Math Methods I
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Calculus of Complex functions. Laurent Series and Residue Theorem

Review of complex numbers. A **complex number** is any expression of the form $x+iy$ where x and y are real numbers. x is called the **real** part and y is called the **imaginary** part of the complex number $x+iy$. The complex number $x-iy$ is said to be **complex conjugate** of the number $x+iy$.

Trigonometric Representations. Let us recall the polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. Using this representation, we have that

$$z = x + iy = r\cos\theta + ir\sin\theta.$$

The value r is the distance from the point (x, y) in the plane to the origin. The value r is called the **modulus** or absolute value of z. It is frequently denoted by $|z|$. The angle θ is the angle between the radius vector of (x, y) and the positive part of x-axis. The angle θ is usually called the **argument** or **phase** of z.

Euler's formula.

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This formula is especially useful in the solution of differential equations. Euler's formula was proved (in an obscured form) for the first time by Roger Cotes in 1714, then rediscovered and popularized by Euler in 1748.

Euler proved this formula using power series expansions of exponential, sine and cosine functions (and this proof can be subject of your project). This formula allows the following simplification

$$z = x + iy = r\cos\theta + ir\sin\theta = re^{\theta i}.$$

Using the trigonometric representation, the formulas for multiplication and division of two complex numbers become easier than when the Cartesian form of complex numbers is used. If z_1 is a complex number with modulus r_1 and phase θ_1 and z_2 is a number with modulus r_2 and phase θ_2, then

$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) = r_1 r_2 e^{i(\theta_1+\theta_2)}.$$

with the agreement that if $\theta_1 + \theta_2$ is larger than 2π, then 2π is subtracted from this sum.

This gives us an easy formula for the n-th power of a complex number $z = r(\cos(\theta) + i\sin(\theta))$,

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)) = r^n e^{n\theta i}.$$

Complex functions. A complex valued function of complex variable is a function $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ where u, v are real functions of two real variables x, y. For example $f(z) = z^2 = (x + iy)^2 = x^2 + 2xy - y^2$ is one such function. Its real part is $u = x^2 - y^2$ and its imaginary part is $v = 2xy$.

The Euler's formula can be used to define various complex functions that allow the same manipulations as their real-valued counterparts. For example, the exponential function e^z is defined as follows.

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$$

Thus, e^z represents complex function with real part $e^x \cos y$ and the imaginary part $e^x \sin y$.

The exponential function with arbitrary (real) base is defined via the exponential function as

$$a^z = e^{z \ln a}$$

The power function can also be defined via the exponential function. In this course we will work just with integer powers of z. So, let n be an integer.

$$z^n = (re^{i\theta})^n = r^n e^{in\theta}$$

Note that the usual formulas for exponentiation work with this definition.

The complex-valued basic trigonometric functions are also defined via exponential function as follows

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) \qquad \cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

The other trigonometric function can be defined via these two, for example $\tan z = \frac{\sin z}{\cos z}$.

Example 1. Verify the following identities.

(a) $(e^z)^n = e^{nz}$ for n positive integer (b) $\sin^2 z + \cos^2 z = 1$.

Solutions. (a) $(e^z)^n = (e^x e^{iy})^n$. Using the formula for z^n, we have that this is equal to $e^{nx}e^{iny} = e^{n(x+iy)} = e^{nz}$.

(b) $\sin^2 z + \cos^2 z = \frac{-1}{4}(e^{iz} - e^{-iz})^2 + \frac{1}{4}(e^{iz} + e^{-iz})^2$. Square the terms in parenthesis to get $\frac{-1}{4}(e^{2iz} - 2 + e^{-2iz}) + \frac{1}{4}(e^{2iz} + 2 + e^{-2iz}) = \frac{1}{4}(-e^{2iz} + 2 - e^{-2iz} + e^{2iz} + 2 + e^{-2iz}) = \frac{1}{4}(4) = 1$.

When it comes to the inverse of these elementary functions, things can become a bit more complicated. In this course we will not be going into branches of complex logarithms or the inverses of complex trigonometric functions, but we will be looking into n-values of complex n-th root. This interest is motivated by the need for finding all n solutions of the algebraic equations of the form $z^n = a$ where a is a given complex number $a = r(\cos(\theta) + i \sin(\theta))$. The n solutions of this equation are given by the formula

$$\sqrt[n]{r} e^{\frac{(\theta+2k\pi)i}{n}} = \sqrt[n]{r}(\cos \frac{\theta + 2k\pi}{n} + i \sin \frac{\theta + 2k\pi}{n})$$

for $k = 0, 1, \ldots n-1$.

Example 2. Find all solutions of the following equations.

$$(1) \quad z^3 = 8, \qquad\qquad (2) \quad z^3 = i$$

Solutions. (1) $z = \sqrt[3]{8e^{0i}} = 2e^{\frac{2k\pi i}{3}}$ for $k = 0, 1, 2 \Rightarrow z_0 = 2e^{0i} = 2$, $z_1 = 2e^{\frac{2\pi}{3}i} = 2(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}) = 2(\frac{-1}{2} + \frac{\sqrt{3}}{2}i) = -1 + \sqrt{3}i$, $z_2 = 2e^{\frac{4\pi}{3}i} = 2(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}) = 2(\frac{-1}{2} - \frac{\sqrt{3}}{2}i) = -1 - \sqrt{3}i$.

(2) $z = \sqrt[3]{1e^{\frac{\pi}{2}i}} = 1e^{\frac{\frac{\pi}{2}+2k\pi i}{3}}$ for $k = 0, 1, 2 \Rightarrow z_0 = e^{\frac{\pi}{6}i} = \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$, $z_1 = e^{\frac{5\pi}{6}i} = \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} = \frac{-\sqrt{3}}{2} + \frac{1}{2}i$, $z_2 = e^{\frac{9\pi}{6}i} = e^{\frac{3\pi}{2}i} = \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} = -i$.

Derivatives of Complex Functions

Consider $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ to be a complex valued function of complex variable. If $\frac{df}{dz}$ is a continuous function on the domain of f, then f is said to be differentiable. If f is differentiable at all points of its domain, we say that f is **analytic**. If f is analytic at all but the finitely many points of the domain, then these finitely many points at which f' may be discontinuous are called **singularities**.

If f is differentiable, all the usual rules of differentiation for real valued functions apply to the derivative $\frac{df}{dz}$. This seemingly obvious fact holds thanks to the convenient way how complex functions are defined. So, in practice, you can differentiate a complex function just like you would a real one.

Example 3. Find derivatives of the following functions.

$$(1) \quad f(z) = (z^2+5)^4 \qquad (2) \quad f(z) = e^{zi-2} \qquad (3) \quad f(z) = \sin(4z^3+5).$$

Solutions. (1) $f'(z) = 4(z^2+5)^3(2z) = 8z(z^2+5)^3$, (2) $f'(z) = ie^{zi-2}$, (3) $f'(z) = 12z^2\cos(4z^3+5)$.

The differentiation may be trickier if a function cannot be represented in terms of $z = x+iy$. For example $f(x+iy) = 3x+5yi$. For such function, the set of equations called **Cauchy-Riemann equations** are useful criterion of being analytic.

Namely, if f is an analytic function, then $\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$ so

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

These two equations are called **Cauchy-Riemann equations**. Thus, if Cauchy-Riemann equations fail for a complex function $f(z)$, then it is not analytic.

Conversely, it can be shown that if the partial derivatives of u and v are continuous and the Cauchy-Riemann equations hold, then f is an analytic function.

Example 4. Check if the following functions are analytic or not.

$$\text{(a) } f(x+iy) = 3y - 3xi, \qquad \text{(b) } f(x+iy) = 3y - 5xi.$$

Solution. (a) Here $u = 3y$ and $v = -3x$. The first Cauchy-Riemann equation holds since both $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ are equal to 0. The second equation holds also since $\frac{\partial u}{\partial y} = 3$ and $\frac{\partial v}{\partial x} = -3$ and so $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

(b) Here $u = 3y$ and $v = -5x$. The first Cauchy-Riemann equation holds since both $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ are equal to 0. However, the second equation fails since $\frac{\partial u}{\partial y} = 3$ and $\frac{\partial v}{\partial x} = -5$. So it fails since $3 = \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x} = 5$. Thus, this function is not analytic.

If f is an analytic function and u and v have continuous second partial derivatives, then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{-\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2} \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Similarly,

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{-\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{-\partial v}{\partial y}\right) = -\frac{\partial^2 v}{\partial y^2} \Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

The equations

$$u_{xx} + u_{yy} = 0 \quad \text{and} \quad v_{xx} + v_{yy} = 0$$

are called the **Laplace equations**. So, if f is analytic, both the real and the imaginary part satisfy the Laplace equations. This fact can be used in situations like in the next example.

Example 5. Check if analytic functions with real part equal to the given functions u exist. If so, find all analytic functions that have real parts equal to u.

(a) $u = xe^{3y}$ (b) $u = x^2 + 3x - y^2 + 5y$.

Solution. (a) To see if such an analytic function exists, check if it satisfies Laplace equation. $\frac{\partial u}{\partial x} = e^{3y} \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0$ and $\frac{\partial u}{\partial y} = 3xe^{3y} \Rightarrow \frac{\partial^2 u}{\partial y^2} = 9xe^{3y}$. Since $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 - 9xe^{3y} \neq 0$, no such analytic function exists.

(b) Note that the given u satisfies Laplace equation: $\frac{\partial u}{\partial x} = 2x+3 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 2$ and $\frac{\partial u}{\partial y} = -2y+5 \Rightarrow \frac{\partial^2 u}{\partial y^2} = -2$. Since $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$.

Since $2x+3 = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ you can find v as $v = \int(2x+3)dy = 2xy+3y+g(x)$. Determine the function g from the second Cauchy-Riemann equation $2y - 5 = \frac{-\partial u}{\partial y} = \frac{\partial v}{\partial x} = 2y + g'(x) \Rightarrow g'(x) = -5 \Rightarrow g(x) = -5x+c$. Thus, $v = 2xy+3y-5x+c$. In this case $f(z) = x^2+3x-y^2+5y+i(2xy+3y-5x+c)$. This function turns out to be equal to $x^2 + 2ixy + y^2 + 3(x+iy) + 5(-ix+y) + ic = (x+iy)^2 + 3(x+iy) - 5i(x+iy) + ic = z^2 + 3z - 5iz + ic$.

Integrals of Complex Functions

Let C be a curve in the xy-plane. In order evaluate the integral $\int_C f(z)dz$ of the complex function $f(z)$ over C, parametrize the curve C as $x = x(t)$ and $y = y(t)$ and obtain the bounds for t, $a \leq t \leq b$.

Then represent z as $x(t) + y(t)i$ and note that then $dz = dx + dyi = x'(t)dt + y'(t)dti = z'(t)dt$. Thus,

$$\int_L f(z)dz = \int_a^b f(z(t))z'(t)dt$$

Example 6. Evaluate the following integrals.

1. $\int z^2 dz$ over quarter of the unit circle in the first quadrant traversed counterclockwise.
2. $\int$ Re(z) dz over the line segment $x = 1$ from (1,0) to (1,1) and the line segment $y = 1$ from (1,1) to (0,1). Recall that Re(z) denotes the real part of $z = x + iy$. So, Re$(z) = x$.

Solutions.

1. We can parametrize the unit circle as $x = \cos t$, $y = \sin t$. In the first quadrant with positive orientation the t-values are such that $0 \leq t \leq \frac{\pi}{2}$. With this parametrization

   $\int z^2 dz = \int_0^{\pi/2} e^{2it} e^{it} i dt = i \int_0^{\pi/2} e^{3it} dt = \frac{i}{3i} e^{3it}|_0^{\pi/2} = \frac{1}{3}(e^{\frac{3\pi}{2}i} - e^{0i}) = \frac{1}{3}(\cos\frac{3\pi}{2} + i \sin\frac{3\pi}{2} - 1) = \frac{1}{3}(-i-1) = \frac{-1}{3}(1+i)$.

2. The first line segment has the parameterization $x = 1, y = y$ and the bounds for y are $0 \leq y \leq 1$. In this case, z is $z = 1 + iy$ so Rez=1 and $dz = 0dx + idy = idy$. The integral over this first part is $\int_0^1 idy = iy|_0^1 = i$.

   The second line segment has the parameterization $x = x$, $y = 1$ and the x-values are decreasing from 1 to 0. Then $z = x + i$ and Re $z = x$, $dz = dx + 0i = dx$. The integral over this part is $\int_1^0 xdx = \frac{x^2}{2}|_1^0 = -\frac{1}{2}$. So, the final answer is $i - \frac{1}{2}$.

Cauchy's Theorem. Let C be a closed piecewise smooth curve and f an analytic function defined on an simply connected domain that contains the interior of the curve C. In this course, we will not need to go into precise meaning of "simply connected". Intuitively, the domain of f is simply connected if it consists of one piece without any holes. For example, a circle is simply connected while an annulus is not. To find more about this concept, you can explore wikipedia.org.

With those assumptions,

$$\oint_C f(z)dz = 0.$$

This statement is known as **Cauchy's Theorem.**

To prove this statement, recall Green's theorem from Calculus 3. If C is a positive oriented, smooth closed curve and P and Q have continuous derivatives, then the line integral $\oint_C Pdx + Qdy$ can be reduced to a double integral over the interior D of C.

$$\oint_C Pdx + Qdy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Thus, if $f(z) = u + iv$, and if D denotes the interior of C, then

$$\oint_C f(z)dz = \oint_C (u + iv)(dx + idy) = \oint_C (udx - vdy) + i\oint_C (vdx + udy)$$

Using Green's theorem, we obtain that this last expression is equal to the following.

$$\int\int_D (\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y})\ dxdy + i\ \int\int_D (\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})\ dxdy = 0.$$

The last equality uses the fact that f is analytic so that the Cauchy-Riemann equations hold.

Example 7. Evaluate

$$\int (z^3 + 5z - \sin z)dz$$

over a unit circle in xy-plane.

Solutions. A unit circle is a closed curve and $f(z) = z^3 + 5z - \sin z$ is an analytic function (derivative $3z^2 + 5 - \cos z$ is continuous) defined for every z. Thus, this curve and function f satisfy the assumptions of Cauchy's Theorem. Thus, the integral is equal to zero.

Cauchy's Integral Formula. If f is an analytic function defined on a simple connected region that contains the interior of the closed piecewise smooth curve C, (note that these are the same assumptions as in Cauchy's Theorem) and a is a point in the interior of C, then

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\ dz$$

where $n = 0, 1, \ldots$. This formula is known as **Cauchy's differentiation formula**. Its proof is one of the project topics you can choose from. Note that for $n = 0$ this gives the formula known as the **Cauchy's integral formula**.

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz \quad \Rightarrow \quad \oint_C \frac{f(z)}{z-a}\, dz = 2\pi i f(a)$$

This formula is frequently used when evaluation integrals of functions that may not be analytic.

Example 8. Evaluate

$$\oint_C \frac{1}{z} dz$$

over the square C with sides (1,0), (0,1), (-1,0), (0,-1).

Solution. Use the Cauchy's integral formula for $f(z) = 1$ and $a = 0$. The formula gives you $2\pi i f(0) = 2\pi i$. Note that evaluating this integral directly would be much more time consuming since you would have to evaluate four integrals, each using different parametrization of a side of the square.

Laurent's Series

Recall that a real function $f(x)$ that with continuous derivatives of any order near a point $x = a$ has the power series expansion

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \text{ where } a_n = \frac{f^{(n)}(a)}{n!}$$

Assume now that $f(z)$ is analytic in a domain that contains a point a. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{where} \quad a_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz$$

This last formula follows by the Cauchy's differentiation formula. The series converges inside of a circle centered at a of radius equal to the distance from a to the nearest singularity of $f(z)$.

Recall the following power series expansions of real functions.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Because of the way how the corresponding complex functions are defined (and using a theorem usually referred to as Uniqueness Theorem) it can be shown that the same expansions hold for complex functions as well. Thus

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \qquad \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

These elementary function expansions can be used to obtain the expansions of functions obtained combining the basic functions as the next example illustrates.

Example 9. Find the power series expansions of the following functions at 0 and determine the radius of convergence.

(a) $f(z) = e^{iz}$, (b) $f(z) = \frac{1}{1-z^2}$.

Solutions. (a) $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \Rightarrow f(z) = e^{iz} = \sum_{n=0}^{\infty} \frac{i^n z^n}{n!}$. This function has no singularities so this series converges for all values of z (i.e. the radius of convergence is infinite).

(b) $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \Rightarrow \frac{1}{1-z^2} = \sum_{n=0}^{\infty} z^{2n}$. 1 and -1 are singularities of this function, thus this series converges within the unit circle, that is, the radius of convergence is 1.

Now assume that the point a may potentially be an isolated singularity of $f(z)$. In this case, the function may be written as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \quad \text{where} \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz$$

This last formula now holds for negative values of n as well. We distinguish three cases:

1. All the coefficients a_{-n} with negative subscripts are zero. In this case a is a said to be a **removable singularity**.

   For example, consider $f(z) = \frac{\sin z}{z}$. Since $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ we have that the Laurent series expansion of $f(z)$ is $\frac{\sin z}{z} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots$ Note that there are no terms with negative powers of z. Thus 0 is a removable singularity of this function.

2. Just finitely many coefficients with negative subscripts are not zero. In this case a is said to be a **pole**. If $a_{-k} \neq 0$ and $a_{-k-1} = a_{-k-2} = \ldots = 0$, then a is a **pole of order** k. Note that a is a pole of order k if $(z-a)^k f(z)$ is an analytic function without singularity or with removable singularity at a. $f(z) = \frac{1}{z}$ has a pole of order 1 at $z = 0$. Similarly, $f(z) = \frac{1}{z^n}$ has a pole of order n at $z = 0$.

3. Infinitely many coefficients with negative subscripts are zero. In this case a is said to be a **essential singularity**. For example, the function $e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n! z^n}$ is an example with an essential singularity at $z = 0$.

The coefficient a_{-1} with $(z-a)^{-1}$ has a special significance and is called the **residue** of $f(z)$.

Practice Problems.

1. Find all solutions of the equation $z^8 = 1$.

2. Find the derivative of the following functions (a) $f(z) = ze^{5iz^2}$, (b) $f(z) = \frac{1}{(z^2+4)^3}$.

3. Determine if the following functions are analytic.

   (a) $f(z) = ze^{5iz^2}$, (b) $f(x+iy) = x^2 y + ixy^2$, (c) $f(x+iy) = x^2 - y^2 + 2xyi$.

4. Find an analytic function $f(z)$, if such function exists, so that the imaginary part of $f(z)$ is equal to

   (a) $v = 3x^2 + 4y - 3y^2$, (b) $v = e^{xy}$ (c) $v = 3e^x \cos y$.

5. Evaluate $\int z^4 dz$

(a) over the upper-half of the unit circle traversed counterclockwise;
(b) over the unit circle traversed counterclockwise.

6. Let $f(z) = z^3 - 2z + e^{z-2}$ and let C be the circle of radius 3 in xy-plane. Evaluate

$$\text{(a)} \int_C f(z)\,dz \qquad\qquad \text{(b)} \int_C \frac{f(z)}{z-2}\,dz$$

7. Find the power series expansions of the following functions at the given z-value. If the given z-value is an isolated singularity, determine the type of singularity and find the residue.

(a) $f(z) = \frac{z}{1-z^2}$, $z = 0$, (b) $f(z) = \frac{e^{z-1}}{(z-1)^2}$ $z = 1$, (c) $f(z) = \frac{1-\cos z}{z^2}$ $z = 0$.

8. Classify all the singularities and find their residues for the following functions.

(a) $f(z) = \frac{1}{(z-2)^2(4+z)}$ (b) $f(z) = \frac{e^z}{(z-1)^5}$, (c) $f(z) = z\cos\frac{1}{z}$.

Solutions.

1. $z^8 = 1 \Rightarrow z = \sqrt[8]{1e^{0i}} = 1e^{\frac{2k\pi i}{8}}$ for $k = 0, 1, \ldots, 7 \Rightarrow z = \pm 1, \pm i, \frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i, -\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i.$

2. (a) $\frac{d}{dz}\left(ze^{5iz^2}\right) = e^{5iz^2} + 10iz^2e^{5iz^2}$. (b) $\frac{d}{dz}\left((z^2+4)^{-3}\right) = -3(z^2+4)^{-4}(2z) = \frac{-6z}{(z^2+4)^4}$.

3. (a) By the previous problem, the function $f(z) = ze^{5iz^2}$ has derivative $f'(z) = e^{5iz^2} + 10iz^2e^{5iz^2}$ which is a continuous function. Thus, f is analytic.

(b) For $f(x+iy) = x^2y + ixy^2$, $u = x^2y$ and $v = xy^2$. Check if the Cauchy-Riemann equations hold. $\frac{\partial u}{\partial x} = 2xy = \frac{\partial v}{\partial y}$ so the first equation holds. $\frac{\partial u}{\partial y} = x^2$ and $\frac{\partial v}{\partial x} = y^2$ so the second equation fails. Thus, f is not analytic.

(c) For $f(x+iy) = x^2 - y^2 + 2xyi$, $u = x^2 + y^2$ and $v = 2xy$. Check if the Cauchy-Riemann equations hold. $\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -2y = \frac{\partial v}{\partial x}$ so both equations hold. Thus, f is analytic.

4. (a) Check first if Laplace equation is satisfied. $v = 3x^2 + 4y - 3y^2 \Rightarrow \frac{\partial v}{\partial x} = 6x \Rightarrow \frac{\partial^2 v}{\partial x^2} = 6$ and $\frac{\partial v}{\partial y} = 4 - 6y \Rightarrow \frac{\partial^2 v}{\partial y^2} = -6$. Thus, $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 6 - 6 = 0$.

Then find f using Cauchy-Riemann equations. $\frac{-\partial v}{\partial x} = -6x = \frac{\partial u}{\partial y} \Rightarrow u = \int -6xdy = -6xy + g(x)$. Using the second equation, we have that $4 - 6y = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = -6y + g'(x) \Rightarrow g'(x) = 4 \Rightarrow g(x) = 4x + c$. Thus $u = -6xy + 4x + c$. Hence $f(z) = -6xy + 4x + c + i(3x^2 + 4y - 3y^2) = 3(x-iy)^2 + 4(x+iy) + c = 3\overline{z}^2 + 4z + c$.

(b) Check Laplace equation: $v = e^{xy} \Rightarrow \frac{\partial v}{\partial x} = ye^{xy} \Rightarrow \frac{\partial^2 v}{\partial x^2} = y^2e^{xy}$ and $\frac{\partial v}{\partial y} = xe^{xy} \Rightarrow \frac{\partial^2 v}{\partial y^2} = x^2exy$. Thus $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = (y^2 + x^2)e^{xy} \neq 0$. So, no analytic function f with the imaginary part equal to v exists.

(c) Check first that Laplace equation is satisfied. Then use the Cauchy-Riemann equations. $v = 3e^x \cos y \Rightarrow \frac{-\partial v}{\partial x} = -3e^x \cos y = \frac{\partial u}{\partial y} \Rightarrow u = \int -3e^x \cos ydy = -3e^x \sin y + g(x)$. Using the second equation, we have that $-3e^x \sin y = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = -3e^x \sin y + g'(x) \Rightarrow g'(x) = 0 \Rightarrow g(x) = c$. Thus $u = -3e^x \sin y + c$. Hence $f(z) = -3e^x \sin y + c + 3ie^x \cos y = 3ie^x(\cos y + i \sin y) + c = 3ie^x e^{iy} + c = 3ie^z + c$.

5. (a) Parametrization of the unit circle is $x = \cos t$, $y = \sin t$, so $z = \cos t + i \sin t = e^{it}$. Thus, $z^4 = e^{4it}$ and $dz = e^{it} i dt$. The bounds for t are $0 \le t \le \pi$. The integral is $\int z^4 dz = \int_0^\pi e^{4it} e^{it} i dt = i \int_0^\pi e^{5it} dt = \frac{i}{5i} e^{5it}|_0^\pi = \frac{1}{5}(e^{5\pi i} - e^{0i}) = \frac{1}{5}(\cos 5\pi + i \sin 5\pi - 1) = \frac{1}{5}(-2) = \frac{-2}{5}$.
   (b) The function $f(z) = z^4$ is analytic because it has derivative $4z^3$ which is a continuous function. Thus, the integral is zero since the integral of an analytic function over a close curve is zero (Cauchy's Theorem).

6. $f(z) = z^3 - 2z + e^{z-2}$ is analytic (derivative $3z^2 - 2 + e^{z-2}$ is continuous) so the integral in part (a) is zero by Cauchy's Theorem. By Cauchy's integral formula, the integral in part (b) is equal to $2\pi i f(2) = 2\pi i(8 - 4 + 1) = 10\pi i$. Alternatively, you can take $g(z) = \frac{f(z)}{z-2}$ and note that g has one singularity, 2, it is a pole of the first order, and it is in the interior of C. The residue of g at 2 is $\lim_{z\to 2}(z-2)\frac{f(z)}{z-2} = \lim_{z\to 2} f(z) = f(2) = 5$. Hence $\int_C g(z)dz = 2\pi i(5) = 10\pi i$.

7. (a) $\frac{1}{1-z} = \sum_{n=0}^\infty z^n \Rightarrow \frac{1}{1-z^2} = \sum_{n=0}^\infty z^{2n} \Rightarrow f(z) = \frac{z}{1-z^2} = \sum_{n=0}^\infty z^{2n+1}$. The only singularities of $f(z)$ are ± 1. So, the radius of the convergence is the distance from 0 to any of these two points and so it is 1. The function is analytic at $z = 0$ (also note that there are no terms with negative exponents in the series expansion).
   (b) $e^z = \sum_{n=0}^\infty \frac{z^n}{n!} \Rightarrow e^{z-1} = \sum_{n=0}^\infty \frac{(z-1)^n}{n!} \Rightarrow$
   $$f(z) = \frac{e^{z-1}}{(z-1)^2} = \frac{1}{(z-1)^2}\sum_{n=0}^\infty \frac{(z-1)^n}{n!} = \sum_{n=0}^\infty \frac{(z-1)^{n-2}}{n!} = \frac{1}{(z-1)^2} + \frac{1}{z-1} + \frac{1}{2!} + \frac{z-1}{3!} + \ldots.$$
   The only singularity is $z = 1$ and, from the power series expansion, we can see that it is a pole of the order 2. The coefficient with the term with $\frac{1}{z-1}$ is 1 so the residue is 1. The series converges at all points except $z = 1$.
   (c) $f(z) = \frac{1-\cos z}{z^2} = \frac{1}{z^2} - \frac{1}{z^2}\cos z = \frac{1}{z^2} - \frac{1}{z^2}\sum_{n=0}^\infty \frac{(-1)^n z^{2n}}{(2n)!} = \frac{1}{z^2} - \sum_{n=0}^\infty \frac{(-1)^n z^{2n-2}}{(2n)!} = \frac{1}{z^2} - \frac{1}{z^2} + \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \ldots = \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \ldots$ Thus, there are no terms with negative exponents. So, the only singularity $z = 0$ is a removable singularity.

8. (a) $f(z) = \frac{1}{(z-2)^2(4+z)}$ has two singularities: 2 is a pole of order 2 and -4 is a pole of order 1. The residue at 2 can be computed as
   $$\lim_{z\to 2}\frac{d}{dz}\left((z-2)^2 f(z)\right) = \lim_{z\to 2}\frac{d}{dz}\left(\frac{1}{4+z}\right) = \lim_{z\to 2}\frac{-1}{(4+z)^2} = \frac{-1}{36}.$$
   The residue at -4 is $\lim_{z\to -4}(z+4)f(z) = \lim_{z\to -4}\frac{1}{(z-2)^2} = \frac{1}{36}$.
   (b) $f(z) = \frac{e^z}{(z-1)^5}$. The only singularity is 1 and it is a pole of order 5. The residue at 1 is $\frac{1}{4!}\lim_{z\to 1}\frac{d^4}{dz^4}\left((z-1)^5 f(z)\right) = \frac{1}{24}\lim_{z\to 1}\frac{d^4}{dz^4}\left(e^z\right) = \frac{1}{24}\lim_{z\to 1} e^z = \frac{e^1}{24} = \frac{e}{24}$.
   (c) The only singularity is 0 and it is an essential singularity. Note that the formula with the limit does not apply to this case since 0 is not a pole. We can find the residue by looking at the power series expansion $f(z) = z\cos\frac{1}{z} = z\sum_{n=0}^\infty \frac{(-1)^n}{(2n)! z^{2n}} = \sum_{n=0}^\infty \frac{(-1)^n}{(2n)! z^{2n-1}} = z - \frac{1}{2! z} + \frac{1}{4! z^3} - \ldots$. So, the residue is $\frac{-1}{2}$.

The Residue Theorem

If $f(z)$ is a complex function which is analytic everywhere except possibly at $z = a$, the coefficient a_{-1} in the Laurent series expansion of $f(z)$ can be computed as

$$a_{-1} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{-1+1}}\, dz = \frac{1}{2\pi i} \oint_C f(z)\, dz.$$

Thus,

$$\oint_C f(z)\, dz = 2\pi i a_{-1} = 2\pi i(\text{ coefficient of the term with} \frac{1}{z-a}\ ).$$

Note that if f is analytic at $z = a$, then the residue $a_{-1} = 0$ since the Laurent series has no terms with negative powers. In this case, the last formula boils down to Cauchy's Theorem $\oint_C f(z)\, dz = 0$.

Assume now that f is an analytic function with isolated singularities $z_1, z_2, \ldots z_n$ and that C is a closed, piecewise smooth, positive oriented curve whose interior contains the singularities $z_1, z_2, \ldots z_n$. If $R_1, R_2, \ldots R_n$ are the residues at $z_1, z_2, \ldots z_n$, then

$$\oint_C f(z)\, dz = 2\pi i(R_1 + R_2 + \ldots + R_n).$$

This statement is known as **Residue Theorem**.

Since residues of a function are important for evaluating integrals, the following formula may be helpful when computing the residues of poles. If a is a **pole of order** k, and a_{-1} denotes the residue at a, then

$$a_{-1} = \frac{1}{(k-1)!} \lim_{z\to a} \frac{d^{k-1}}{dz^{k-1}} \left((z-a)^k f(z)\right).$$

Thus, if a is a **pole of the first order**, then

$$a_{-1} = \lim_{z\to a}(z-a)f(z).$$

Example 1. Let $f(z) = \frac{1}{z^2(z^2+1)}$.
(a) Find the residues of all isolated singularities of f.
(b) Evaluate $\oint_C f(z)dz$ where C is the circle of radius 2 centered at 0.

Solution. (a) Note that $z^2 + 1 = (z-i)(z+i)$. Thus, f has three singularities $0, i$ and $-i$. 0 is a pole of order 2 and $\pm i$ are poles of the first order.
The residue at 0 can be computed as $\lim_{z\to 0} \frac{d}{dz}\left(z^2 f(z)\right) = \lim_{z\to 0} \frac{d}{dz}\left(\frac{1}{z^2+1}\right) = \lim_{z\to 0} \frac{-2z}{(z^2+1)^2} = 0$.
The residue at i can be computed as $\lim_{z\to i}(z-i)f(z) = \lim_{z\to i} \frac{1}{z^2(z+i)} = \frac{1}{-1(i+i)} = \frac{-1}{2i} = \frac{i}{2}$.
The residue at $-i$ is $\lim_{z\to -i}(z+i)f(z) = \lim_{z\to i} \frac{1}{z^2(z-i)} = \frac{1}{-1(-i-i)} = \frac{1}{2i} = \frac{-i}{2}$.
(b) Using the Residue Theorem, the integral is equal to $2\pi i(0 + \frac{i}{2} - \frac{i}{2}) = 0$.

The following example illustrate how some real value integrals can be evaluated using Residue Theorem.

Example 2. Evaluate

$$\oint_C \frac{e^z}{z^2-4} dz$$

over the circle C of radius 5 using Residue Theorem.

Solution. The function $f(z) = \frac{e^z}{z^2-4} = \frac{e^z}{(z-2)(z+2)}$ has two singularities 2 and -2 and each is a pole of the first order. Both are contained in the interior of C. Thus, the integral is equal to the product of $2\pi i$ and the sum of the two residues.

The residue at 2 is $\lim_{z\to 2}(z-2)\frac{e^z}{(z-2)(z+2)} = \lim_{z\to 2}\frac{e^z}{(z+2)} = \frac{e^2}{4}$.

The residue at -2 is $\lim_{z\to -2}(z+2)\frac{e^z}{(z-2)(z+2)} = \lim_{z\to -2}\frac{e^z}{(z-2)} = \frac{e^{-2}}{-4}$.

Thus,

$$\oint_C \frac{e^z}{(z-2)(z+2)}dz = 2\pi i(\frac{e^2}{4} + \frac{e^{-2}}{-4}) = \frac{\pi i}{2}(e^2 - e^{-2}).$$

Example 3. Let a be any real number. Note that the integral

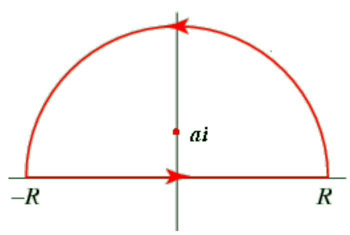
$$\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2}dx$$

would be difficult to evaluate directly. Evaluate using complex integration and Residue Theorem. Use your result to evaluate

$$\int_0^{\infty} \frac{1}{(x^2+a^2)^2}dx$$

Solution. Consider the function $f(z) = \frac{1}{(z^2+a^2)^2} = \frac{1}{(z-ai)^2(z+ai)^2}$. It has two poles of the second order $\pm ai$.

Let us now consider a contour C_R that is the boundary of an upper semicircular region of a circle of radius R large enough to include the pole ai. On the bottom part, we have a line segment parametrized as $z = x + 0i = x$ and $dz = dx$ so the integral over the bottom is $\int_{-R}^{R}\frac{1}{(x^2+a^2)^2}dx$. If we let $R \to \infty$ we will obtain the integral we need to evaluate.



On the semi-circular part $z = Re^{it}$ for $t : 0 \to \pi$ and $dz = Rie^{it}dt$. The integral over this part is $\int_0^{\pi}\frac{Rie^{it}}{(R^2e^{2it}+a^2)^2}dt$. We shall show that the integral of the modulus of f converges to 0 for $R \to \infty$. Then the integral of f converges to 0 as well.

Consider the modulus of this function is $\frac{R}{|R^2e^{2it}+a^2|^2}$. The denominator $|R^2e^{2it}+a^2|^2$ is larger than $|R^2e^{2it}|^2 = R^4$. So, the whole function is smaller than $\frac{1}{R^3}$. So, when $R \to \infty$, this function converges to 0. Thus, the integral is equal to 0.

On the other hand, the integral $\int_{C_R}\frac{1}{(z^2+a^2)^2}dz$ is equal to the product of $2\pi i$ and the residue at ai. Note that we do not need to consider the residue at $-ai$ since it is not in the contour C_R. The residue at ai is $\lim_{z\to ai}\frac{d}{dz}\left((z-ai)^2 f(z)\right) = \lim_{z\to ai}\frac{d}{dz}\left(\frac{1}{(z+ai)^2}\right) = \lim_{z\to ai}\frac{-2}{(z+ai)^3} = \frac{-2}{(2ai)^3} = \frac{-1}{4a(-i)} = \frac{1}{4ai} = \frac{-i}{4a}$.

Thus, $\int_{C_R}\frac{1}{(z^2+a^2)^2}dz = 2\pi i\frac{-i}{4a} = \frac{\pi}{4a}$. When we let $R \to \infty$, the integral on the left converges to the sum over the lower part that converges to $\int_{-\infty}^{\infty}\frac{1}{(x^2+a^2)^2}dx$ and the integral of the top part that converges to 0. Thus, we obtain that this integral is equal to $\frac{\pi}{4a}$.

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2} dx = \frac{\pi}{4a}$$

We can use $\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2} dx$ to evaluate $\int_{0}^{\infty} \frac{1}{(x^2+a^2)^2} dx$ as well. Note that the function under the integral is even, thus $\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2} dx = \int_{-\infty}^{0} \frac{1}{(x^2+a^2)^2} dx + \int_{0}^{\infty} \frac{1}{(x^2+a^2)^2} dx = 2\int_{0}^{\infty} \frac{1}{(x^2+a^2)^2} dx$. Thus,

$$\int_{0}^{\infty} \frac{1}{(x^2+a^2)^2} dx = \frac{\pi}{8a}.$$

Probably the trickiest part of this argument was showing that the integral over the semi-circle converges to 0. Note that the following argument can be made for any rational function $f(x) = \frac{p_n(x)}{q_m(x)} = \frac{a_n x^n + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0}$ over the contour with $z = Re^{it} \Rightarrow dz = Rie^{it} dt$.

Since the terms of the form e^{kti} where k is a positive integer have modulus 1, the modulus of the numerator $|p_n(z)|$ is smaller or equal to $(|a_n|R^n + \ldots + |a_1|R + |a_0|) \leq |a_n|R^n$ and the modulus of the denominator $|q_m(z)|$ is larger than or equal to $(|b_m|R^m - \ldots - |b_1|R - |a_0|)$. Also note that the modulus of dz is equal to Rdt. Thus the integral over the semicircle of the modulus of $f(z)$ is less than or equal to to

$$\int_0^{\pi} \frac{|a_n|R^n}{(|b_m|R^m - \ldots - |b_1|R - |a_0|)} Rdt = \frac{|a_n|R^{n+1}}{(|b_m|R^m - \ldots - |b_1|R - |a_0|)} \int_0^{\pi} dt =$$

$$= \frac{|a_n|\pi R^{n+1}}{(|b_m|R^m - \ldots - |b_1|R - |a_0|)} \to 0 \text{ when } R \to \infty \text{ and } m > n+1.$$

So the integral of the modulus of $f(z)$ converges to 0 and hence the integral of f converges to 0 as well.

In similar considerations, it is also useful to note that the absolute value of a complex number of the form e^{it} is equal to 1.

$$|e^{it}| = 1$$

One way to see this is to note that this number lies on the unit circle. Another way is to note that

$$|e^{it}| = |\cos t + i \sin t| = \sqrt{\cos^2 t + \sin^2 t} = \sqrt{1} = 1.$$

Practice Problems.

1. Evaluate the complex integrals of given function $f(z)$ over the given contour.

   (a)
   $$f(z) = \frac{1}{(z-2)^2(4+z)} \qquad \text{over the circle of radius 3.}$$

   (b)
   $$f(z) = \frac{1}{(z-2)^2(4+z)} \qquad \text{over the circle of radius 5.}$$

(c)

$$f(z) = \frac{e^z}{(z-1)^5} \quad \text{over the boundary of the right half of the disc with radius 2.}$$

(d)

$$f(z) = z\cos\frac{1}{z} \quad \text{over the square with vertices (1,0), (0,1), (-1, 0) and (0,-1).}$$

2. Evaluate the following integrals using the Residue Theorem and a suitable contour in complex plane.

   (a) $\int_{-\infty}^{\infty} \frac{1}{(x^2-2x+2)^2} dx$

   (b) $\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+1} dx$. You can assume that $\int_0^\pi e^{-R\sin t} dt \to 0$ when $R \to \infty$. Hint: consider the complex function $f(z) = \frac{ze^{iz}}{z^2+1}$ and note that the given function is equal to the imaginary part of $f(z)$ when $z = x$.

Solutions.

1. (a) Note that just one of the two singularities is in the given contour: 2 is in and -4 is not. So, the integral is the product of $2\pi i$ and the residue at 2 which can be computed as follows.

$$\lim_{z\to 2} \frac{d}{dz}\left((z-2)^2 f(z)\right) = \lim_{z\to 2} \frac{d}{dz}\left(\frac{1}{4+z}\right) = \lim_{z\to 2} \frac{-1}{(4+z)^2} = \frac{-1}{36}.$$

   Thus, the integral is equal to $\frac{-\pi i}{18}$

   (b) Since both singularities 2 and -4 are in the given contour, the integral is the product of $2\pi i$ and the sums of residues at 2 and at -4. The residue at 2 is $\frac{-1}{36}$ by the previous problem. The residue at 4 is $\lim_{z\to -4}(z+4)f(z) = \lim_{z\to -4} \frac{1}{(z-2)^2} = \frac{1}{36}$. Thus, the integral is equal to $2\pi i(\frac{-1}{36} + \frac{1}{36}) = 0$.

   (c) The only singularity of $f(z) = \frac{e^z}{(z-1)^5}$ is 1 and it is a pole of order 5. The residue is $\frac{1}{4!}\lim_{z\to 1} \frac{d^4}{dz^4}\left((z-1)^5 f(z)\right) = \frac{1}{24}\lim_{z\to 1} \frac{d^4}{dz^4}\left(e^z\right) = \frac{1}{24}\lim_{z\to 1} e^z = \frac{e^1}{24} = \frac{e}{24}$. Thus, the integral is $2\pi i \frac{e}{24} = \frac{e\pi i}{12}$.

   (d) The only singularity is 0 and it is an essential singularity. Note that the formula with the limit does not apply to this case since 0 is not a pole. We can find the residue by looking at the power series expansion $f(z) = z\cos\frac{1}{z} = z\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!z^{2n}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!z^{2n-1}} = z - \frac{1}{2!z} + \frac{1}{4!z^3} - \ldots$. So, the residue is $\frac{-1}{2}$. Thus, the integral is equal to $-\pi i$.

2. (a) Consider the function $f(z) = \frac{1}{(z^2-2z+2)^2}$. The zeros of the denominator are $z = \frac{2\pm\sqrt{4-8}}{2} = \frac{2\pm 2i}{2} = 1 \pm i$. Thus $f(z) = \frac{1}{(z-(i+1))^2(z-(1-i))^2}$. So, there are two poles the second order $1 \pm i$.

   Let us now consider a contour C_R that is the same as in Example 12: the boundary of an upper semicircular region of a circle of radius R large enough to include the pole $1+i$. Note that the second pole will be out of this contour so it will not be relevant.

   On the bottom part $z = x + 0i = x$ and $dz = dx$ so the integral over the bottom produces $\int_{-R}^{R} \frac{1}{(x^2-2x+2)^2} dx$. If we let $R \to \infty$ we will obtain the integral we need to evaluate.

On the top part $z = Re^{it}$ for $t : 0 \to \pi$ and $dz = Rie^{it}dt$. The integral over the top part is $\int_0^\pi \frac{Rie^{it}}{(R^2e^{2it}-2Re^{it}+2)^2}dt$. Using the reasoning as in discussion following Example 12, the absolute value of this integral is less or equal to $\frac{\pi R}{(R^2-2R-2)^2}$ that converges to 0 when $R \to \infty$. Thus, the integral over the top part converges to 0 as well.

On the other hand, the integral $\int_{C_R} \frac{1}{(z^2-2z+2)^2}dz$ is equal to the product of $2\pi i$ and the residue at $1+i$. The residue at $1+i$ is $\lim_{z\to 1+i} \frac{d}{dz}\left((z-(1+i))^2 f(z)\right)$. The term $(z-(1+i))^2$ cancels with the same term in the denominator of $f(z)$ and so we have

$$\lim_{z\to 1+i} \frac{d}{dz}\left(\frac{1}{(z-(1-i))^2}\right) = \lim_{z\to 1+i} \frac{-2}{(z-1+i)^3} = \frac{-2}{(2i)^3} = \frac{-2}{8(-i)} = \frac{1}{4i} = \frac{-i}{4}.$$

Thus, $\int_{C_R} \frac{1}{(z^2-2z+2)^2}dz = 2\pi i \frac{-i}{4} = \frac{\pi}{2}$. When we let $R \to \infty$, the integral on the left converges to the sum over the lower part that converges to $\int_{-\infty}^{\infty} \frac{1}{(z^2-2z+2)^2}dz$ and the integral of the top part that converges to 0. Thus, we obtain that this integral is equal to $\frac{\pi}{2}$.

(b) Following the hint, consider the complex function $f(z) = \frac{ze^{iz}}{z^2+1}$. When $z = x$, $f(z) = f(x) = \frac{xe^{ix}}{x^2+1} = \frac{x\cos x}{x^2+1} + i\frac{x\sin x}{x^2+1}$. Thus, the function under the integral is the imaginary part of $f(z)$ when z is real.

To evaluate the integral, consider the integral of $f(z)$ over the same contour C_R as in previous problem and Example 12. The zeros of the denominator $\pm i$ are the only singularities, they are both poles of the first order and just i is in the given contour. The residue at i is $\lim_{z\to i}(z-i)\frac{ze^{iz}}{(z-i)(z+i)} = \lim_{z\to i} \frac{ze^{iz}}{z+i} = \frac{ie^{-1}}{2i} = \frac{1}{2e}$.

On the bottom part of the contour, $z = x + 0i = x$ and $dz = dx$ so the integral over the bottom produces $\int_{-R}^{R} \frac{xe^{ix}}{x^2+1}dx$. If we let $R \to \infty$, the imaginary part of this integral is the integral we need to evaluate.

On the top part of the contour $z = Re^{it}$ for $t : 0 \to \pi$ and $dz = Rie^{it}dt$. The integral over the top part is $\int_0^\pi \frac{R^2ie^{2it}e^{Re^{it}}}{R^2e^{2it}+1}dt$. Note that the absolute value of the numerator is $|R^2ie^{2it}e^{Rie^{it}}| = R^2|e^{Ri(\cos t+i\sin t)}| = R^2|e^{Ri\cos t}|e^{-R\sin t} = R^2e^{-R\sin t}$. Thus, the absolute value of this integral is less or equal to the following integral

$$\int_0^\pi \frac{R^2e^{-R\sin t}}{R^2-1}dt = \frac{R^2}{R^2-1}\int_0^\pi e^{-R\sin t}dt \to 0$$

for $R \to \infty$. Thus, the integral over the top part converges to 0 as well.

Thus, $\int_{C_R} \frac{ze^{iz}}{z^2+1}dz = 2\pi i\frac{1}{2e} = \frac{\pi i}{e}$. When we let $R \to \infty$, the integral on the left converges to $\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+1}dx$. Thus,

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+1}dx = \int_{-\infty}^{\infty} \frac{x\cos x}{x^2+1}dx + i\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+1}dx = \frac{\pi i}{e}.$$

Equating the real parts on both sides and the imaginary parts on both sides, we obtain that

$$\int_{-\infty}^{\infty} \frac{x\cos x}{x^2+1}dx = 0 \text{ and } \int_{-\infty}^{\infty} \frac{x\sin x}{x^2+1}dx = \frac{\pi}{e}.$$

Thus, the integral we needed to find is equal to $\frac{\pi}{e}$.