Math Methods 1
Lia Vas

Formulas for Exam 1

1. **Derivatives.**

y	x^n	e^x	b^x	$\ln x$	$\log_b x$	$\sin x$	$\cos x$	$\sin^{-1} x$	$\tan^{-1} x$	$\sec^{-1} x$
y'	nx^{n-1}	e^x	$b^x \ln b$	$\frac{1}{x}$	$\frac{1}{x} \cdot \frac{1}{\ln b}$	$\cos x$	$-\sin x$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{1+x^2}$	$\frac{1}{x\sqrt{x^2-1}}$

2. **Integrals.**

y	x^n	e^x	b^x	$\frac{1}{x}$	$\sin x$	$\cos x$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{1+x^2}$
$\int y\, dx$	$\frac{1}{n+1} x^{n+1}$	e^x	$\frac{1}{\ln b}\, b^x$	$\ln \lvert x \rvert$	$-\cos x$	$\sin x$	$\sin^{-1} x$	$\tan^{-1} x$

3. **Vectors.** If $\overrightarrow{a} = (a_1, a_2, a_3)$ and $\overrightarrow{b} = (b_1, b_2, b_3)$, then
   (a) The **length** of $\overrightarrow{a}$ is $|\overrightarrow{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.
   (b) The **dot product** is $\overrightarrow{a} \cdot \overrightarrow{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$.
   (c) The **cross product** is $\overrightarrow{a} \times \overrightarrow{b} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

4. **Line Integrals.** The **work** done by the force $\mathbf{f} = (P, Q, R)$ in moving the particle along the curve C is the line integral

$$W = \int_C \mathbf{f} \cdot d\mathbf{r} = \int_C P dx + Q dy + R dz.$$

5. **Parametric Surfaces. Surface and Flux Integrals.**
   Parametric surface is $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$.
   (a) The tangent plane has the **normal vector** $\mathbf{r}_u \times \mathbf{r}_v = (x_u, y_u, z_u) \times (x_v, y_v, z_v)$.

$$dS = |\mathbf{r}_u \times \mathbf{r}_v|\, dudv \qquad\qquad d\mathbf{S} = \pm(\mathbf{r}_u \times \mathbf{r}_v)\, dudv$$

   (b) The **surface integral** of $f(x, y, z)$ is

$$\boxed{\int\int_S f(x, y, z)\; dS = \int\int_S f(\mathbf{r}(u, v))\; |\mathbf{r}_u \times \mathbf{r}_v|\; dudv}$$

   If $f = 1$, the integral computes the surface area of the surface $\mathbf{r}(u, v)$, then $S = \int\int_S dS$.
   If $\rho(x, y, z)$ is the density function of a thin sheet S, then the mass m is given by $m = \int\int_S \rho(x, y, z)\; dS$.

(c) The **flux integral** of $\mathbf{f} = (P, Q, R)$ is

$$\boxed{\int\int_S \mathbf{f} \cdot d\mathbf{S} = \int\int_S \mathbf{f} \cdot \mathbf{n}\, dS = \pm \int\int_S \mathbf{f} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dudv}$$

If $\mathbf{f}$ is a force field, then this integral computes the flux of the field – the total effect of the field acting over the region S.

6. **Gradient, Curl, and Divergence.**

   The gradient operator is $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$. If $\mathbf{f} = (P, Q, R)$ is a vector function then the **divergence** is

$$\text{div } \mathbf{f} = \nabla \cdot \mathbf{f} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

   and the **curl** is

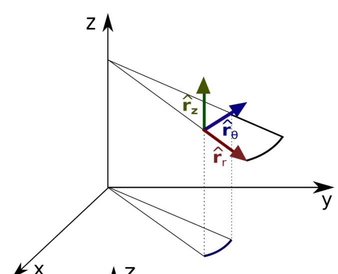
$$\text{curl } \mathbf{f} = \nabla \times \mathbf{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

7. **Stokes' and Divergence Theorems.**

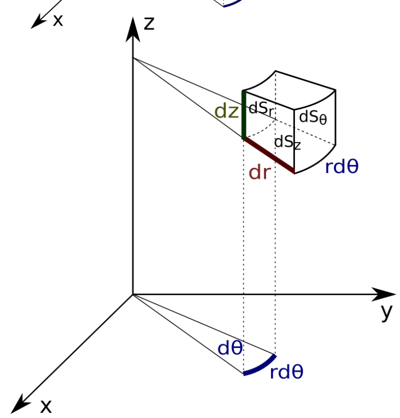
$$\boxed{\oint_C \mathbf{f} \cdot d\mathbf{r} = \int\int_S \text{curl}\mathbf{f} \cdot d\mathbf{S} \qquad\qquad \int\int_S \mathbf{f} \cdot d\mathbf{S} = \int\int\int_V \text{div}\mathbf{f}\ \ dV}$$

8. **Cylindrical coordinates.** $\mathbf{r} = (r\cos\theta, r\sin\theta, z)$. Then $x^2 + y^2 = r^2$, the Jacobian is $J = r$, and so $dV = dx\, dy\, dz = r dr\, d\theta\, dz$.

$$\begin{aligned} \hat{\mathbf{r}}_r &= \mathbf{r}_r = (\cos\theta, \sin\theta, 0), \\ \hat{\mathbf{r}}_\theta &= \tfrac{1}{r}\mathbf{r}_\theta = (-\sin\theta, \cos\theta, 0), \quad \text{and} \\ \hat{\mathbf{r}}_z &= \mathbf{r}_z = (0, 0, 1). \end{aligned}$$



$$\begin{aligned} d\mathbf{r} &= \hat{\mathbf{r}}_r\, dr + r\, \hat{\mathbf{r}}_\theta\, d\theta + \mathbf{r}_z\, dz \\ dV &= r\ dr\, d\theta\, dz \\ d\mathbf{S} &= r\, d\theta\, dz\, \hat{\mathbf{r}}_r\ +\ dr\, dz\, \hat{\mathbf{r}}_\theta\ +\ r\, dr\, d\theta\, \hat{\mathbf{r}}_z \end{aligned}$$



Conversion equations: $\hat{\mathbf{r}}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$, $\hat{\mathbf{r}}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$, and $\hat{\mathbf{r}}_z = \mathbf{k}$. $\mathbf{i} = \cos\theta\hat{\mathbf{r}}_r - \sin\theta\hat{\mathbf{r}}_\theta$, $\mathbf{j} = \sin\theta\hat{\mathbf{r}}_r + \cos\theta\hat{\mathbf{r}}_\theta$, and $\mathbf{k} = \hat{\mathbf{r}}_z$.

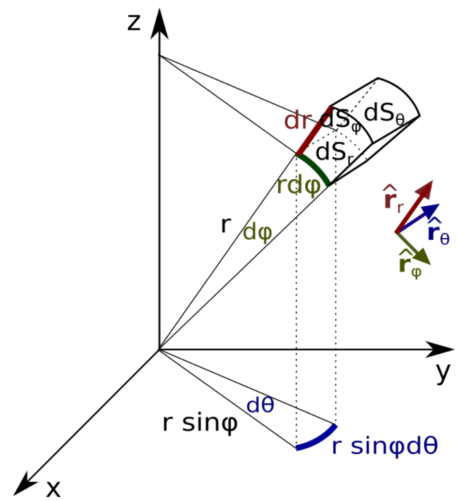
If $\mathbf{f} = P\hat{\mathbf{r}}_r + Q\hat{\mathbf{r}}_\theta + R\hat{\mathbf{r}}_z$, then

$$\operatorname{div}\mathbf{f} = \nabla \cdot \mathbf{f} = \frac{1}{r}\frac{\partial(rP)}{\partial r} + \frac{1}{r}\frac{\partial Q}{\partial \theta} + \frac{\partial R}{\partial z} \qquad \operatorname{curl}\mathbf{f} = \nabla \times \mathbf{f} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}}_r & r\hat{\mathbf{r}}_\theta & \hat{\mathbf{r}}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ P & rQ & R \end{vmatrix}$$

9. **Spherical coordinates.** $\mathbf{r} = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$. Then $x^2 + y^2 + z^2 = r^2$, the Jacobian J is $r^2\sin\phi$ and so $dV = dx\,dy\,dz = r^2\sin\phi\,dr\,d\theta\,d\phi$

$\hat{\mathbf{r}}_r = \mathbf{r}_r = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$,
$\hat{\mathbf{r}}_\phi = \frac{1}{r}\mathbf{r}_\phi = (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi)$, and
$\hat{\mathbf{r}}_\theta = \frac{1}{r\sin\phi}\mathbf{r}_\theta = (-\sin\theta, \cos\theta, 0)$.

$d\mathbf{r} = \hat{\mathbf{r}}_r\,dr + r\,\mathbf{r}_\phi\,d\phi + r\sin\phi\,\hat{\mathbf{r}}_\theta\,d\theta$
$dV = r^2\sin\phi\;dr\,d\theta\,d\phi$
$d\mathbf{S} = r^2\sin\phi\,d\phi\,d\theta\,\hat{\mathbf{r}}_r \;+\; r\sin\phi\,dr\,d\theta\,\hat{\mathbf{r}}_\phi \;+\; r\,dr\,d\phi\,\hat{\mathbf{r}}_\theta$



Conversion equations:
$\hat{\mathbf{r}}_r = \cos\theta\sin\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\phi\mathbf{k}$,
$\hat{\mathbf{r}}_\phi = \cos\theta\cos\phi\mathbf{i} + \sin\theta\cos\phi\mathbf{j} - \sin\phi\mathbf{k}$, and
$\hat{\mathbf{r}}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$.

If $\mathbf{f} = P\hat{\mathbf{r}}_r + Q\hat{\mathbf{r}}_\phi + R\hat{\mathbf{r}}_\theta$, then

$$\operatorname{div}\mathbf{f} = \nabla \cdot \mathbf{f} = \frac{1}{r^2}\frac{\partial(r^2P)}{\partial r} + \frac{1}{r\sin\phi}\frac{\partial(\sin\phi\,Q)}{\partial\phi} + \frac{1}{r\sin\phi}\frac{\partial R}{\partial\theta}$$

$$\operatorname{curl}\mathbf{f} = \nabla \times \mathbf{f} = \frac{1}{r^2\sin\phi}\begin{vmatrix} \hat{\mathbf{r}}_r & r\hat{\mathbf{r}}_\phi & r\sin\phi\hat{\mathbf{r}}_\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial\theta} \\ P & rQ & r\sin\phi\,R \end{vmatrix}$$