + \int_0^{\infty}\frac{1}{(x^2+a^2)^2}dx = 2\int_0^{\infty}\frac{1}{(x^2+a^2)^2}dx$. Thus,

$$\int_0^{\infty}\frac{1}{(x^2+a^2)^2}dx = \frac{\pi}{4a^3}.$$

Probably the trickiest part of this argument was showing that the integral over the semi-circle converges to 0. Note that the following argument can be made for any rational function $f(x) = \frac{p_n(x)}{q_m(x)} = \frac{a_nx^n+\ldots+a_1x+a_0}{b_mx^m+\ldots+b_1x+b_0}$ over the contour with $z = Re^{it}\Rightarrow dz = Rie^{it}dt$.

Since the terms of the form e^{kti} where k is a positive integer have modulus 1, the modulus of the numerator $|p_n(z)|$ is smaller or equal to $(|a_n|R^n+\ldots+|a_1|R+|a_0|)\le|a_n|R^n$ and the modulus of the denominator $|q_m(z)|$ is larger than or equal to $(|b_m|R^m-\ldots-|b_1|R-|a_0|)$. Also note that the modulus of dz is equal to Rdt. Thus the integral over the semicircle of the modulus of $f(z)$ is less than or equal to to

$$\int_0^{\pi}\frac{|a_n|R^n}{(|b_m|R^m-\ldots-|b_1|R-|a_0|)}Rdt = \frac{|a_n|R^{n+1}}{(|b_m|R^m-\ldots-|b_1|R-|a_0|)}\int_0^{\pi}dt =$$

$$= \frac{|a_n|\pi R^{n+1}}{(|b_m|R^m-\ldots-|b_1|R-|a_0|)}\to 0 \text{ when } R\to\infty \text{ and } m>n+1.$$

So the integral of the modulus of $f(z)$ converges to 0 and hence the integral of f converges to 0 as well.

In similar considerations, it is also useful to note that the absolute value of a complex number of the form e^{it} is equal to 1.

$$|e^{it}| = 1$$

One way to see this is to note that this number lies on the unit circle. Another way is to note that

$$|e^{it}| = |\cos t + i \sin t| = \sqrt{\cos^2 t + \sin^2 t} = \sqrt{1} = 1.$$

Practice Problems.

1. Evaluate the complex integrals of given function $f(z)$ over the given contour.

   (a)
   $$f(z) = \frac{1}{(z-2)^2(4+z)} \qquad \text{over the circle of radius 3.}$$

   (b)
   $$f(z) = \frac{1}{(z-2)^2(4+z)} \qquad \text{over the circle of radius 5.}$$

   (c)
   $$f(z) = \frac{1}{(z^2-2z+2)^2} \qquad \text{over the contour as in Example 3 with } R > 2.$$

   (d)
   $$f(z) = \frac{z+4}{(z-3)^2(z^2-6z+10)} \qquad \text{over the circle of radius } \tfrac{1}{2} \text{ centered at (3,0).}$$

   (e)
   $$f(z) = \frac{z+4}{(z-3)^2(z^2-6z+10)} \qquad \text{over the circle of radius 2 centered at (3,0).}$$

   (f)
   $$f(z) = \frac{e^z}{(z-1)^5} \qquad \text{over the boundary of the right half of the disc with radius 2.}$$

   (g)
   $$f(z) = z \cos\frac{1}{z} \qquad \text{over the square with vertices (1,0), (0,1), (-1, 0) and (0,-1).}$$

2. **Extra credit level.** Evaluate the following integrals using the Residue Theorem and a suitable contour in complex plane.

   (a) $\int_{-\infty}^{\infty} \frac{1}{(x^2-2x+2)^2} dx$ Problem 1c above is related to this.

   (b) $\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1} dx$. You can assume that $\int_0^{\pi} e^{-R \sin t} dt \to 0$ when $R \to \infty$. Hint: consider the complex function $f(z) = \frac{z e^{iz}}{z^2+1}$ and note that the given function is equal to the imaginary part of $f(z)$ when $z = x$.

Solutions.

1. (a) Note that just one of the two singularities is in the given contour: 2 is in and -4 is not. So, the integral is the product of $2\pi i$ and the residue at 2 which can be computed as follows.

$$\lim_{z\to 2}\frac{d}{dz}\left((z-2)^2 f(z)\right)=\lim_{z\to 2}\frac{d}{dz}\left(\frac{1}{4+z}\right)=\lim_{z\to 2}\frac{-1}{(4+z)^2}=\frac{-1}{36}.$$

Thus, the integral is equal to $\frac{-\pi i}{18}$

(b) Since both singularities 2 and -4 are in the given contour, the integral is the product of $2\pi i$ and the sums of residues at 2 and at -4. The residue at 2 is $\frac{-1}{36}$ by the previous problem. The residue at 4 is $\lim_{z\to -4}(z+4)f(z)=\lim_{z\to -4}\frac{1}{(z-2)^2}=\frac{1}{36}$. Thus, the integral is equal to $2\pi i(\frac{-1}{36}+\frac{1}{36})=0$.

(c) The denominator z^2-2z+2 has zeros $z=\frac{2\pm\sqrt{4-8}}{2}=\frac{2\pm 2i}{2}=1\pm i$. Thus $f(z)=\frac{1}{(z-(i+1))^2(z-(1-i))^2}$. So, there are two poles, $1\pm i$, of the second order. Since $1-i$ is not in the contour, only the residue at $1+i$ is relevant and the integral is equal to the product of $2\pi i$ and the residue at $1+i$.

The residue at $1+i$ is $\lim_{z\to 1+i}\frac{d}{dz}\left((z-(1+i))^2 f(z)\right)$. The term $(z-(1+i))^2$ cancels with the same term in the denominator of $f(z)$ and so we have

$$\lim_{z\to 1+i}\frac{d}{dz}\left(\frac{1}{(z-(1-i))^2}\right)=\lim_{z\to 1+i}\frac{-2}{(z-1+i)^3}=\frac{-2}{(2i)^3}=\frac{-2}{8(-i)}=\frac{1}{4i}=\frac{-i}{4}.$$

Hence, the integral is $2\pi i\frac{-i}{4}=\frac{\pi}{2}$.

(d) $z^2-6z+10=0\Rightarrow z=\frac{6\pm\sqrt{36-40}}{2}=\frac{6\pm 2i}{2}=3\pm i$. Thus

$$f(z)=\frac{z+4}{(z-3)^2(z^2-6z+10)}=\frac{z+4}{(z-3)^2(z-3-i)(z-3+i)}$$

has three singularities: 3 is a pole of order 2 and $3\pm i$ are two poles of order 1. Just $z=3$ is inside the given circle, so the residue at 3 is the only relevant residue.

The residue at 3 can be computed as $\lim_{z\to 3}\frac{d}{dz}\left((z-3)^2 f(z)\right)=\lim_{z\to 3}\frac{d}{dz}\left(\frac{z+4}{z^2-6z+10}\right)=\lim_{z\to 3}\frac{1(z^2-6z+10)-(2z-6)(z+4)}{(z^2-6z+10)^2}=\frac{1(9-18+10)-0}{(9-18+10)^2}=1$. So, the integral is equal to $2\pi i$.

(e) All three singularities 3, $3+i$, and $3-i$ are inside the given circle.

The residue at $3+i$ is $\lim_{z\to 3+i}(z-3-i)\frac{z+4}{(z-3)^2(z-3-i)(z-3+i)}=\lim_{z\to 3+i}\frac{z+4}{(z-3)^2(z-3+i)}=\frac{3+i+4}{(3+i-3)^2(3+i-3+i)}=\frac{7+i}{-2i}=\frac{(7+i)i}{2}=\frac{-1}{2}+\frac{7}{2}i$.

The residue at $3-i$ is $\lim_{z\to 3-i}(z-3+i)\frac{z+4}{(z-3)^2(z-3-i)(z-3+i)}=\lim_{z\to 3-i}\frac{z+4}{(z-2)^2(z-3-i)}=\frac{3-i+4}{(3-i-3)^2(3-i-3-i)}=\frac{7-i}{2i}=\frac{(7-i)(-i)}{2}=\frac{-1}{2}-\frac{7}{2}i$.

The residue at 3 has been find it the previous problem to be 1. Thus, the integral is $2\pi i(1+\frac{-1}{2}+\frac{7}{2}i+\frac{-1}{2}-\frac{7}{2}i)=0$.

(f) The only singularity of $f(z)=\frac{e^z}{(z-1)^5}$ is 1 and it is a pole of order 5. The residue is $\frac{1}{4!}\lim_{z\to 1}\frac{d^4}{dz^4}\left((z-1)^5 f(z)\right)=\frac{1}{24}\lim_{z\to 1}\frac{d^4}{dz^4}\left(e^z\right)=\frac{1}{24}\lim_{z\to 1}e^z=\frac{e^1}{24}=\frac{e}{24}$. Thus, the integral is $2\pi i\frac{e}{24}=\frac{e\pi i}{12}$.

(g) The only singularity is 0 and it is an essential singularity. Note that the formula with the limit does not apply to this case since 0 is not a pole. We can find the residue by looking at the power series expansion $f(z) = z\cos\frac{1}{z} = z\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!z^{2n}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!z^{2n-1}} = z - \frac{1}{2!z} + \frac{1}{4!z^3} - \ldots$. So, the residue is $\frac{-1}{2}$. Thus, the integral is equal to $-\pi i$.

2. (a) Let us consider a contour C that is the same as in Example 3: the boundary of an upper semicircular region of a circle of radius R large enough to include the pole $1+i$. Note that the second pole will be out of this contour so it will not be relevant.

On the bottom part $z = x + 0i = x$ and $dz = dx$ so the integral over the bottom produces $\int_{-R}^{R}\frac{1}{(x^2-2x+2)^2}dx$. If we let $R\to\infty$ we will obtain the integral we need to evaluate.

On the top part $z = Re^{it}$ for $t : 0 \to \pi$ and $dz = Rie^{it}dt$. The integral over the top part is $\int_0^{\pi}\frac{Rie^{it}}{(R^2e^{2it}-2Re^{it}+2)^2}dt$. Using the reasoning as in discussion following Example 3, the absolute value of this integral is less or equal to $\frac{\pi R}{(R^2-2R-2)^2}$ that converges to 0 when $R\to\infty$. Thus, the integral over the top part converges to 0 as well.

Thus, $\int_{C_R}\frac{1}{(z^2-2z+2)^2}dz = \frac{\pi}{2}$. When we let $R\to\infty$, the integral on the left converges to the sum over the lower part that converges to $\int_{-\infty}^{\infty}\frac{1}{(z^2-2z+2)^2}dz$ and the integral of the top part that converges to 0. Thus, we obtain that this integral is equal to $\frac{\pi}{2}$.

(b) Following the hint, consider the complex function $f(z) = \frac{ze^{iz}}{z^2+1}$. When $z = x$, $f(z) = f(x) = \frac{xe^{ix}}{x^2+1} = \frac{x\cos x}{x^2+1} + i\frac{x\sin x}{x^2+1}$. Thus, the function under the integral is the imaginary part of $f(z)$ when z is real.

To evaluate the integral, consider the integral of $f(z)$ over the same contour C_R as in the previous problem and in Example 3. The zeros of the denominator $\pm i$ are the only singularities, they are both poles of the first order and just i is in the given contour. The residue at i is $\lim_{z\to i}(z-i)\frac{ze^{iz}}{(z-i)(z+i)} = \lim_{z\to i}\frac{ze^{iz}}{z+i} = \frac{ie^{-1}}{2i} = \frac{1}{2e}$.

On the bottom part of the contour, $z = x+0i = x$ and $dz = dx$ so the integral over the bottom produces $\int_{-R}^{R}\frac{xe^{ix}}{x^2+1}dx$. If we let $R\to\infty$, the imaginary part of this integral is the integral we need to evaluate.

On the top part of the contour $z = Re^{it}$ for $t : 0 \to \pi$ and $dz = Rie^{it}dt$. The integral over the top part is $\int_0^{\pi}\frac{R^2ie^{2it}e^{Re^{it}}}{R^2e^{2it}+1}dt$. Note that the absolute value of the numerator is $|R^2ie^{2it}e^{Rie^{it}}| = R^2|e^{Ri(\cos t+i\sin t)}| = R^2|e^{Ri\cos t}|e^{-R\sin t} = R^2e^{-R\sin t}$. Thus, the absolute value of this integral is less or equal to the following integral

$$\int_0^{\pi}\frac{R^2e^{-R\sin t}}{R^2-1}dt = \frac{R^2}{R^2-1}\int_0^{\pi}e^{-R\sin t}dt \to 0$$

for $R\to\infty$. Thus, the integral over the top part converges to 0 as well.

Thus, $\int_{C_R}\frac{ze^{iz}}{z^2+1}dz = 2\pi i\frac{1}{2e} = \frac{\pi i}{e}$. When we let $R\to\infty$, the integral on the left converges to $\int_{-\infty}^{\infty}\frac{xe^{ix}}{x^2+1}dx$. Thus,

$$\int_{-\infty}^{\infty}\frac{xe^{ix}}{x^2+1}dx = \int_{-\infty}^{\infty}\frac{x\cos x}{x^2+1}dx + i\int_{-\infty}^{\infty}\frac{x\sin x}{x^2+1}dx = \frac{\pi i}{e}.$$

Equating the real parts on both sides and the imaginary parts on both sides, we obtain that

$$\int_{-\infty}^{\infty} \frac{x \cos x}{x^2+1} dx = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1} dx = \frac{\pi}{e}.$$

Thus, the integral we needed to find is equal to $\frac{\pi}{e}$.