

Review for Exam 2

1. Analytic Functions. Complex integrals. Cauchy's Integral Formula.

(a) Check if the following functions are analytic.

(i) $f(z) = ze^{5iz^2}$, (ii) $f(x + iy) = x^2y + ixy^2$, (iii) $f(x + iy) = x^2 - y^2 + 2xyi$.

(b) Check if analytic functions with real part equal to the given functions u exist. If so, find all analytic functions that have real parts equal to u .

(i) $u = xe^{3y}$ (ii) $u = x^2 + 3x - y^2 + 5y$.

(c) Evaluate $\int \operatorname{Re}(z) dz$ over the line segment $x = 1$ from $(1,0)$ to $(1,1)$ and the line segment $y = 1$ from $(1,1)$ to $(0,1)$.

(d) Evaluate $\int z^4 dz$ (i) over the upper-half of the unit circle traversed counterclockwise, (ii) over the unit circle traversed counterclockwise.

(e) Let $f(z) = z^3 - 2z + e^{z-2}$ and let C be the circle of radius 3 in xy -plane. Evaluate $\int_C f(z) dz$.

2. Laurent Series. Find the power series expansions of the functions $f(z)$ centered at indicated point a . Then, determine all the singularities of $f(z)$, classify their types and find the residues.

(a) $f(z) = \frac{z}{1-z^2}$, $z = 0$

(b) $f(z) = \frac{e^{z-1}}{(z-1)^2}$, $z = 1$

(c) $f(z) = \frac{1-\cos z}{z^2}$, $z = 0$

(d) $f(z) = z \cos \frac{1}{z}$, $z = 0$

3. Complex integrals with Residue Theorem. Use the Residue Theorem to evaluate the complex integrals of given function $f(z)$ over the given contour C . Express your answers in $a + bi$ form.

(a) $f(z) = \frac{1}{z^2(z^2+1)}$, C is the circle of radius 1 centered at $\frac{i}{2}$.

(b) $f(z) = \frac{e^z}{z^2-4}$, C is the circle of radius 5 centered at the origin.

(c) $f(z) = \frac{1}{(z^2-2z+2)^2}$, C is the boundary of the upper half of the disc with radius R where $R > 2$.

(d) $f(z) = \frac{z+4}{(z-3)^2(z^2-6z+10)}$, C is the circle of radius $\frac{1}{2}$ centered at $(3,0)$.

(e) $f(z) = \frac{z+4}{(z-3)^2(z^2-6z+10)}$, C is the circle of radius 2 centered at $(3,0)$.

(f) $f(z) = \frac{e^z}{(z-1)^5}$, C is the boundary of the right half of the disc with radius 2.

(g) $f(z) = z \cos \frac{1}{z}$, C is the square with vertices $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$.

Solutions.

More detailed solutions can be found on the class handout.

1. Analytic Functions. Complex integrals. Cauchy's Integral Formula.

- (a) (i) $f(z)$ has derivative $f'(z) = e^{5iz^2} + 10iz^2e^{5iz^2}$ which is continuous. Thus, f is analytic.
 (ii) For $f(x + iy) = x^2y + ixy^2$, $u = x^2y$ and $v = ixy^2$. Check the Cauchy-Riemann equations. $\frac{\partial u}{\partial x} = 2xy = \frac{\partial v}{\partial y}$ but $\frac{\partial u}{\partial y} = x^2$ and $\frac{\partial v}{\partial x} = y^2$ so the second equation fails. Thus, f is not analytic.
 (iii) For $f(x + iy) = x^2 - y^2 + 2xyi$, $u = x^2 - y^2$ and $v = 2xy$. Check if the Cauchy-Riemann equations hold. $\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -2y = \frac{\partial v}{\partial x}$ so both equations hold. Thus, f is analytic.
- (b) (i) Check if u satisfies Laplace equation. $\frac{\partial u}{\partial x} = e^{3y} \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0$ and $\frac{\partial u}{\partial y} = 3xe^{3y} \Rightarrow \frac{\partial^2 u}{\partial y^2} = 9xe^{3y}$. Since $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 - 9xe^{3y} \neq 0$, no such analytic function exists. (ii) Check the Laplace equation: $\frac{\partial u}{\partial x} = 2x + 3 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 2$ and $\frac{\partial u}{\partial y} = -2y + 5 \Rightarrow \frac{\partial^2 u}{\partial y^2} = -2$. Since $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$.
 Since $2x + 3 = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ you can find v as $v = \int (2x + 3)dy = 2xy + 3y + g(x)$ and you can find g from the second Cauchy-Riemann equation $2y - 5 = \frac{-\partial u}{\partial y} = \frac{\partial v}{\partial x} = 2y + g'(x) \Rightarrow g'(x) = -5 \Rightarrow g(x) = -5x + c$. Thus, $v = 2xy + 3y - 5x + c$. In this case $f(z) = x^2 + 3x - y^2 + 5y + i(2xy + 3y - 5x + c) = x^2 + 2ixy + y^2 + 3(x + iy) + 5(-ix + y) = (x + iy)^2 + 3(x + iy) - 5i(x + iy) = z^2 + 3z - 5iz$.
- (c) On the first line segment $x = 1$, $y = y$ and $0 \leq y \leq 1$. $z = 1 + iy$ so $\text{Re}z=1$ and $dz = 0dx + idy = idy$. $\int \text{Re}(z)dz = \int_0^1 idy = iy|_0^1 = i$. On the second line segment $x = x$, $y = 1$ and the x -values are decreasing from 1 to 0. $z = x + i$ and $\text{Re}z = x$, $dz = dx + 0i = dx$. $\int \text{Re}(z)dz = \int_1^0 xdx = \frac{x^2}{2}|_1^0 = -\frac{1}{2}$. So, the total integral is $i - \frac{1}{2}$.
- (d) (i) On the unit circle $x = \cos t$, and $y = \sin t$, so $z = \cos t + i \sin t = e^{it}$. Thus, $z^4 = e^{4it}$ and $dz = e^{it}idt$. The bounds for t are $0 \leq t \leq \pi$ so $\int z^4 dz = \int_0^\pi e^{4it} e^{it} i dt = i \int_0^\pi e^{5it} dt = \frac{1}{5}(e^{5\pi i} - e^{0i}) = \frac{1}{5}(\cos 5\pi + i \sin 5\pi - 1) = \frac{-1}{5}$. (ii) $f(z) = z^4$ is analytic ($f'(z) = 4z^3$ is continuous). Thus, the integral is zero by Cauchy's Theorem.
- (e) $f(z) = z^3 - 2z + e^{z-2}$ is analytic (derivative $3z^2 - 2 + e^{z-2}$ is continuous) so the integral in part (i) is zero by Cauchy's Theorem.

2. Laurent Series

- (a) $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \Rightarrow \frac{1}{1-z^2} = \sum_{n=0}^{\infty} z^{2n} \Rightarrow f(z) = \frac{z}{1-z^2} = \sum_{n=0}^{\infty} z^{2n+1}$. The only singularities of $f(z)$ are ± 1 and they are poles of order 1. The residues are $\lim_{z \rightarrow 1} (z-1) \frac{z}{(1-z)(1+z)} = \lim_{z \rightarrow 1} \frac{z}{-(1+z)} = -\frac{1}{2}$ and $\lim_{z \rightarrow -1} (z+1) \frac{z}{(1-z)(1+z)} = \lim_{z \rightarrow -1} \frac{z}{1-z} = \frac{-1}{2}$.
- (b) $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \Rightarrow e^{z-1} = \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \Rightarrow \frac{e^{z-1}}{(z-1)^2} = \frac{1}{(z-1)^2} \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} = \sum_{n=0}^{\infty} \frac{(z-1)^{n-2}}{n!} = \frac{1}{(z-1)^2} + \frac{1}{z-1} + \frac{1}{2!} + \frac{z-1}{3!} + \dots$. The only singularity is $z = 1$ and it is a pole of the order 2. The coefficient with the term with $\frac{1}{z-1}$ is 1 so the residue is 1.
- (c) $f(z) = \frac{1-\cos z}{z^2} = \frac{1}{z^2} - \frac{1}{z^2} \cos z = \frac{1}{z^2} - \frac{1}{z^2} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = \frac{1}{z^2} - \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n-2}}{(2n)!} = \frac{1}{z^2} - \frac{1}{z^2} + \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \dots = \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \dots$. Thus, there are no terms with negative exponents. So, the only singularity $z = 0$ is a removable singularity.
- (d) $f(z) = z \cos \frac{1}{z} = z \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! z^{2n}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! z^{2n-1}} = z - \frac{1}{2!z} + \frac{1}{4!z^3} - \dots$. The only singularity is 0 and it is an essential singularity (infinitely many terms with z in denominator). From the power series expansion, the coefficient with $\frac{1}{z}$ is $\frac{-1}{2}$ so that is the residue.

3. Complex integrals with Residue Theorem. More detailed solutions can be found on the class handout.

- (a) Since $z^2 + 1 = (z - i)(z + i)$, f has three singularities $0, i$ and $-i$. 0 is a pole of order 2 and $\pm i$ are poles of the first order. Just 0 and i are inside curve C . The residue at 0 is $\lim_{z \rightarrow 0} \frac{d}{dz} (z^2 f(z)) = \lim_{z \rightarrow 0} \frac{d}{dz} \left(\frac{1}{z^2 + 1} \right) = \lim_{z \rightarrow 0} \frac{-2z}{(z^2 + 1)^2} = 0$. The residue at i is $\lim_{z \rightarrow i} (z - i) f(z) = \lim_{z \rightarrow i} \frac{1}{z^2(z + i)} = \frac{1}{-1(i + i)} = \frac{-1}{2i} = \frac{i}{2}$. Using the Residue Theorem, the integral is equal to $2\pi i(0 + \frac{i}{2}) = -\pi$.
- (b) $\frac{e^z}{z^2 - 4} = \frac{e^z}{(z - 2)(z + 2)}$ has two poles of order 1, 2 and -2 and both are in C . The residue at 2 is $\lim_{z \rightarrow 2} (z - 2) f(z) = \lim_{z \rightarrow 2} \frac{e^z}{(z + 2)} = \frac{e^2}{4}$. The residue at -2 is $\lim_{z \rightarrow -2} (z + 2) f(z) = \lim_{z \rightarrow -2} \frac{e^z}{(z - 2)} = \frac{-e^{-2}}{4}$. Thus, the integral is $2\pi i(\frac{e^2}{4} - \frac{e^{-2}}{4}) = \frac{\pi i}{2}(e^2 - e^{-2})$.
- (c) The zeros of the denominator of $f(z) = \frac{1}{(z^2 - 2z + 2)^2}$ are $z = \frac{2 \pm \sqrt{4 - 8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$. Thus $f(z) = \frac{1}{(z - (1 + i))^2(z - (1 - i))^2}$. So, there are two poles the second order $1 \pm i$ and just $1 + i$ is inside C . The residue at $1 + i$ is $\lim_{z \rightarrow 1 + i} \frac{d}{dz} ((z - (1 + i))^2 f(z)) = \lim_{z \rightarrow 1 + i} \frac{d}{dz} \left(\frac{1}{(z - (1 - i))^2} \right) = \lim_{z \rightarrow 1 + i} \frac{-2}{(z - 1 - i)^3} = \frac{-2}{(2i)^3} = \frac{-2}{8(-i)} = \frac{1}{4i} = \frac{-i}{4}$. Thus the integral is equal to $2\pi i \frac{-i}{4} = \frac{\pi}{2}$.
- (d) $z^2 - 6z + 10 = 0 \Rightarrow z = \frac{6 \pm \sqrt{36 - 40}}{2} = \frac{6 \pm 2i}{2} = 3 \pm i$. Thus

$$f(z) = \frac{z + 4}{(z - 3)^2(z^2 - 6z + 10)} = \frac{z + 4}{(z - 3)^2(z - 3 - i)(z - 3 + i)}$$

has three singularities: 3 is a pole of order 2 and $3 \pm i$ are two poles of order 1. Just $z = 3$ is inside the given circle, so the residue at 3 is the only relevant residue.

The residue at 3 can be computed as $\lim_{z \rightarrow 3} \frac{d}{dz} ((z - 3)^2 f(z)) = \lim_{z \rightarrow 3} \frac{d}{dz} \left(\frac{z + 4}{z^2 - 6z + 10} \right) = \lim_{z \rightarrow 3} \frac{1(z^2 - 6z + 10) - (z + 4)(2z - 6)}{(z^2 - 6z + 10)^2} = \frac{1(9 - 18 + 10) - 0}{(9 - 18 + 10)^2} = 1$. So, the integral is equal to $2\pi i$.

- (e) All three singularities $3, 3 + i$, and $3 - i$ are inside the given circle.

The residue at $3 + i$ is $\lim_{z \rightarrow 3 + i} (z - 3 - i) \frac{z + 4}{(z - 3)^2(z - 3 - i)(z - 3 + i)} = \lim_{z \rightarrow 3 + i} \frac{z + 4}{(z - 3)^2(z - 3 + i)} = \frac{3 + i + 4}{(3 + i - 3)^2(3 + i - 3 + i)} = \frac{7 + i}{-2i} = \frac{(7 + i)i}{2} = \frac{-1}{2} + \frac{7}{2}i$.

The residue at $3 - i$ is $\lim_{z \rightarrow 3 - i} (z - 3 + i) \frac{z + 4}{(z - 3)^2(z - 3 - i)(z - 3 + i)} = \lim_{z \rightarrow 3 - i} \frac{z + 4}{(z - 3)^2(z - 3 - i)} = \frac{3 - i + 4}{(3 - i - 3)^2(3 - i - 3 - i)} = \frac{7 - i}{2i} = \frac{(7 - i)(-i)}{2} = \frac{-1}{2} - \frac{7}{2}i$.

The residue at 3 has been find it the previous problem to be 1. Thus, the integral is $2\pi i(1 + \frac{-1}{2} + \frac{7}{2}i + \frac{-1}{2} - \frac{7}{2}i) = 0$.

- (f) $f(z) = \frac{e^z}{(z - 1)^5}$. The only singularity is 1 and it is a pole of order 5. The residue at 1 is $\frac{1}{4!} \lim_{z \rightarrow 1} \frac{d^4}{dz^4} ((z - 1)^5 f(z)) = \frac{1}{24} \lim_{z \rightarrow 1} \frac{d^4}{dz^4} (e^z) = \frac{1}{24} \lim_{z \rightarrow 1} e^z = \frac{e^1}{24} = \frac{e}{24}$ so the integral is $2\pi i \frac{e}{24} = \frac{e\pi i}{12}$.
- (g) $f(z) = z \cos \frac{1}{z} = z \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! z^{2n}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! z^{2n-1}} = z - \frac{1}{2!z} + \frac{1}{4!z^3} - \dots$. The only singularity is 0 and it is an essential singularity (infinitely many terms with z in denominator). From the power series expansion, the coefficient with $\frac{1}{z}$ is $\frac{-1}{2}$ so that is the residue. Hence, the integral is equal to $2\pi i \frac{-1}{2} = -\pi i$.