Math Methods I
Lia Vas

Fourier Series and Fourier Transform

Fourier Series

Recall that a function differentiable any number of times at $x = a$ can be represented as a power series

$$\sum_{n=0}^{\infty} a_n (x-a)^n \quad \text{where the coefficients are given by} \quad a_n = \frac{f^{(n)}(a)}{n!}$$

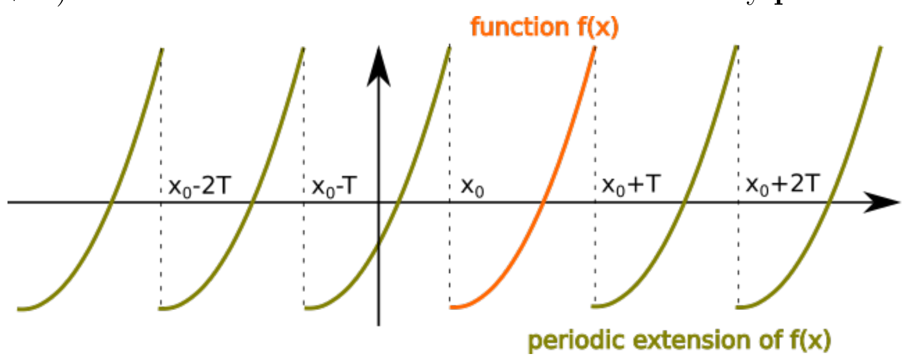
Thus, the function can be approximated by a polynomial. Since this formula involves the n-th derivative, the function f should be differentiable n-times at a. So, any function which is discontinuous (has jumps or breaks) or not differentialbe (has corners or sharp turns) cannot be represented as a power series. This significantly restricts functions we could expand in a power series. Some functions frequently considered in signal processing, electrical circuits and other applications can be discontinuous or not differentiable. For those functions, one considers a different kind of series: the type of series which can handle discontinuous functions.

One such the type of series is the **Fourier Series**. It represents a **periodic function** with a period T in the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{2\pi n x}{T} + b_n \sin\frac{2\pi n x}{T})$$

The coefficients a_n and b_n are called **Fourier coefficients**.

Fourier series can be used for non-periodic functions defined on a finite interval. Say that a function $f(x)$ is defined on an interval which we denote by $(x_0, x_0 + T)$. Extend the function outside this interval by copying the graph on intervals (x_0+T, x_0+2T), (x_0+2T, x_0+3T), $(x_0+3T, x_0+4T), \ldots$ right from (x_0, x_0+T) as well as to intervals (x_0-T, x_0), (x_0-2T, x_0-T), $(x_0-3T, x_0-2T), \ldots$ left from from (x_0, x_0+T). Note that this makes the extension automatically **periodic** with the period T.



To find the Fourier series of such extension, $f(x)$ has to (1) have just a finite number of maxima and minima and just a finite number of discontinuities on (x_0, x_0+T), (2) the integral of $|f(x)|$ over (x_0, x_0+T) must converge. In our examples, these conditions are always going to be satisfied.

If these conditions are satisfied and one period of $f(x)$ is given on an interval (x_0, x_0+T), the Fourier coefficients a_n and b_n can be computed using the formulas

$$a_n = \frac{2}{T}\int_{x_0}^{x_0+T} f(x)\cos\frac{2n\pi x}{T}\,dx \qquad\qquad b_n = \frac{2}{T}\int_{x_0}^{x_0+T} f(x)\sin\frac{2n\pi x}{T}\,dx$$

Obtaining the formulas for coefficients. If $f(x)$ has a Fourier series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos\frac{2\pi nx}{T} + b_n\sin\frac{2\pi nx}{T})$$

one can prove the formulas for Fourier series coefficients a_n by multiplying this formula by $\cos\frac{2\pi nx}{T}$ and integrating over one period, say (x_0, x_0+T)). Not to confuse n we picked here from the n appearing in the sum, make the sum have terms dependent of m where the sum start with $m=1$. For the first term (in front of the sum), think that the value of m is zero.

$$\int_{x_0}^{x_0+T} f(x)\cos\frac{2\pi nx}{T}dx \;=\; \int_{x_0}^{x_0+T}\frac{a_0}{2}\cos\frac{2\pi nx}{T}dx +$$
$$\sum_{m=1}^{\infty}\left(\int_{x_0}^{x_0+T} a_m\cos\frac{2\pi mx}{T}\cos\frac{2\pi nx}{T}dx \;+\; \int_{x_0}^{x_0+T} b_m\sin\frac{2\pi mx}{T}\cos\frac{2\pi nx}{T}dx\right)$$

All the integrals with $m \neq n$ are zero and the integrals with both sin and cosine functions are zero as well. Thus,

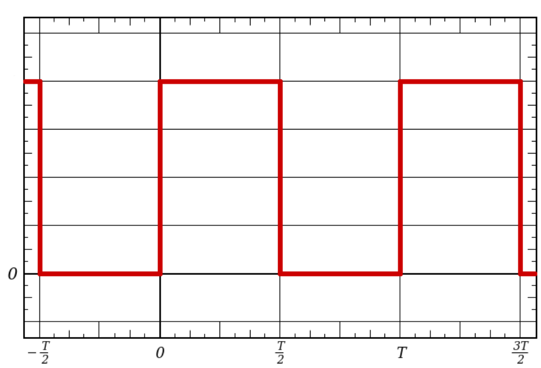
$$\int_{x_0}^{x_0+T} f(x)\cos\frac{2\pi nx}{T}dx = \int_{x_0}^{x_0+T} a_n\cos\frac{2\pi nx}{T}\cos\frac{2\pi nx}{T}dx$$

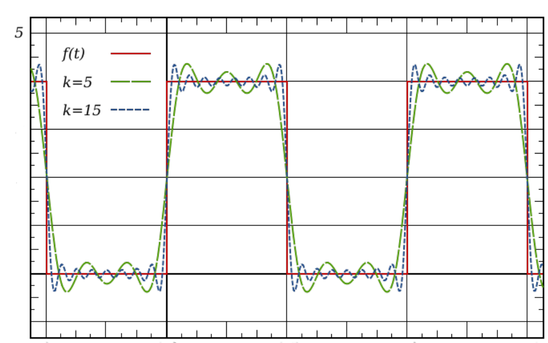
This last integral can be calculated to be $a_n\frac{T}{2}$ using the trigonometric identity $\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$. Hence

$$\int_{x_0}^{x_0+T} f(x)\cos\frac{2\pi nx}{T}dx = a_n\frac{T}{2} \Rightarrow a_n = \frac{2}{T}\int_{x_0}^{x_0+T} f(x)\cos\frac{2\pi nx}{T}dx.$$

The formula for b_n is proved similarly, multiplying by $\sin\frac{2\pi nx}{T}$ instead of $\cos\frac{2\pi nx}{T}$.

At first it may seem strange that a discontinuous function can be equal to a sum of continuous sine and cosine functions. The following figure illustrates how adding more terms of the series makes the series match a discontinuous function better and better.





When studying phenomena that are periodic in time, the term $\frac{2\pi}{T}$ in the above formula is usually replaced by ω and t is used to denote the independent variable. Thus,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$$

If the interval $(x_0, x_0 + T)$ is of the form $(-L, L)$ (thus $T = 2L$), then the above formulas become the following.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \quad f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L})$$

Symmetry considerations. Note that if $f(x)$ is even (that is $f(-x) = f(x)$), then $b_n = 0$. Since $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{-L}^{0} f(x)\sin\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$. Using the substitution $u = -x$ for the first integral we obtain that it is equal to $\frac{-1}{L}\int_{L}^{0} f(-u)\sin\frac{-n\pi u}{L}\,du$. Using that $f(-x) = f(x)$, that $\sin\frac{-n\pi u}{L} = -\sin\frac{n\pi u}{L}$, and that $-\int_{L}^{0} = \int_{0}^{L}$, we obtain $\frac{1}{L}\int_{0}^{L} f(u)(-\sin\frac{n\pi u}{L})\,du = \frac{-1}{L}\int_{0}^{L} f(u)\sin\frac{n\pi u}{L}\,du$. Note that this is exactly the negative of the second integral. Thus, the first and the second integral cancel and we obtain that $b_n = 0$.

Similarly, $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{-L}^{0} f(x)\cos\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$. Using the substitution $u = -x$ for the first integral we obtain that it is equal to $\frac{-1}{L}\int_{L}^{0} f(-u)\cos\frac{-n\pi u}{L}\,du$. Using that $f(-x) = f(x)$, that $\cos\frac{-n\pi u}{L} = \cos\frac{n\pi u}{L}$, and that $-\int_{L}^{0} = \int_{0}^{L}$, we obtain $\frac{1}{L}\int_{0}^{L} f(u)\cos\frac{n\pi u}{L}$. Note that this is equal to the second integral. Thus, the two integrals can be combined and so

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \quad \text{and} \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L}.$$

If $f(x)$ is odd, using analogous arguments, we obtain that $a_n = 0$ and

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \quad \text{and} \quad f(x) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{L}.$$

Example 1. The input to an electrical circuit that switches between a high and a low state with time period 2π can be represented by the **boxcar function**.

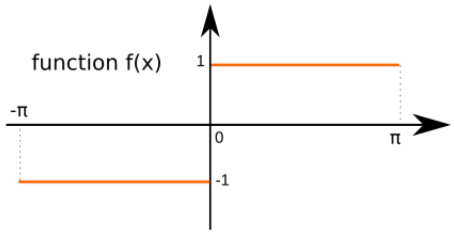
$$f(x) = \begin{cases} 1 & 0 \le x < \pi \\ -1 & -\pi \le x < 0 \end{cases}$$

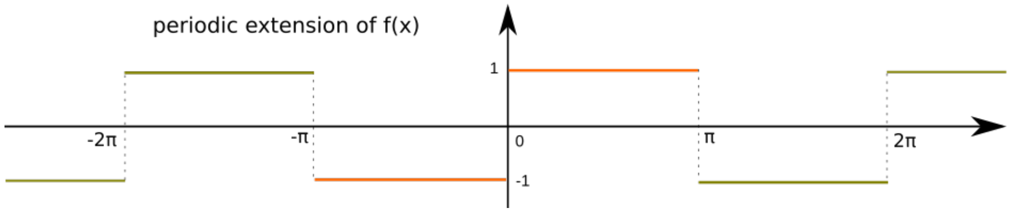
More generally, the input to an electrical circuit that switches from a high to a low state with time period T can be represented by the **general square wave function** with the following formula on the basic period. $f(x) = \begin{cases} 1 & 0 \le x < \frac{T}{2} \\ -1 & \frac{-T}{2} \le x < 0 \end{cases}$

Find the Fourier series of the given square wave and the general square wave. Use the expansion of the given square wave function to find the sum of the series

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}.$$

Solutions. Graph the square wave function, then graph its periodic extension. The graphs are on the figure below.





The periodic expansion of this function is called the **square wave function**.

Note that the function is odd. Thus, the coefficients a_n of the cosine terms are zero. Since $L = \pi$ (and the period T is 2π), the coefficients b_n can be computed as

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_0^{\pi}\sin nx dx = \frac{-2}{n\pi}\cos nx\Big|_0^{\pi} = \frac{-2}{n\pi}((-1)^n - 1).$$

Note that $(-1)^n - 1 = 1 - 1 = 0$ if n is even (say $n = 2k$) and $(-1)^n - 1 = -1 - 1 = -2$ if n is odd (say $n = 2k+1$). Thus, $b_{2k} = 0$ and $b_{2k+1} = \frac{-2}{n\pi}(-2) = \frac{4}{(2k+1)\pi}$. Hence,

$$f(x) = \frac{4}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{1}{n}\sin nx = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\sin(2k+1)x = \frac{4}{\pi}(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots).$$

For the general square wave, analogously to this previous consideration you obtain that $a_n = 0$, $b_{2k} = 0$ and $b_{2k+1} = \frac{4}{(2k+1)\pi}\sin\frac{2(2k+1)\pi x}{T}$. Thus, $f(x) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\sin\frac{2(2k+1)\pi x}{T}$.

Let us use the expansion $f(x) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin(2n+1)x$ where $f(x)$ is the square wave function to obtain the sum $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$. To match the desired series, we need to to choose an x-value which would make $\sin(2n+1)x$ have value $(-1)^n$. Such x-value is $x = \frac{\pi}{2}$. Since $f(\frac{\pi}{2}) = 1$, plugging $x = \frac{\pi}{2}$ in both sides of the equation $f(x) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin(2n+1)x$ produces

$$1 = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}(-1)^n = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} \quad\Rightarrow\quad \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = \frac{\pi}{4}.$$

Useful formulas. In the previous example, we have seen that the formulas simplifying $\cos n\pi$ and $\sin\frac{n\pi}{2}$ were useful. To simplify the answers, sometimes the following identities may also be useful.

$\sin n\pi = 0$	$\cos n\pi = (-1)^n$

In particular, $\cos 2n\pi = 1$ and $\cos(2n+1)\pi = -1$.

If $n = 2k+1$ is an odd number,

$\sin\frac{n\pi}{2} = \sin\frac{(2k+1)\pi}{2} = (-1)^k$	$\cos\frac{n\pi}{2} = \cos\frac{(2k+1)\pi}{2} = 0$

Even and odd extensions. If an arbitrary function $f(x)$, not necessarily even or odd, is defined on the interval $(0, L)$, we can extend it to an even function

$$g(x) = \begin{cases} f(x) & 0 < x < L \\ f(-x) & -L < x < 0 \end{cases}$$

and consider its Fourier cosine expansion:

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \quad \text{and} \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}.$$

Similarly, we can extend $f(x)$ to an odd function

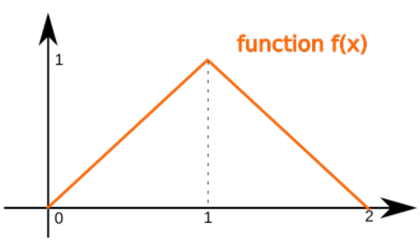
$$h(x) = \begin{cases} f(x) & 0 < x < L \\ -f(-x) & -L < x < 0 \end{cases}$$

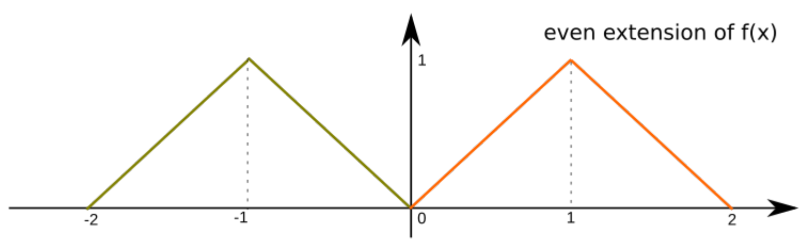
and consider its Fourier sine expansion:

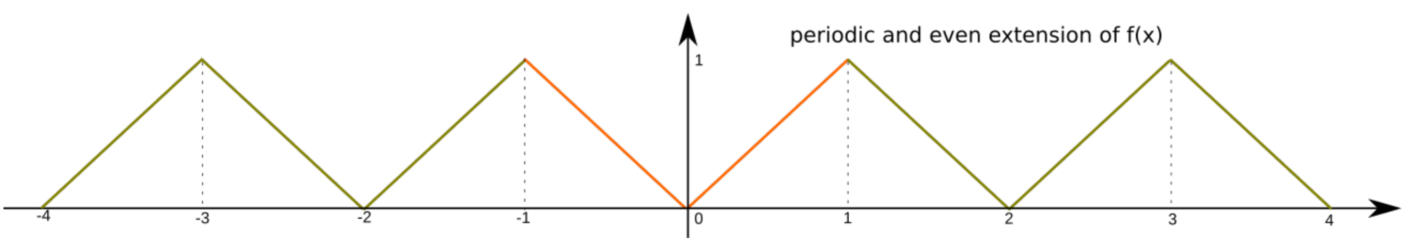
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \quad \text{and} \quad f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}.$$

Example 2. Find the Fourier cosine expansion of $f(x) = \begin{cases} x & 0 < x \leq 1 \\ 2-x & 1 < x < 2 \end{cases}$ Use it to find the sum of the series $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

Solution. Graph the function first. Then, extend it symmetrically with respect to y-axis and then extend it further so that it is periodic. Note that the extension is even and that repeating the pattern on $(-1, 1)$ produces the entire periodic extension. So, you can take $T = 2$ and $L = 1$.







Since the function is even, the coefficients b_n are zero and the coefficients a_n can be computed as follows. $a_n = \frac{2}{1}\int_0^1 f(x)\cos\frac{n\pi x}{1}dx = 2\int_0^1 x\cos n\pi x\ dx$. Using integration by parts with $u = x$, $v = \frac{1}{n\pi}\sin n\pi x$, you obtain $a_n = \frac{2x}{n\pi}\sin n\pi x\Big|_0^1 + \frac{2}{n^2\pi^2}\cos n\pi x\Big|_0^1 = 0 + \frac{2}{n^2\pi^2}(\cos n\pi - 1) = \frac{2}{n^2\pi^2}((-1)^n - 1)$. If $n = 2k$ is even, $a_n = 0$. If $n = 2k+1$ is odd, $a_n = \frac{2}{(2k+1)^2\pi^2}(-1-1) = \frac{-4}{(2k+1)^2\pi^2}$.

Since we are dividing by n already when finding v, the above formula does not apply to $n = 0$ so a_0 must be computed separately as $a_0 = \frac{2}{1}\int_0^1 x\cos 0 dx = 2\int_0^1 x dx = 2\left.\frac{x^2}{2}\right|_0^1 = 1$.

So, $f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}\cos(2k+1)\pi x$.

Alternatively, you can consider the function as defined on the basic period [-2,2] and take $T = 4$ and $L = 2$. In this case, $a_n = \int_0^2 f(x)\cos\frac{n\pi x}{2}dx = \int_0^1 x\cos\frac{n\pi x}{2}dx + \int_1^2 (2-x)\cos\frac{n\pi x}{2}dx$. Using integration by parts, obtain the same formula for a_n as above and the same Fourier series as above.

To find the sum of $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$, note that when $x = 0$ the function $f(x)$ is equal to $f(0) = 0$ and its Fourier cosine expansion is equal to $\frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos(0) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$. Rename the index k to be n if you want and obtain the equation. Then solve for the required sum.

$$0 = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \quad\Rightarrow\quad \frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{1}{2} \quad\Rightarrow\quad \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

Example 3. Consider the function $f(x) = x^2$ for $0 \le x \le 2$.

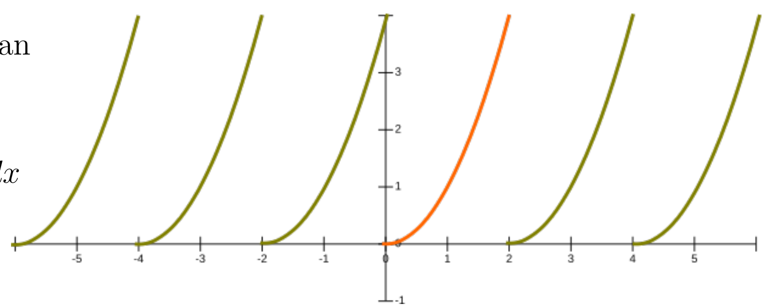
(a) Sketch the graphs of the following (1) the periodic extension of $f(x)$, (2) the even periodic extension of $f(x)$, (3) the odd periodic extension of $f(x)$ and write the integrals computing the coefficients of the corresponding Fourier series in all three cases.

(b) Find the Fourier cosine expansion for $f(x)$.

(c) Using part (b), show the following

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} \qquad \text{and} \qquad \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Solution. (a) The *periodic* extension of $f(x)$ is neither even nor odd. You can obtain the graph of it by replicating $f(x)$ on intervals $\ldots[-4,-2],\ [-2,0],\ [0,2],\ [2,4],\ [4,6]\ldots$ of length $T = 2$. The coefficients of the corresponding Fourier series can be calculated by

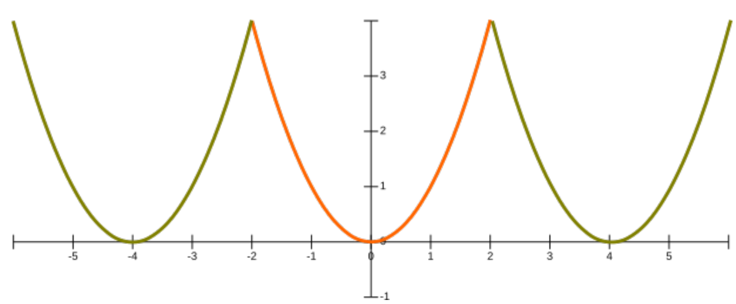


$$a_n = \int_0^2 x^2\cos n\pi x\ dx, \quad b_n = \int_0^2 x^2\sin n\pi x\ dx$$

and $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos n\pi x + b_n\sin n\pi x$.

The *even* extension of $f(x)$ is obtained by extending $f(x)$ from $[0,\ 2]$ to $[-2, 2]$ by defining $f(x) = (-x)^2 = x^2$ on $[-2, 0]$. Thus, the result is the function x^2 defined on interval $[-2, 2]$.

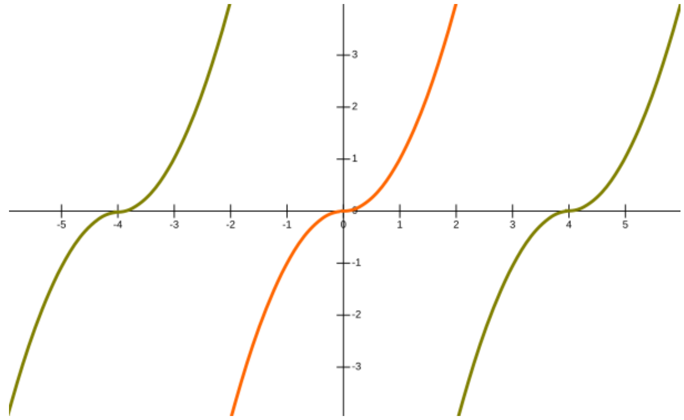
Then, replicate this function on intervals $\ldots[-6,-2],[-2,2],[2,6],[6,10],\ldots$ of length $T = 4$. Thus, $T = 4$ and $L = 2$. The coefficients of the cosine Fourier series can be calculated by



$$a_n = \int_0^2 x^2\cos\frac{n\pi x}{2}dx \qquad b_n = 0.$$

The *odd* extension of $f(x)$ is obtained by extending $f(x)$ from $[0, 2]$ to $[-2, 2]$ by defining $f(x) = -(x)^2 = -x^2$ on $[-2, 0]$. Thus, the result is the function

$$\begin{cases} x^2 & 0 \leq x \leq 2 \\ -x^2 & -2 \leq x < 0 \end{cases}$$



defined on interval $[-2, 2]$. Replicate this function on intervals $\ldots [-6, -2], [-2, 2], [2, 6], [6, 10], \ldots$ of length $T = 4$. Thus, $T = 4$ and $L = 2$. The coefficients of the corresponding since Fourier series can be calculated by

$$b_n = \int_0^2 x^2 \sin \frac{n\pi x}{2}\, dx \qquad a_n = 0.$$

(b) By part (a), the coefficients a_n can be calculated by $a_n = \int_0^2 x^2 \cos \frac{n\pi x}{2} dx$. Using integration by parts with $u = x^2$ and $v = \int \cos \frac{n\pi x}{2} dx = \frac{2}{n\pi} \sin \frac{n\pi x}{2}$, we obtain that $a_n = \frac{2}{n\pi} x^2 \sin \frac{n\pi x}{2}\Big|_0^2 - \frac{4}{n\pi} \int_0^2 x \sin \frac{n\pi x}{2} dx$. The first term is zero and the second term requires another integration by parts, this time with $u = x$ and $v = \int \sin \frac{n\pi x}{2} dx = \frac{-2}{n\pi} \cos \frac{n\pi x}{2}$. Thus $a_n = \frac{8}{n^2\pi^2} x \cos \frac{n\pi x}{2}\Big|_0^2 - \frac{16}{n^3\pi^3} \sin \frac{n\pi x}{2}\Big|_0^2$. The first term is zero in the lower bound and the second term is zero at both bounds. Thus $a_n = \frac{16}{n^2\pi^2} \cos n\pi = \frac{16(-1)^n}{n^2\pi^2}$. Note that this formula works just for $n > 0$ so a_0 has to be computed separately by $a_0 = \int_0^2 x^2 dx = \frac{x^3}{3}\Big|_0^2 = \frac{8}{3}$.

Thus, the Fourier series is $x^2 = \frac{4}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{2}$.

(c) For both sums, we need to find convenient values of x which reduce the right side of the formula

$$x^2 = \frac{4}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{2}$$

to the series we need to sum. For the first one, it seems that the value of cosine should be 1 so take $x = 0$. Plugging $x = 0$ in the formula above produces

$$0 = \frac{4}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \quad \Rightarrow \quad -\frac{4}{3} = \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \quad \Rightarrow \quad -\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

Multiply both sides with -1 to have

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)(-1)^n}{n^2} \quad \Rightarrow \quad \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

For the second sum, note that the value of cosine should be $(-1)^n$ so that, multiplied by another $(-1)^n$, the numerator becomes $(-1)^{2n} = 1$. The value $x = 2$ would produce $\cos \frac{n\pi(2)}{2} = \cos n\pi = (-1)^n$. Plugging $x = 2$ in the expansion in part (b) produces

$$2^2 = \frac{4}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (-1)^n \quad \Rightarrow \quad 4 = \frac{4}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2} \quad \Rightarrow \quad \frac{8}{3} = \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \Rightarrow \quad \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Complex Fourier Series. The complex form of Fourier series is the following:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2n\pi i x}{T}} = \sum_{n=-\infty}^{\infty} c_n \left(\cos\frac{2n\pi x}{T} + i \sin\frac{2n\pi x}{T} \right) \text{ where } c_n = \frac{1}{T}\int_{x_0}^{x_0+T} f(x) e^{\frac{-2n\pi i x}{T}}\, dx.$$

If $f(x)$ is a real function, the coefficients c_n satisfy the relations $c_n = \frac{1}{2}(a_n - ib_n)$ and $c_{-n} = \frac{1}{2}(a_n + ib_n)$ for $n > 0$. Thus, $c_{-n} = \overline{c_n}$ for all $n > 0$. In addition,

$$a_n = c_n + c_{-n} \text{ and } b_n = i(c_n - c_{-n}) \text{ for } n > 0 \text{ and } a_0 = 2c_0.$$

These coefficients c_n are further associated to $f(x)$ by **Parseval's Theorem**. This theorem is related to conservation law and states that

$$\frac{1}{T}\int_{x_0}^{x_0+T} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

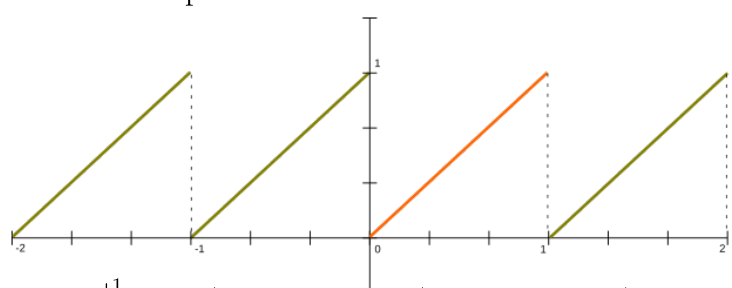
Note that the integral on the left side computes the average value of $|f(x)|^2$ over one period. The proof of this theorem can be your project topic.

Symmetry Considerations. If $f(x)$ is either even or odd function defined on interval $(-L, L)$, the value of $|f(x)|^2$ on $(-L, 0)$ is the same as the value of $|f(x)|^2$ on $(0, L)$. Thus, $\frac{1}{2L}\int_{-L}^{L} |f(x)|^2 dx = \frac{1}{L}\int_0^L |f(x)|^2 dx$. In this case, Parseval's Theorem has the following form.

$$\frac{1}{L}\int_0^L |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

Example 4. The voltage in an electronic oscillator is represented as a sawtooth function $f(t) = t$ for $0 \le t \le 1$ that keeps repeating with the period of 1. (a) Sketch this function and represent it using a complex Fourier series. (b) Use the Fourier series expansion and Parseval's Theorem to find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$



Solution. Graph the function first, note its periodic extension and the period $T = 1$. Thus, $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2n\pi i t}$ where

$$c_n = \int_0^1 t e^{-2n\pi i t} = \frac{t}{-2n\pi i} e^{-2n\pi i t}\Big|_0^1 + \frac{1}{4n^2\pi^2} e^{-2n\pi i t}\Big|_0^1 = \frac{1}{-2n\pi i} e^{-2n\pi i} + \frac{1}{4n^2\pi^2} e^{-2n\pi i} - \frac{1}{4n^2\pi^2}.$$

Note that $e^{-2n\pi i} = \cos(-2n\pi) + i\sin(-2n\pi) = 1$. Thus $c_n = \frac{1}{-2n\pi i} + 0 = \frac{i}{2n\pi}$. This formula applies just to $n \ne 0$ so compute c_0 separately as $c_0 = \int_0^1 t\, dt = \frac{1}{2}$. Hence, $f(t) = \frac{1}{2} + \sum_{n=-\infty, n\ne 0}^{\infty} \frac{i}{2n\pi} e^{2n\pi i t}$.

This sum can be expressed using formulas with a_n and b_n also. Since $c_{-n} = \frac{-i}{2n\pi} = \overline{c_n}$, we have that $a_n = 0$ for $n > 0$, $a_0 = 2c_0 = 1$, and $b_n = \frac{-1}{n\pi}$.

By Parseval's Theorem,

$$\int_0^1 x^2 dx = \frac{1}{4} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} \quad \Rightarrow \quad \frac{1}{3} = \frac{1}{4} + \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} \quad \Rightarrow \quad \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Example 5. Using the Fourier cosine series expansion for $f(x) = x^2$ for $0 < x < 2$ from Example 3 and Parseval's Theorem, find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

Solution. Recall that $T = 4$, $L = 2$ and that the function is even. In Example 3 we have find that $a_n = \frac{16(-1)^n}{n^2\pi^2}$ for $n > 0$, $a_0 = \frac{8}{3}$. Thus, Parseval's Theorem applied to this function results in the following

$$\frac{1}{2}\int_0^2 x^4 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{16}{9} + \frac{16^2}{2\pi^4}\sum_{n=1}^{\infty}\left(\frac{1}{n^4} + 0\right).$$

Note that integral on the left side is $\frac{1}{2}\int_0^2 x^4 dx = \frac{16}{5}$. Dividing the equation above by 16 produces

$$\frac{1}{5} = \frac{1}{9} + \frac{8}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4} \quad \Rightarrow \quad \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

Practice Problems.

1. Note that the boxcar function from Example 1 represents the odd extension of the function $f(x) = 1$ for $0 \leq x < \pi$. Consider the even extension of this function and find its Fourier cosine expansion.

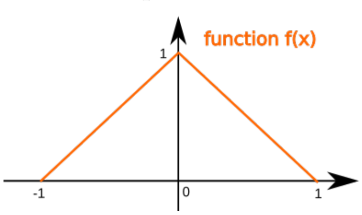
2. Find the Fourier series of $f(x) = \begin{cases} 1 - x & 0 \leq x < 1 \\ 1 + x & -1 < x < 0 \end{cases}$

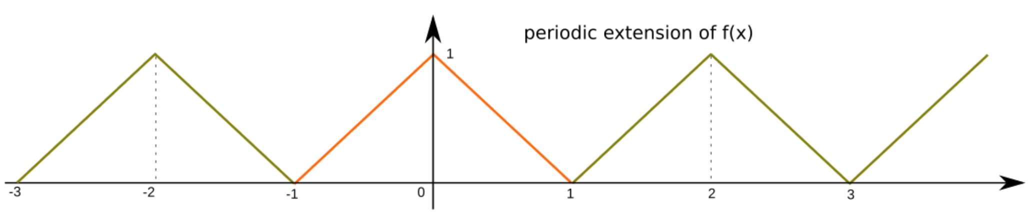
3. Find the Fourier sine expansion of $f(x) = \begin{cases} x & 0 < x \leq 1 \\ 2 - x & 1 < x < 2 \end{cases}$. Use it to find the sum of the series
$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}.$$

Solutions.

1. The even extension of $f(x)$ is $f(x) = 1$ for $-\pi < x < \pi$. The periodic extension of this function is constant function equal to 1 for every x value. Note that this is already in the form of a Fourier series with $b_n = 0$ for every n, $a_n = 0$ for $n > 1$ and $a_0 = 2$. Computing the Fourier coefficients would give you the same answer: $a_n = \frac{2}{\pi}\int_0^{\pi}\cos nx\ dx = 0$ if $n > 1$ and $a_0 = \frac{2}{\pi}\int_0^{\pi} dx = 2$. Thus, $f(x) = \frac{2}{2} + \sum_{n=1}^{\infty} 0 = 1$. This answer should not be surprising since this function is already in the form of a Fourier series $1 = 1 + \sum_{n=0}^{\infty}(0\cos nx + 0\sin nx)$.

2. Graph the function and note it is even.

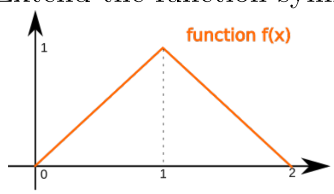


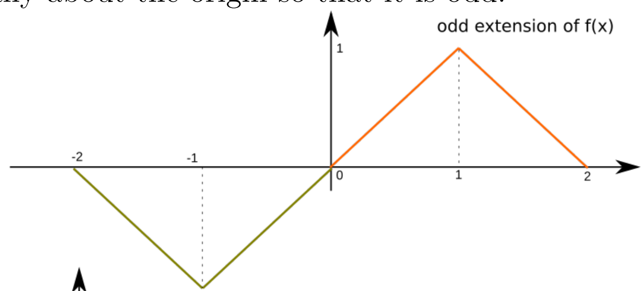


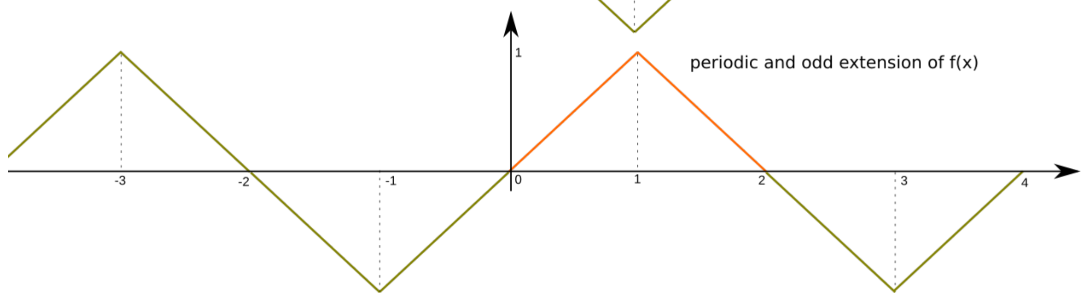
Thus, $b_n = 0$. Since $T = 2$ and $L = 1$ $a_n = 2\int_0^1(1-x)\cos(n\pi x)\ dx$. Using the integration by parts with $u = 1 - x$ and $v = \int\cos(n\pi x)dx = \frac{1}{n\pi}\sin(n\pi x)$, obtain that $a_n = \frac{2}{n\pi}(1-x)\sin(n\pi x)\Big|_0^1 + \frac{2}{n\pi}\int_0^1\sin(n\pi x)\ dx = 0 - \frac{2}{n^2\pi^2}\cos(n\pi x)\Big|_0^1 = -\frac{2}{n^2\pi^2}((-1)^n - 1)$. This last expression is 0 if n is even and equal to $\frac{2}{n^2\pi^2} = \frac{2}{(2k+1)^2\pi^2}$ if $n = 2k+1$ is odd. Note that the formula $-\frac{2}{n^2\pi^2}((-1)^n - 1)$ does not compute a_0 because of the n in the denominator so calculate a_0 from the formula $a_0 = 2\int_0^1 (1-x)\ dx = x - \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$. This gives you the Fourier series expansion

$$f(x) = \frac{1}{4} + \sum_{k=0}^{\infty} \frac{4}{(2k+1)^2\pi^2} \cos(2k+1)\pi x.$$

3. Extend the function symmetrically about the origin so that it is odd.







Thus $T = 4$ and $L = 2$. Since this extension is odd, $a_n = 0$. Compute b_n as

$$b_n = \int_0^2 f(x) \sin\frac{n\pi x}{2} dx = \int_0^1 x \sin\frac{n\pi x}{2} dx + \int_1^2 (2-x)\sin\frac{n\pi x}{2} dx =$$

$$\left(\frac{-2x}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\Bigg|_0^1 + \left(\frac{-2(2-x)}{n\pi}\cos\frac{n\pi x}{2} - \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\Bigg|_1^2 =$$

$$\underline{\frac{-2}{n\pi}\cos\frac{n\pi}{2}} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} + \underline{\frac{2}{n\pi}\cos\frac{n\pi}{2}} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} = \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

This last expression is 0 if n is even. If $n = 2k+1$ is odd, this is $\frac{8(-1)^k}{(2k+1)^2\pi^2}$. So,

$$f(x) = \frac{8}{\pi^2}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sin\frac{(2k+1)\pi x}{2}.$$

To find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$, note that when $x = 1$ the function $f(1)$ is equal to 1 and its Fourier sine expansion is equal to $\frac{8}{\pi^2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin\frac{(2n+1)\pi}{2} = \frac{8}{\pi^2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}(-1)^n = \frac{8}{\pi^2}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$ so

$$\frac{8}{\pi^2}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = 1 \quad \Rightarrow \quad \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

Fourier Transformation

The Fourier transform is an integral operator meaning that it is defined via an integral and that it maps one function to the other. If you took a differential equations course, you may have encountered the Laplace transform which is another integral operator.

Recall that the complex Fourier series is $f(t)=\sum_{n=-\infty}^{\infty} c_n e^{\frac{2n\pi it}{T}}$. The sequence c_n can be regarded as a function of n and is called **Fourier spectrum of** $f(t)$. We can think of $c(n)$ being another representation of $f(t)$, meaning that $f(t)$ and $c(n)$ are different representations of the same object. Indeed: given $f(t)$ the coefficients $c(n)$ can be computed and, conversely, given $c(n)$, the Fourier series with coefficients $c(n)$ defines a function $f(t)$. We can plot $c(n)$ as a function of n (and get a set of infinitely many equally spaced points). In this case we think of c as a function of n, the **wave number**.

We can also think of c as a function of $\omega=\frac{2n\pi}{T}$, the **frequency**. If T is large, then ω is small and the function Tc_n becomes a continuous function of ω for $T\to\infty$. If we let $T\to\infty$, the requirement that $f(t)$ is periodic can be waved since the period becomes $(-\infty,\infty)$.

The **Fourier Transform** $F(\omega)$ of $f(t)$ is the limit of the continuous function $\frac{1}{\sqrt{2\pi}}Tc_n$ when $T\to\infty$. Since

$$\frac{1}{\sqrt{2\pi}}Tc_n=\frac{1}{\sqrt{2\pi}}\int_{-T/2}^{T/2} f(t)e^{\frac{-2n\pi it}{T}}\,dt=\frac{1}{\sqrt{2\pi}}\int_{-T/2}^{T/2} f(t)e^{-i\omega t}\,dt,$$

we have that

$$\frac{1}{\sqrt{2\pi}}Tc_n\to F(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$$

Let us consider now the Fourier series expansion of $f(t)$ with the substitution $\omega=\frac{2n\pi}{T}$. Two consecutive n values are length 1 apart so $dn=1$. Thus, $d\omega=\frac{2\pi}{T}dn\Rightarrow d\omega=\frac{2\pi}{T}$ and $\frac{Td\omega}{2\pi}=1$. Hence

$$f(t)=\sum_{n=-\infty}^{\infty} c_n e^{\frac{2n\pi it}{T}}=\sum_{T\omega/(2\pi)=-\infty}^{\infty} c_n e^{\omega it}\frac{Td\omega}{2\pi}=\frac{1}{\sqrt{2\pi}}\sum_{T\omega/(2\pi)=-\infty}^{\infty}\frac{Tc_n}{\sqrt{2\pi}}e^{\omega it}d\omega\to\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(\omega)e^{\omega it}d\omega.$$

Thus, for a given function $F(\omega)$ of ω, the above expression computes a function $f(t)$ of t. This formula computes the **Inverse Fourier Transform** $f(t)$ for a given function $F(\omega)$.

We can still think of $f(t)$ and $F(\omega)$ being the same representations of the same object:

$F(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$	$F(\omega)$ is the **Fourier transform** of $f(t)$ $F(\omega)\leftrightsquigarrow$ the coefficients of the Fourier series for $f(t)$
$f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$	$f(t)$ is the **inverse Fourier transform** of $F(\omega)$ $f(t)\leftrightsquigarrow$ the expansion having coefficients $F(\omega)$

The constant 2π in the numerator of the formula $\omega=\frac{2n\pi}{T}$ can be written as $2\pi=\sqrt{2\pi}\cdot\sqrt{2\pi}$ so that it can be evenly distributed in both formulas above resulting in each of the two formulas having the same constant $\frac{1}{\sqrt{2\pi}}$ in front of the integral. This makes the formulas for Fourier and inverse Fourier transforms symmetric and certain calculations easier.

One of the most important applications of Fourier transform is in **signal processing** with various applications in cryptography, acoustics, optics and other areas. In this case, one can think of $f(t)$ as of a signal which is measured in time and is represented as a function of time t but which needs to be represented as a function of frequency, not time. The Fourier transform produces exactly that – a representation of $f(t)$ as a function $F(\omega)$ of frequency ω. The function $F(\omega)$ is also known as the **frequency spectrum** of the signal.

Moreover, Fourier transform provides information on the amplitude and phase of a source signal at various frequencies. The transform $F(\omega)$ of a signal $f(t)$ can be written in polar coordinates as $F = |F|e^{\theta i}$. The modulus $|F|$ represents the **amplitude** of the signal at respective frequency ω, while θ computes the phase shift at frequency ω.

The Fourier transform is not limited to functions of time and temporal frequencies. It can be used to analyze spatial frequencies. If the independent variable in $f(t)$ stands for space instead of time, x is usually used instead of t. In some cases, the independent variable of the inverse transform is denoted by k.

$$t \longleftrightarrow x \qquad \text{and} \qquad \omega \longleftrightarrow k$$

We consider some applications in **magnetic resonance imaging** in the last section of this note.

Besides generalizing Fourier series, the mathematical importance of the Fourier transform lies in the following.

1. If the initial function $f(t)$ has the properties that are not desirable in a particular application (e.g. discontinuous, not smooth), we can consider the function $F(\omega)$ instead which is possible better behaved.

2. Fourier transform represents a function that is not necessarily periodic and that is defined on an infinite interval. The only requirement for the Fourier transform to exist is that the integral $\int_{-\infty}^{\infty} |f(t)|dt$ is convergent.

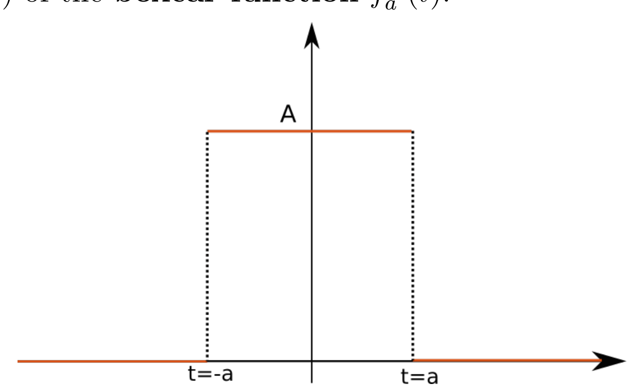
3. Fourier transform can be used similarly to Laplace transform: in converts a differential equation into an ordinary (algebraic) equation that is easier to solve. After solving it, the solution of the original differential equation can be obtained by using the inverse Fourier transform.

Example 1. Find the Fourier transform $F_a^A(\omega)$ of the **boxcar function** $f_a^A(t)$.

$$f_a^A(t) = \begin{cases} 0 & t \le -a \\ A & -a < t < a \\ 0 & t \ge a \end{cases}$$

Express your answer as real function.

The boxcar function $f_a^A(t)$ is said to be **normalized** if $A = \frac{1}{2a}$ so that the total area under the function is 1. We use f_a to denote the normalized boxcar function and F_a for its Fourier transform. Consider how changes in value of a impact the shape and values of $F_a(\omega)$.

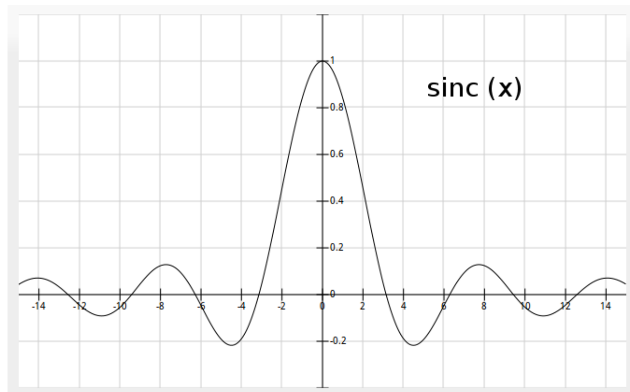


Solution. $F_a^A(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_a^A(t) e^{-i\omega t} dt$. Write the integral $\int_{-\infty}^{\infty}$ as a sum of the integrals $\int_{-\infty}^{-a} + \int_{-a}^{a} + \int_{a}^{\infty}$ to match the three branches of the function. Since $f_a^A(t) = 0$ for $t < -a$ and $t > a$, the first and the last integral of $f_a^A(t)e^{-i\omega t}$ are zero. For the middle branch, $f_a^A(t) = A$ so we have that

$$F_a^A(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} A e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}} \frac{A}{-i\omega} e^{-i\omega t}\Big|_{-a}^{a} = \frac{-A}{\sqrt{2\pi} i\omega} (e^{-ai\omega} - e^{ai\omega}).$$

Use the Euler's formula to obtain $e^{-ai\omega} - e^{ai\omega} = (\cos(-a\omega) + i \sin(-a\omega) - \cos a\omega - i \sin a\omega)$. Using that $\cos(-a\omega) = \cos a\omega$ since cosine is even and that $\sin(-a\omega) = -\sin a\omega$ since sine is an odd function, the cosine terms cancel and we obtain that

$$F_a^A(\omega) = \frac{-A}{\sqrt{2\pi} i\omega} (-2i \sin a\omega) = \frac{2A}{\sqrt{2\pi}} \frac{\sin a\omega}{\omega}.$$



The function $\frac{\sin x}{x}$ is known as the **sinc function**.

$$\operatorname{sinc} x = \frac{\sin x}{x}$$

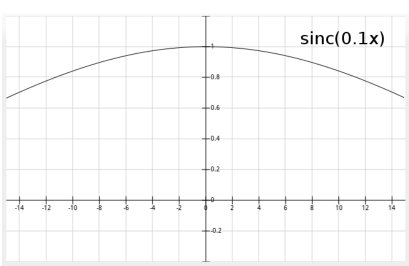
Note that $\lim_{x\to 0} \operatorname{sinc} x = 1$ and that $\lim_{x\to\infty} \operatorname{sinc} x = 0$.

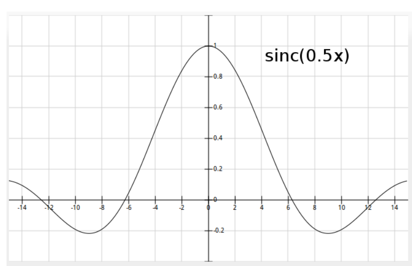
Since $\frac{\sin a\omega}{\omega} = a \frac{\sin a\omega}{a\omega}$, we can represent

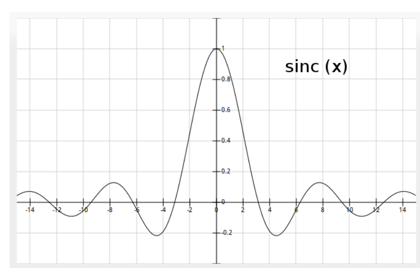
> the Fourier transform of the boxcar function $f_a^A(t)$ as the sinc function $F_a^A(\omega) = \frac{2aA}{\sqrt{2\pi}} \operatorname{sinc} a\omega$.

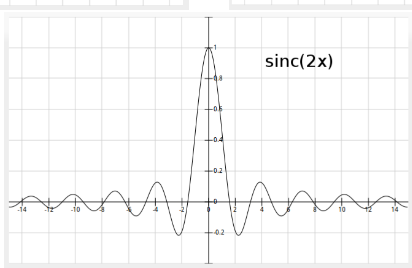
If the boxcar function is normalized, $A = \frac{1}{2a}$ so $F_a(\omega) = \frac{1}{\sqrt{2\pi}} \operatorname{sinc} a\omega$. Since sinc(0)=1, the peak of this function has value $\frac{1}{\sqrt{2\pi}}$.

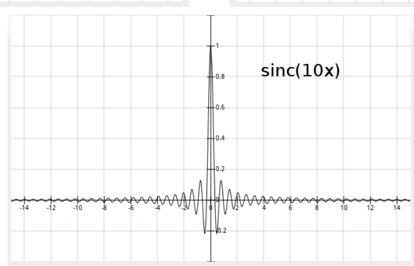
Let us compare the graphs of f_a and F_a for several different values of a. On the graphs below, you can notice that larger values of a correspond to f_a having smaller height and being more spread out. In this case, F_a has a narrow, sharp peak at $\omega = 0$ and converges to 0 faster.











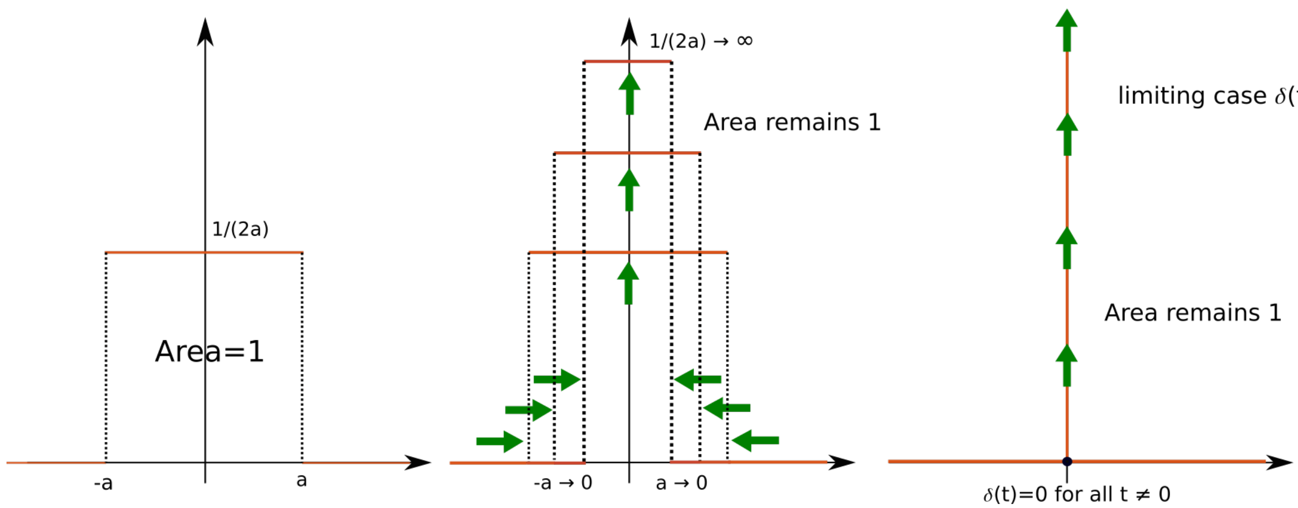
If a is small, f_a has large height and very narrow base. In this case, F_a has very spread out peak around $\omega = 0$ and the convergence to 0 is much slower.

In the limiting cases $a \to 0$ represents a constant function and $a \to \infty$ represents **the Dirac delta function** $\delta(t)$. This function, known also as the **impulse function** is used to represent phenomena of an impulsive nature. For example, voltages that act over a very short period of time. It is defined by

$$\delta(t) = \lim_{a\to 0} f_a(t).$$

Since the area under $f_a(t)$ is 1, the area under $\delta(t)$ is 1 as well. Thus, $\delta(t)$ is characterized by the following properties:

$$(1) \quad \delta(t) = 0 \text{ for all values of } t \neq 0 \qquad (2) \quad \int_{-\infty}^{\infty} \delta(t)dt = 1.$$



Since no ordinary function satisfies both of these properties, δ is not a function in the usual sense of the word. It is an example of a generalized function or a distribution. An alternate definition of the Dirac delta function can be found on Wikipedia.

We have seen that F_a for a small a is very spread out and almost completely flat. So, in the limiting case when $a = 0$, it becomes a constant. Thus, the Fourier transform of $\delta(t)$ can be obtained as limit of $F_a(\omega)$ when $a \to 0$. We have seen that this is a constant function passing $\frac{1}{\sqrt{2\pi}}$. Thus,

the Fourier transform of the delta function $\delta(t)$ is the constant function $F(\omega) = \frac{1}{\sqrt{2\pi}}$.

Conversely, when $a \to \infty$ $f_a \to \frac{1}{\sqrt{2\pi}}$. The Fourier transform of the constant function $\frac{1}{\sqrt{2\pi}}$ is the limit of F_a for $a \to \infty$. We have seen that this limit function is zero at all nonzero values of ω. So, we can relate this limit to $\delta(\omega)$. The exact relation will be clear after the next example.

Example 2. Find the inverse Fourier transforms of the boxcar function
$G_a^A(\omega) = \begin{cases} A, & -a < \omega < a \\ 0, & \text{otherwise} \end{cases}$ and of the Dirac delta function $\delta(\omega)$.

Solution. The inverse Fourier transform $g_a^A(t)$ of $G_a^A(\omega)$ can be computed by

$$g_a^A(t) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} Ae^{i\omega t} d\omega = \frac{A}{\sqrt{2\pi}it}(e^{iat} - e^{-iat})$$

To express this answer as a real function, use the Euler's formula and symmetries of sine and cosine functions similarly as in Example 1 to obtain the following.

$g_a^A(t) = \frac{A}{\sqrt{2\pi}it}(\cos at + i\sin at - \cos(-at) - i\sin(-at)) = \frac{A}{\sqrt{2\pi}it}(\cos at + i\sin at - \cos at + i\sin at) = \frac{A}{\sqrt{2\pi}it}(2i\sin at) = \frac{2A}{\sqrt{2\pi}t}\sin at = \frac{2aA}{\sqrt{2\pi}t}\frac{\sin at}{a} = \frac{2aA}{\sqrt{2\pi}}\frac{\sin at}{at} = \frac{2aA}{\sqrt{2\pi}}\text{sinc}\, at$. Thus, the inverse transform of the general boxcar function is the sinc function above. This also illustrates that

> the Fourier transform of the sinc function above is the general boxcar function.

Taking the limit when $a \to 0$ in the normalized case, we obtain that the inverse Fourier of the delta function is the limit of $\frac{1}{\sqrt{2\pi}}\text{sinc}\, at$ when $a \to 0$ which is the constant function $\frac{1}{\sqrt{2\pi}}$. Thus, the inverse Fourier transform of $\delta(\omega)$ to $\frac{1}{\sqrt{2\pi}}$. Hence, we have that

> the Fourier transform of the constant function $f(t) = \frac{1}{\sqrt{2\pi}}$ is the delta function $\delta(\omega)$.

Example 3. Consider the **Gaussian probability function** (a.k.a. the "bell curve") $f(t) = Ne^{-at^2}$ where N and a are constants. N determines the height of the peak and a determines how fast it decreases after reaching the peak. The Fourier transform of $f(t)$ can be found to be $F(\omega) = \frac{N}{\sqrt{2a}}e^{-\omega^2/4a}$. This is another Gaussian probability function. Examine how changes in a impact the graph of the transform. This consideration is related to Heisenberg uncertainty principle and has applications in quantum mechanics.

Solution. If a is small, f is flattened. In this case the presence of a in the denominator of the exponent of $F(\omega)$ will cause the F to be sharply peaked and the presence of a in the denominator of $\frac{N}{\sqrt{2a}}$, will cause F to have high peak value. If a is large, f is sharply peaked and F is flattened and with small peak value.

The following table summarizes our current conclusions.

Function in time domain	Fourier Transform in frequency domain
boxcar function	sinc function
sinc function	boxcar function
delta function	constant function
constant function	delta function
Gaussian function	Gaussian function

Symmetry considerations

The table on the right displays the formulas of Fourier and inverse Fourier transform for even and odd functions.

$f(t)$ even $F(\omega)$ even	$F(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \omega t \, dt$ $f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(\omega) \cos \omega t \, d\omega$
$f(t)$ odd $F(\omega)$ odd	$F(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin \omega t \, dt$ $f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(\omega) \sin \omega t \, d\omega$

Fourier sine and cosine functions. Suppose that $f(t)$ is defined just for $t > 0$. We can extend $f(t)$ so that it is even. Then we get the formula for $F(\omega)$ by using the formulas for even function above. $F(\omega)$ is then called **Fourier cosine transformation.**

Similarly, if we extend $f(t)$ so that it is odd, we get the formula for $F(\omega)$ same as for an odd function above. $F(\omega)$ is then called **Fourier sine transformation.**

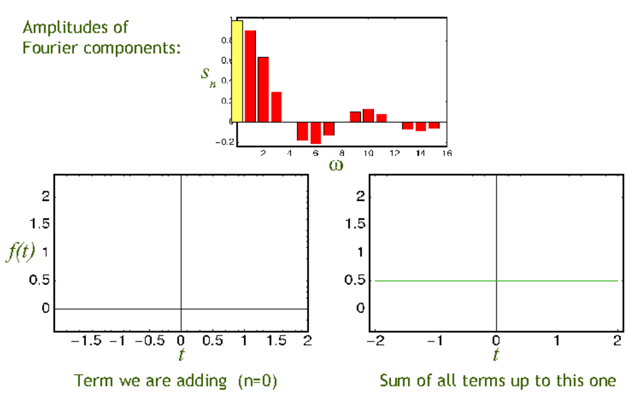
Fourier series and Fourier transform

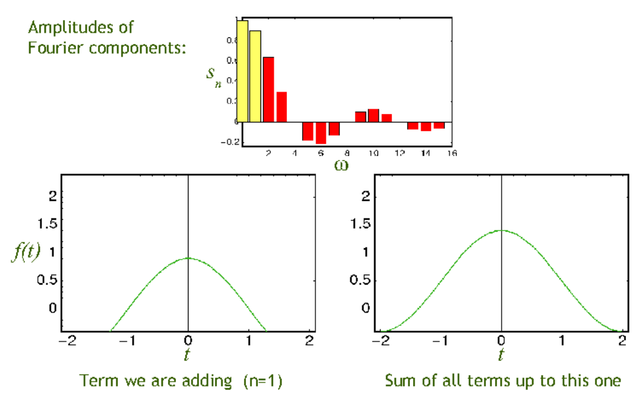
We have seen that the Fourier transform of a boxcar function is a sinc function so that the inverse Fourier transform of a sinc function is a boxcar function. The following series of figures illustrates the connection between a boxcar function $f(t)$, the sinc function $F(\omega)$ (the inverse Fourier transform of $f(t)$), and the coefficients of the Fourier series expansion of the boxcar function $f(t)$.

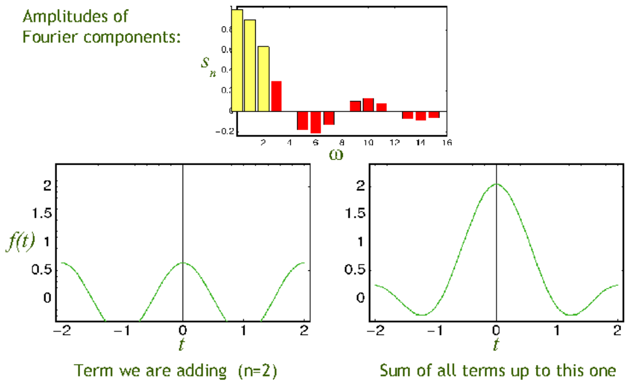
The length of the base of the box determines the sampling period T. The spacing $\omega_0 = \frac{2\pi}{T}$ in the frequency-domain determines the value of ω since $\omega = \frac{2n\pi}{T} \quad \Rightarrow \quad \omega = n\omega_0$.

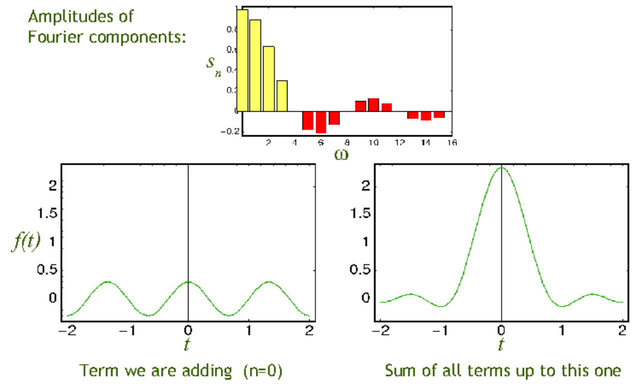
In the following figures, s_n denotes the Fourier coefficient of the complex Fourier series. We illustrate how adding sufficiently many terms of the Fourier series results in the formation of the shape of the initial boxcar function. We consider the boxcar function with $a = \frac{1}{2}$ (so $T = 1$, $\omega_0 = 2\pi$ and $s_n = \text{sinc}\frac{\omega}{2}$). The first graph on each figure displays the Fourier transform, the sinc function, in the frequency-domain. The highlighted frequency on the first graph determines the value of n. The second graph displays the harmonic function corresponding to n-th term of the Fourier series of the function $f(t)$ in the time-domain. The third graph is the sum of the first n terms of the Fourier series of $f(t)$.

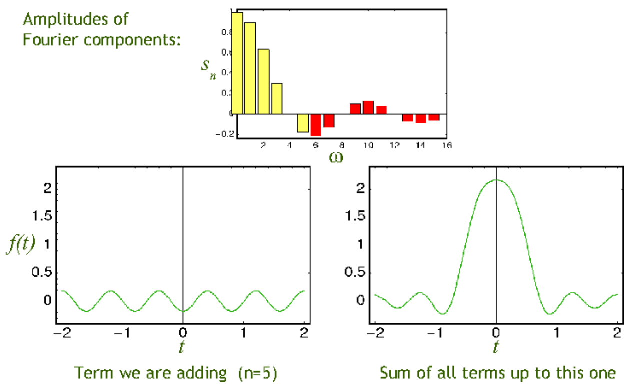
Note that as $n \to \infty$ the sum of Fourier series terms converge to the boxcar function.

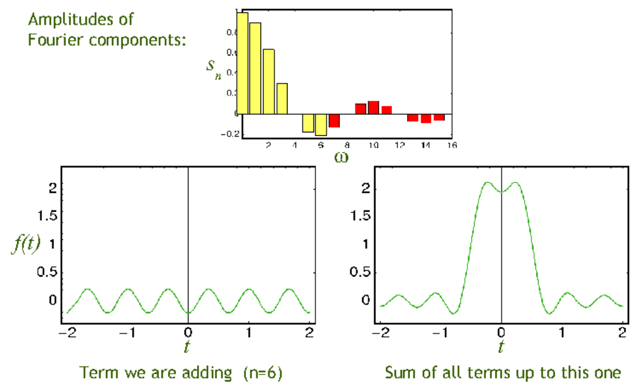


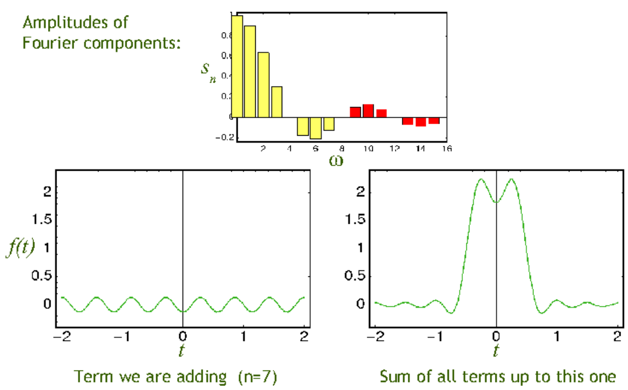


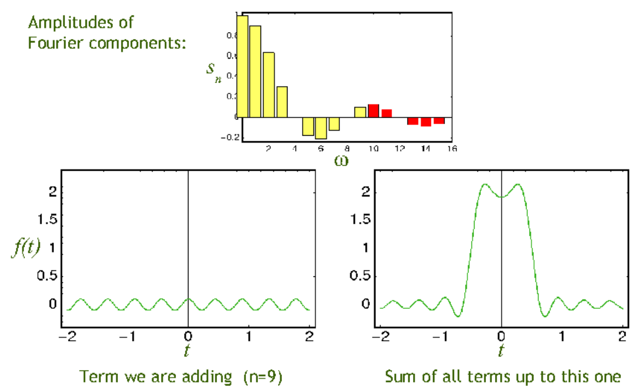

Amplitudes of
Fourier components:
ω
f(t)
t
Term we are adding (n=2)
Sum of all terms up to this one

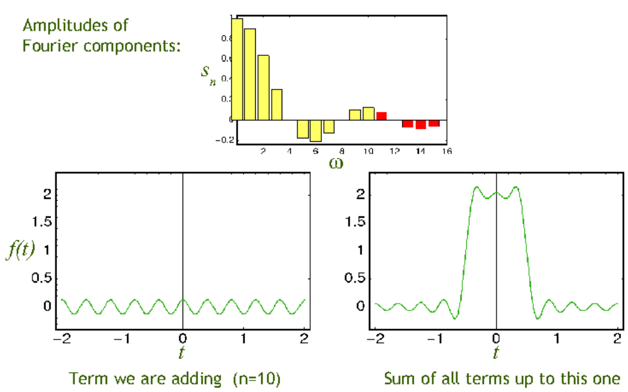

Amplitudes of
Fourier components:
ω
f(t)
t
Term we are adding (n=0)
Sum of all terms up to this one

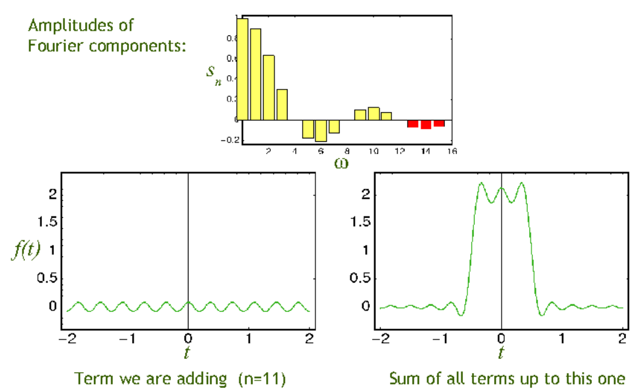

Amplitudes of
Fourier components:
ω
f(t)
t
Term we are adding (n=5)
Sum of all terms up to this one

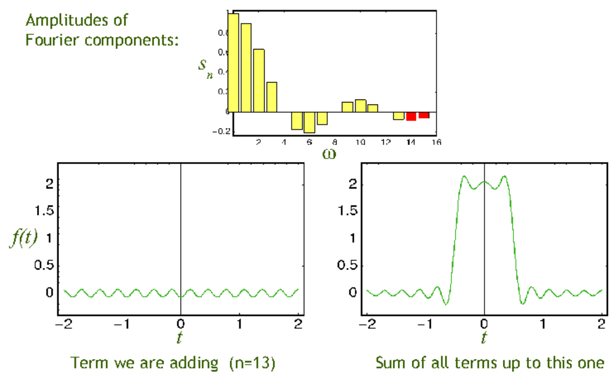

Amplitudes of
Fourier components:
ω
f(t)
t
Term we are adding (n=6)
Sum of all terms up to this one

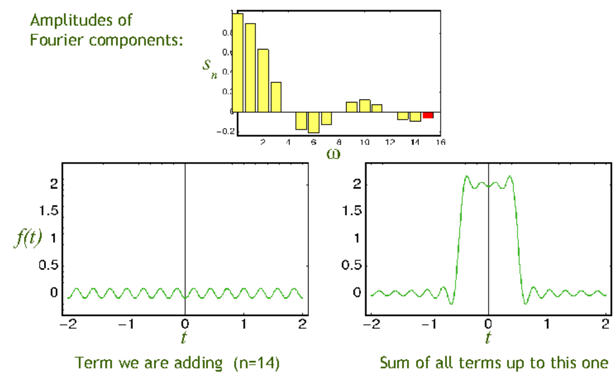

Amplitudes of
Fourier components:
ω
f(t)
t
Term we are adding (n=7)
Sum of all terms up to this one

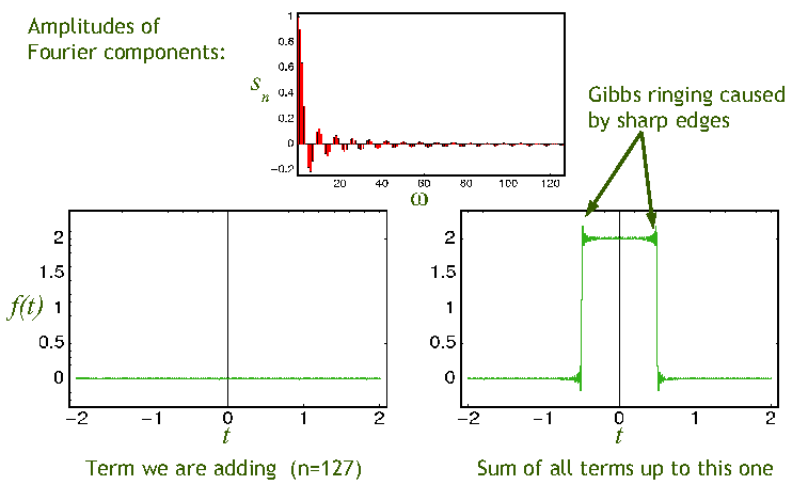

Amplitudes of
Fourier components:
ω
f(t)
t
Term we are adding (n=9)
Sum of all terms up to this one


Amplitudes of
Fourier components:
ω
f(t)
t
Term we are adding (n=10)
Sum of all terms up to this one


Amplitudes of
Fourier components:
ω
f(t)
t
Term we are adding (n=11)
Sum of all terms up to this one





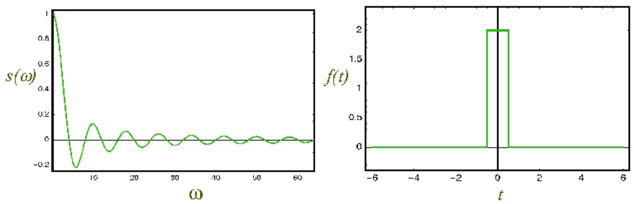


In the previous examples, the function in frequency-domain was a continuous function. In applications, it is impossible to collect infinitely many infinitely dense samples. As a consequence, certain error may come from sampling just finite number of points taken over a finite interval. Thus, it is relevant to keep in mind what effects in time (resp. frequency) domain may have finite and discrete samples of frequencies (resp. time).

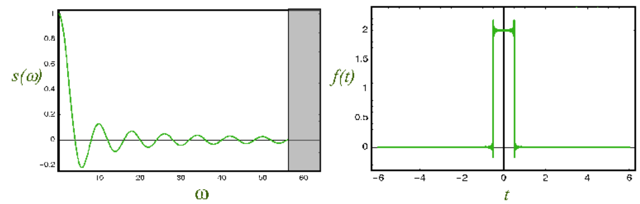
frequency-domain	function in time-domain	effects in time-domain
infinite continuous set of frequencies	non-periodic function	none
finite continuous set of frequencies	non-periodic function	Gibbs ringing, blurring
infinite number of discrete frequencies	periodic function	aliasing
finite number of discrete frequencies	periodic function	aliasing, Gibbs ringing, blurring

Assume that the sample consists of frequencies and that all the values lie on the graph of a sinc function. The goal is to reconstruct the boxcar function in the time-domain. The following four figures illustrate each of the four scenarios from the table.

If we sum an **infinite range of continuous frequencies** we get an exact version of the initial function *f(t)*

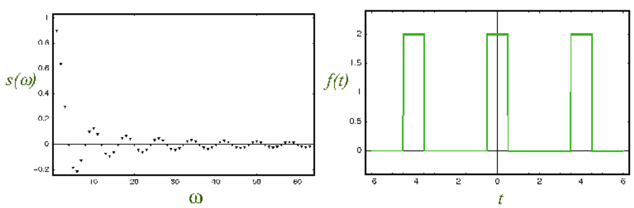


If we sum a **finite range of continous frequencies** we get an approximate version of the function *f(t)* that is **blurred** and has **Gibbs ringing**

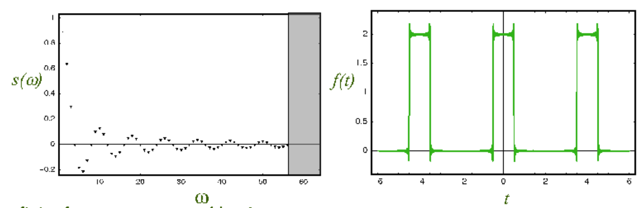


If we sum an **infinite range of discrete frequencies** we get an exact version of the function *f(t)* repeated an infinite number of times

The infinite repetition of *f(t)* is called **aliasing**



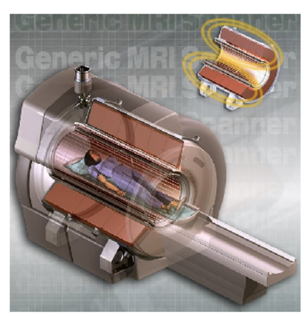
If we sum a **finite number of discrete frequencies** we get an **infinite number** of approximate copies of the function *f(t)* that are **blurred** and have **Gibbs ringing**



finite frequency range → **blurring**
sharp edges → **Gibbs ringing**
discrete frequencies → **aliasing**

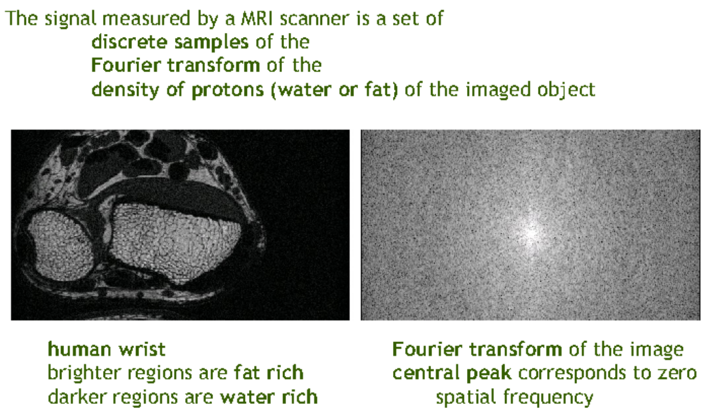
Application in Magnetic Resonance Imaging

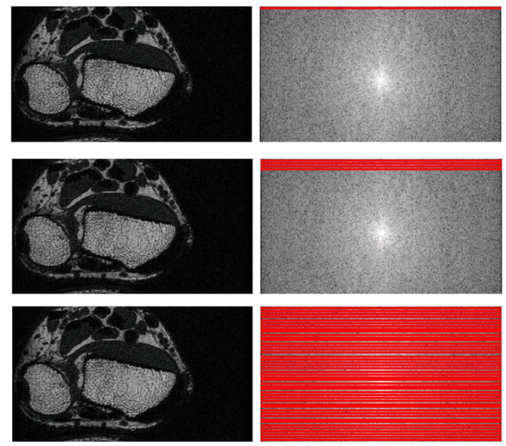
A MRI scanner consists of a large **superconducting magnet** and **coils** that generate **temporally and spatially varying magnetic fields** (no ionizing radiation)

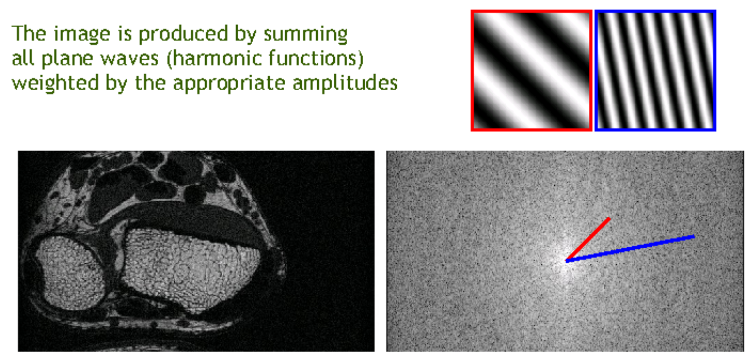


The Fourier transform is prominently used in Magnetic Resonance Imaging (MRI). The fist figure represents a typical MRI scanner. The scanner samples spacial frequencies and creates the Fourier transform of the image we would like to obtain. The inverse Fourier transform converts the measured signal (input) into the reconstructed image (output).

In the following two figures, the image on the right represents the amplitude of the scanned input signal. The image on the left represents the output - the image of a human wrist, more precisely, the density of protons in a human wrist. The whiter area on the image correspond to the fat rich regions. The darker area on the image correspond to the water rich regions.





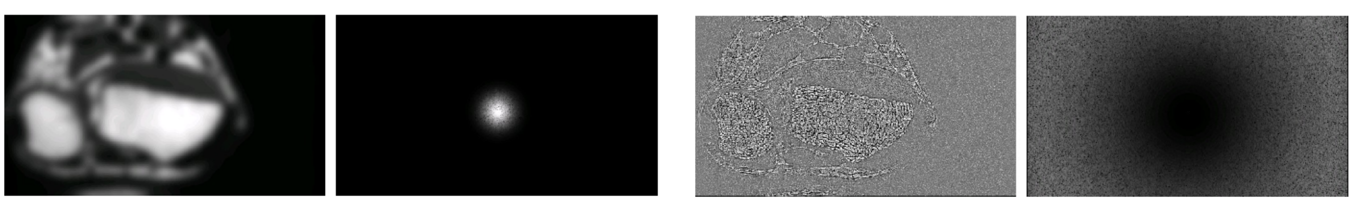


Each point on the input images corresponds to certain frequency. Two smaller figures on the right side represent the components of the inverse transforms at two highlighted frequencies. The output image is created by combining many such images - one for each sampled frequency in fact.

The last figure represent the output with high (first pair of images) and low frequencies (second pair of images) removed.

If high frequencies are removed (low-pass filter), the image becomes blurred and only shows the rough shape of the object.

If low frequencies are removed (high-pass filter), the image is sharp but intensity variations are lost.



Practice Problems.

1. Find the Fourier transform of $f(t) = e^{-t}$, $t > 0$, $f(t) = 0$ otherwise.
2. Find Fourier cosine transformation of the function from the previous problem.
3. Find cosine Fourier transform of $f(t) = 2t - 3$ for $0 < t < 3/2$, $f(x) = 0$ otherwise.

4. If

$$f_a(t) = \frac{a}{a^2+t^2}$$

the graph of f has a peak at 0. Since $f(0) = \frac{1}{a}$, the height is conversely proportional to a. The Fourier transform of f can shown to be

$$F_a(\omega) = \sqrt{\pi/2}e^{-a|\omega|}.$$

Graph $F_a(\omega)$ for several values of a and make conclusion how values of a impact the graph of F_a.

5. Solve the equation $\int_0^\infty f(t)\cos t\omega\, dt = \begin{cases} 1-\omega & 0 \le \omega \le 1 \\ 0 & \omega > 1 \end{cases}$

Solutions.

1. $F(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t}e^{-i\omega t}dt = \frac{-1}{\sqrt{2\pi}(1+i\omega)}e^{-(1+i\omega)t}\Big|_0^\infty = \frac{1}{\sqrt{2\pi}(1+i\omega)}$. Rationalizing the denominator, you obtain $F(\omega) = \frac{1-i\omega}{\sqrt{2\pi}(1+\omega^2)}$

2. $F(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-t}\cos\omega t\; dt$. Using two integration by parts with $u = e^{-t}$, one obtains that $F(\omega) = \sqrt{\frac{2}{\pi}}\frac{1}{\omega}e^{-t}\sin\omega t\Big|_0^\infty + \sqrt{\frac{2}{\pi}}\frac{1}{\omega}\int_0^\infty e^{-t}\sin\omega t\; dt =$
$0 - \sqrt{\frac{2}{\pi}}\frac{1}{\omega^2}e^{-t}\cos\omega t\Big|_0^\infty - \sqrt{\frac{2}{\pi}}\frac{1}{\omega^2}\int_0^\infty e^{-t}\cos\omega t\; dt$. Note that the first term is equal to $\sqrt{\frac{2}{\pi}}\frac{1}{\omega^2}$ and the second is $-\frac{1}{\omega^2}F(\omega)$. Hence,

$$F(\omega) = \sqrt{\frac{2}{\pi}}\frac{1}{\omega^2} - \frac{1}{\omega^2}F(\omega) \Rightarrow F(\omega)(1+\frac{1}{\omega^2}) = \sqrt{\frac{2}{\pi}}\frac{1}{\omega^2}.$$

Multiply by ω^2 to get

$$F(\omega)(\omega^2+1) = \sqrt{\frac{2}{\pi}} \quad \Rightarrow \quad F(\omega) = \sqrt{\frac{2}{\pi}}\,\frac{1}{\omega^2+1}.$$

3. $F(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{3/2}(2t-3)\cos\omega t\; dt = \sqrt{\frac{2}{\pi}}(\frac{2t-3}{\omega}\sin\omega t\Big|_0^{3/2} - \frac{2}{\omega}\int_0^{3/2}\sin\omega t\; dt) =$
$\sqrt{\frac{2}{\pi}}(0 + \frac{2}{\omega^2}\cos\omega t\Big|_0^{3/2}) = \sqrt{\frac{2}{\pi}}\frac{2}{\omega^2}(\cos\frac{3\omega}{2} - 1) = \frac{2\sqrt{2}}{\omega^2\sqrt{\pi}}(\cos\frac{3\omega}{2} - 1)$.

4. If a is small, then f_a has a larger peak. In this case F_a is more spread out and flattened. If a is large, f_a is spread out and the height of the peak is not large. In this case, F_a has a sharp peak and converges to zero fast.

5. Use inverse cosine Fourier transform. First multiply with $\sqrt{\frac{2}{\pi}}$ to match the definition of the transform. Then the left side is exactly the Fourier cosine transform of $f(t)$. So we can get $f(t)$ as the inverse transform of the left side of the equation multiplied by $\sqrt{\frac{2}{\pi}}$.

$f(t) = \sqrt{\frac{2}{\pi}}\sqrt{\frac{2}{\pi}}\int_0^1(1-\omega)\cos\omega t\; d\omega = \frac{2}{\pi}(\frac{1-\omega}{t}\sin\omega t\Big|_0^1 + \frac{1}{t}\int_0^1\sin\omega t\; dt) = \frac{2}{\pi}(0 - \frac{1}{t^2}\cos\omega t\Big|_0^1) = \frac{2}{\pi t^2}(-\cos t + 1) = \frac{2}{\pi t^2}(1-\cos t)$.