Math Methods 1
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Review for Exam 1

1. **Surface Integrals**. Evaluate the given surface integrals.
   (a) $\int\int_S \, dS$ where S is a part of the cone $z = \sqrt{x^2+y^2}$ that lies between the cylinders $x^2+y^2=4$ and $x^2+y^2=9$.
   (b) $\int\int_S yz \, dS$ where S is the part of the plane $z = y+3$ that lies inside the cylinder $x^2+y^2=1$.

2. **Flux integrals with or without using the Divergence Theorem.** Find the flux integrals of the given vector fields over the specified surfaces.
   (a) $\mathbf{f} = (y, x, z)$ over the part of the paraboloid $z = 1 - x^2 - y^2$ above the plane $z = 0$.
   (b) $\mathbf{f} = (y, x, z)$ over the boundary of the region enclosed by the paraboloid $z = 1-x^2-y^2$ and the plane $z = 0$. (i) Without the Divergence Theorem; (ii) Using the Divergence Theorem.
   (c) $\mathbf{f} = (x, 2y, 3z)$ over the cube with vertices $(\pm 1, \pm 1, \pm 1)$. (i) Without using the Divergence Theorem; (ii) Using the Divergence Theorem.

3. **Line integrals with and without using Stokes Theorem.** Find the work done by the given vector field acting over the specified positive oriented curve C.
   (a) $\mathbf{f} = (-y, x, x^2+y^2)$, C is the boundary of the part of the paraboloid $z = 4 - x^2 - y^2$ in the first octant. (i) Without using Stokes' Theorem; (ii) Using Stokes' Theorem.
   (b) $\mathbf{f} = (-y^2, x, z^2)$ and C is the intersection of the plane $y+z = 2$ and the cylinder $x^2+y^2 = 1$. (i) Without using Stokes' Theorem; (ii) Using Stokes' Theorem.
   (c) $\mathbf{f} = (x + z^2, y + x^2, z + y^2)$, C is the boundary of the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies in the first octant. Use Stokes' Theorem.

4. **Flux integrals in cylindrical or spherical coordinates.** Compute the flux done by the given vector field acting over the specified S.
   (a) $\mathbf{f} = r\sin\phi \, \hat{\mathbf{r}}_r + r\cos\phi \, \hat{\mathbf{r}}_\theta$, S is the unit sphere, (i) directly, (ii) using Divergence Theorem.
   (b) $\mathbf{f} = r\sin\theta\hat{\mathbf{r}}_r + 2r\cos\theta\hat{\mathbf{r}}_\theta + 3rz\hat{\mathbf{r}}_z$, S is the boundary of the region inside the cylinder $x^2 + y^2 = 9$ between the planes $z = 0$ and $z = 4$. (i) Without using the Divergence Theorem; (ii) Using the Divergence Theorem.
   (c) $\mathbf{f} = r^2\hat{\mathbf{r}}_r + 2r^2\sin\phi\hat{\mathbf{r}}_\phi + r^4\sin\phi\hat{\mathbf{r}}_\theta$, S is the boundary of the region inside the cone $z = \sqrt{3(x^2+y^2)}$ and the sphere of radius a. Use the Divergence Theorem.

5. **Line integrals in cylindrical or spherical coordinates.** Find the work done by the given vector field acting over the specified positive oriented curve C.
   (a) $\mathbf{f} = 2r\hat{\mathbf{r}}_r + 3r^2z\hat{\mathbf{r}}_\theta - z\hat{\mathbf{r}}_z$, C is the circle $x^2 + y^2 = a^2$ in the horizontal plane $z = b$ where a and b are positive constants, (i) directly, (ii) using Stokes' Theorem.
   (b) $\mathbf{f} = r^2\hat{\mathbf{r}}_r + r^2\sin\phi\hat{\mathbf{r}}_\phi + 2\hat{\mathbf{r}}_\theta$, C is the intersection of the sphere of radius R centered at the origin and the horizontal plane $z = \frac{1}{2}R$, (i) directly, (ii) using Stokes' Theorem.

Solutions.

More detailed solutions can be found on the class handout.

1. Surface integrals.

   (a) $\mathbf{r} = (r\cos t, r\sin t, r) \Rightarrow \mathbf{r}_r = (\cos t, \sin t, 1)$ and $\mathbf{r}_t = (-r\sin t, r\cos t, 0) \Rightarrow \mathbf{r}_r \times \mathbf{r}_t = (-r\cos t,\ -r\sin t, r) \Rightarrow |\mathbf{r}_r \times \mathbf{r}_t| = \sqrt{r^2\cos^2 t + r^2\sin^2 t + r^2} = \sqrt{r^2+r^2} = \sqrt{2r^2} = \sqrt{2}r$. The bounds are $0 \le t \le 2\pi$ and $2 \le r \le 3$. So, $S = \int_0^{2\pi} dt\ \int_2^3 \sqrt{2}rdr = 2\pi\sqrt{2}(\frac{9}{2} - \frac{4}{2}) = 5\pi\sqrt{2}$.

   (b) $\mathbf{r} = (x, y, y+3) \Rightarrow dS = \sqrt{2}dxdy$. $\int\int y(y+3)\sqrt{2}dxdy =$ (using polar coordinates) $= \int_0^{2\pi}\int_0^1\ r\sin t(r\sin t + 3)\ \sqrt{2}rdrdt = \sqrt{2}\int_0^{2\pi}\sin t(\frac{1}{4}\sin t + 1) = \sqrt{2}\frac{\pi}{4}$. Alternatively, $\mathbf{r} = (r\cos t, r\sin t, r\sin t + 3) \Rightarrow |\mathbf{r}_r \times \mathbf{r}_t| = \sqrt{2}r$. The integral is $\int_0^{2\pi}\int_0^1 r\sin t(r\sin t + 3)r\sqrt{2}\ drdt = \sqrt{2}\frac{\pi}{4}$.

2. Flux integrals.

   (a) $\mathbf{r} = (r\cos\theta, r\sin\theta, 1-r^2) \Rightarrow d\mathbf{S} = (2r^2\cos\theta, 2r^2\sin\theta, r)drd\theta$ and $\mathbf{f}\cdot d\mathbf{S} = (4r^3\sin\theta\cos\theta + r - r^3)drd\theta$. The bounds are $0 \le \theta \le 2\pi$ and $0 \le r \le 1$. The integral is $\int\int_S \mathbf{f}d\mathbf{S} = \int_0^{2\pi}\int_0^1(4r^3\sin\theta\cos\theta + r - r^3)drd\theta = \int_0^{2\pi}(\sin\theta\cos\theta + \frac{1}{2} - \frac{1}{4})d\theta = \frac{\pi}{2}$.

   (b) (i) The flux integral is the sum of the integrals over the paraboloid and over the plane $z = 0$. The first integral is $\frac{\pi}{2}$ by the previous problem. The second integral is $\int\int_S(y, x, 0)\cdot(0, 0, -1)dxdy = \int\int 0dxdy = 0$. So, the total is $\frac{\pi}{2}$.

   (ii) $\text{div}\mathbf{f} = 1$. Thus, $\int\int_S \mathbf{f}\cdot d\mathbf{S} = \int\int\int_V 1dxdydz = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} rdzdrd\theta = \int_0^{2\pi}\int_0^1(1 - r^2)rdrd\theta = 2\pi(\frac{1}{2} - \frac{1}{4}) = \frac{\pi}{2}$.

   (c) (i) The cube consists of 6 sides. On top and bottom $z = \pm 1$ and $-1 \le x, y \le 1$ so $\int\int_S(x, 2y, \pm 3)\cdot(0, 0, \pm 1)dxdy = \int_{-1}^1\int_{-1}^1 3dxdy = 12$. On the left and right $y = \pm 1$ and $-1 \le x, z \le 1$ so $\int\int_S(x, \pm 2, 3z)\cdot(0, \pm 1, 0)dxdz = \int_{-1}^1\int_{-1}^1 2dxdz = 8$. On the front and back $x = \pm 1$ and $-1 \le y, z \le 1$ so $\int\int_S(\pm 1, 2y, 3z)\cdot(\pm 1, 0, 0)dydz = \int_{-1}^1\int_{-1}^1 1dydz = 4$. Thus, the total flux is 2(12+8+4)=48.

   (ii) It is **much** easier to evaluate the integral using the Divergence Theorem. $\text{div}\mathbf{f} = 1+2+3 = 6$ and so $\int\int_S \mathbf{f}\cdot d\mathbf{S} = \int\int\int_V 6dxdydz = \int_{-1}^1\int_{-1}^1\int_{-1}^1 6dxdydz = 6x|_{-1}^1 y|_{-1}^1 z|_{-1}^1 = 6(2)^3 = 48$.

3. Line integrals.

   (a) (i) Without Stokes: The boundary of the part of the paraboloid $z = 4 - x^2 - y^2$ in the first octant consists of three curves, C_1 in xy-plane, C_2 in yz-plane, and C_3 in xz-plane.

   C_1 C_1 in is the xy-plane $z = 0 \Rightarrow 0 = 4 - x^2 - y^2 \Rightarrow x^2 + y^2 = 4$ which has parametric equations $x = 2\cos t, y = 2\sin t, z = 0$. The bounds are $0 \le t \le \frac{\pi}{2}$. Thus, $dx = -2\sin tdt, dy = 2\cos tdt, dz = 0$. and $\int_{C_1} -ydx + xdy + (x^2+y^2)dz = \int_0^{\pi/2}(4\sin^2 t + 4\cos^2 t + 4(0))dt = \int_0^{\pi/2} 4dt = 2\pi$.

   C_2 C_2 is in the yz-plane $x = 0 \Rightarrow z = 4 - 0^2 - y^2 \Rightarrow z = 4 - y^2 \Rightarrow x = 0, y = y, z = 4 - y^2 \Rightarrow dx = 0, dy = dy, dz = -2ydy$. The bounds are 2 to 0 and $\int_{C_2} -ydx + xdy + (x^2+y^2)dz = \int_2^0(0 + 0 + (0 + y^2)(-2y)dy = \int_2^0 -2y^3dy = \frac{-y^4}{4}\Big|_2^0 = 8$.

C_3 C_3 is in the xz-plane $y=0 \Rightarrow z = 4-x^2-0^2 \Rightarrow z = 4-x^2 \Rightarrow x = x, y = 0, z = 4-x^2 \Rightarrow dx = dx, dy = 0, dz = -2xdx$. The bounds are 0 to 2 and $\int_{C_3} -ydx + xdy + (x^2 + y^2)dz = \int_0^2 (0+0+(x^2+0^2)(-2x)dx = \int_0^2 -2x^3dx = \frac{-x^4}{4}\Big|_0^2 = -8$

The total work is the sum of work done along C_1, C_2, and C_3. Hence $W = \int_{C_1} + \int_{C_2} + \int_{C_3} \mathbf{f} \cdot d\mathbf{r} = 2\pi + 8 - 8 = 2\pi$.

(ii) Using Stokes: curl$\mathbf{f} = (2y, -2x, 2)$. The paraboloid can be parametrized by $\mathbf{r} = (r\cos\theta, r\sin\theta, 4-r^2) \Rightarrow d\mathbf{S} = (2r^2\cos\theta, 2r^2\sin\theta, r)drd\theta$. The bounds are $0 \le \theta \le \frac{\pi}{2}$ and $0 \le r \le 2$. curl$\mathbf{f}d\mathbf{S} = 2rdrd\theta$. Thus,

$$\int_C \mathbf{f} \cdot d\mathbf{r} = \int\int_S \text{curl}\mathbf{f}d\mathbf{S} = \int_0^{\pi/2}\int_0^2 2rdrd\theta = 2\frac{\pi}{2}\frac{r^2}{2}\Big|_0^2 = 2\frac{\pi}{2}(2) = 2\pi.$$

(b) Without Stokes: C has parametrization $x = \cos t$, $y = \sin t$, $z = 2 - \sin y$, $0 \le t \le 2\pi$. $\int_C \mathbf{f} \cdot d\mathbf{r} = \int_C -y^2dx + xdy + z^2dz = \int_0^{2\pi} \sin^3 tdt + \cos^2 tdt + (2-\sin t)^2 \cos tdt = \pi$.
With Stokes: curl$\mathbf{f} = (0, 0, 1+2y)$. $\mathbf{r} = (x, y, 2-y) \Rightarrow d\mathbf{S} = (0,1,1)dxdy$. $\int\int_S \text{curl}\mathbf{f}d\mathbf{S} = \int\int_S (1+2y)dxdy = \int_0^{2\pi}\int_0^1 (1+2r\sin t)rdrdt = \int_0^{2\pi}(\frac{1}{2} + \frac{2}{3}\sin t)dt = \pi$.

(c) curl$\mathbf{f} = (2y, 2z, 2x)$. In spherical coordinates, $\mathbf{r} = (2\cos\theta\sin\phi, 2\sin\theta\sin\phi, 2\cos\phi) \Rightarrow d\mathbf{S} = (4\sin^2\phi\cos\theta,\ 4\sin^2\phi\sin\theta,\ 4\sin\phi\cos\phi)d\phi d\theta$. $\int\int_S \text{curl}\mathbf{f}d\mathbf{S} = 16\int_0^{\pi/2}\int_0^{\pi/2}(\sin^3\phi\cos\theta\sin\theta + \sin^2\phi\sin\theta\cos\phi + \sin^2\phi\cos\phi\cos\theta)d\phi d\theta = 16\int_0^{\pi/2}(\frac{1}{2}\sin^3\phi + 2\sin^2\phi\cos\phi)\,d\phi\,d\theta = 16$.

4. Flux integrals in cylindrical or spherical coordinates.

(a) (i) $\mathbf{f} \cdot d\mathbf{S} = \sin^2\phi d\phi d\theta$. $F = \int\int_S \mathbf{f} \cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi}\sin^2\phi d\phi d\theta = 2\pi\int_0^{\pi}\sin^2\phi d\phi = 2\pi\,\frac{\pi}{2} = \pi^2 \approx 9.87$. (ii) div$\mathbf{f} = 3\sin\phi$. $F = \int\int\int_V 3\sin\phi\, r^2\sin\phi\, dr\, d\theta\, d\phi = 2\pi\,\frac{\pi}{2} = \pi^2 \approx 9.87$.

(b) (i) $F = F_1 + F_2 + F_3 = 0 + 216\pi + 0 = 216\pi$. (ii) div$\mathbf{f} = 3r$. $F = \int\int\int_V 3r\ r\ dr\, d\theta\, dz = \int_0^{2\pi} d\theta \int_0^3 3r^2\ dr \int_0^4\ dz = 2\pi\ 3^3\, 4 = 216\pi$.

(c) div$\mathbf{f} = 4r(1+\cos\phi)$. $F = \int\int\int_V 4r(1+\cos\phi)\ r^2\sin\phi\ dr\, d\theta\, d\phi = \int_0^{2\pi}\int_0^{\pi/6}\int_0^a 4r^3(1+\cos\phi)\sin\phi\ dr\, d\theta\, d\phi = 2a^4\pi\ \ (-\cos\phi - \frac{\cos^2\phi}{2})\Big|_0^{\pi/6} = 2a^4\pi\left(\frac{9}{8} - \frac{\sqrt{3}}{2}\right) \approx 1.63a^4$

5. Line integrals in cylindrical or spherical coordinates.

(a) (i) On the circle, $r = a, \theta = \theta$, and $z = b$. Since r and z are constant, $dr = 0$ and $dz = 0$ so $d\mathbf{r} = rd\theta\hat{\mathbf{r}}_\theta$. $\mathbf{f} \cdot d\mathbf{r} = (2r\hat{\mathbf{r}}_r + 3r^2z\hat{\mathbf{r}}_\theta - z\hat{\mathbf{r}}_z) \cdot rd\theta\hat{\mathbf{r}}_\theta = 3r^3zd\theta = 3a^3bd\theta$. The bounds for θ are 0 and 2π so $\int_c \mathbf{f} \cdot d\mathbf{r} = \int_0^{2\pi} 3a^3bd\theta = 3a^3b\theta|_0^{2\pi} = 6a^3b\pi$. (ii) curl$\mathbf{f} = -3r^2\hat{\mathbf{r}}_r + 9rz\hat{\mathbf{r}}_z$. If S is the horizontal plane $z = b$ in which the circle lies, then z is constant, so $dz = 0$ and $d\mathbf{S} = rdrd\theta\hat{\mathbf{r}}_z$. Hence $\mathbf{f} \cdot d\mathbf{S} = 9rz\,rdrd\theta = 9b\,r^2drd\theta$. The bounds are $0 \le \theta \le 2\pi$ and $0 \le r \le a$ so $\int_c \mathbf{f} \cdot d\mathbf{r} = \int\int_S \text{curl}\mathbf{f} \cdot d\mathbf{S} = 9b\int_0^{2\pi} d\theta \int_0^a r^2dr = 9b\,2\pi\,\frac{a^3}{3} = 6a^3b\pi$.

(b) (i) $d\mathbf{r} = \frac{\sqrt{3}R}{2}d\theta\hat{\mathbf{r}}_\theta \Rightarrow \mathbf{f} \cdot d\mathbf{r} = \sqrt{3}Rd\theta$. $W = \oint_C \mathbf{f} \cdot d\mathbf{r} = \sqrt{3}R\int_0^{2\pi} d\theta = 2\sqrt{3}R\pi$. (ii) $d\mathbf{S} = r^2\sin\phi d\theta d\phi\hat{\mathbf{r}}_r$, curl$\mathbf{f} = \frac{1}{r^2\sin\phi}(2r\cos\phi\hat{\mathbf{r}}_r - 2r\sin\phi\hat{\mathbf{r}}_\phi + 3r^3\sin^2\phi\hat{\mathbf{r}}_\theta) \Rightarrow$ curl$\mathbf{f} \cdot d\mathbf{S} = 2R\cos\phi d\theta d\phi$. $W = \int\int$ curl$\mathbf{f} \cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\frac{\pi}{3}} 2R\cos\phi d\theta d\phi = 4R\pi\int_0^{\pi/3}\cos\phi d\phi = 2\sqrt{3}R\pi$.