Special Topics Course
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Cyclic Groups

Definition. Let G be a group. G is said to be a cyclic group if there exists an element a in G, such that every element in G is of the form a^n for some integer n.

In other words, a group is cyclic if it is generated by a single element.

There are basically two types of cyclic groups : finite and infinite.

Finite Cyclic groups. A cyclic group generated by a is finite if there exist a positive integer n such that $a^n = 1$. Such integer n is not unique. For example, if $a^3 = 1$, then $a^6, a^9 \ldots$ are also 1. If n is the smallest positive integer such that $a^n = 1$, then the group has exactly n elements $1, a, a^2, \ldots a^{n-1}$. The presentation of such group is

$$\langle a \mid a^n = 1 \rangle.$$

Prove that every cyclic group with n elements is isomorphic with the remainders of division with n under addition. Recall that this is the group of remainders with division with $n : 0, 1, 2, \ldots, n-1$. This group is denoted by C_n or Z_n. Show that the map $i \mapsto a^i$ is the isomorphism of this group to $\langle a \mid a^n = 1 \rangle$. Because of this isomorphism we identify *every* cyclic group with C_n.

Infinite Cyclic Groups. If G is generated by a and a^n is not 1 for any integer n, then the elements $a, a^2, a^3, ldots$ are all distinct. So, G is infinite. Note that the inverse a^{-i} of a^i is also an element of such group so the group contains all the elements $\ldots a^{-3}, a^{-2}, a^{-1}, 1 - a^0, a^1 = a, a^2, a^3, \ldots$ The presentation of such group is

$$\langle a \rangle.$$

We can identify any such infinite cyclic group with the group of integers (Z={...-3, -2, -1, 0, 1, 2, 3,...}) by mapping i to a^i. Show that this map is isomorphism. Because of this isomorphism, we identify every infinite cyclic group with Z.

Every cyclic group is **abelian**. Recall that we say that a group G is abelian if it satisfies $xy = yx$ for every elements x, y of G. Every cyclic group is abelian. To prove that, let G be a cyclic group generated by a. Any x, y from G have to have the form a^n and a^m then, for some integers n and m. $xy = yx$ since $a^n a^m = a^{n+m} = a^{m+n} = a^m a^n$.

Product of cyclic groups. If $C_m = \langle a \mid a^m = 1 \rangle$ and $C_n = \langle b \mid b^n = 1 \rangle$ are two cyclic groups, their product $C_m \times C_n$ has the presentation

$$\langle a, b \mid a^m = 1, b^n = 1, ab = ba \rangle.$$

Point Groups. When considering point groups, we found the following cyclic groups

C_2 when having C_i, C_s and C_2 as point groups.

C_n when having $C_n,$ as point group.

$C_n \times C_2$ when having C_{nh} as point group.

Note that $C_{nh} = C_n \times C_2$ is different than $C_{nv} = D_n$ because the first one is abelian with the presentation

$$C_n \times C_2 = \langle a, b \mid a^n = 1, b^2 = 1, ba = ab \rangle.$$

while the second one is not abelian and has the presentation

$$D_n = \langle a, b \mid a^n = 1, b^2 = 1, ba = a^{n-1}b \rangle.$$

This can be a basis for your presentation. If you want to include some further material on cyclic groups, look at

1. http://mathworld.wolfram.com/CyclicGroup.html Contains similar exposition as this given here but also has a nice graphical representation of cyclic groups.

2. http://en.wikipedia.org/wiki/Cyclic_group Similar as 1. Has also nice graphical representation.

3. http://web.usna.navy.mil/~wdj/tonybook/gpthry/node27.html This site contains more detailed exposition on cyclic groups that the one I gave you here. You can add anything from this site.