Math Methods 1
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Review for Exam 2

1. **Flux integral in cylindrical or spherical coordinates.**

   (a) Compute the flux of the vector field $\mathbf{f} = r\sin\phi\ \hat{\mathbf{r}}_r + r\cos\phi\ \hat{\mathbf{r}}_\theta$ over the unit sphere (i) directly, (ii) using Divergence Theorem.

   (b) Compute the flux of the vector field $\mathbf{f} = r\sin\theta\hat{\mathbf{r}}_r + 2r\cos\theta\hat{\mathbf{r}}_\theta + 3rz\hat{\mathbf{r}}_z$ over the boundary of the region inside the cylinder $x^2 + y^2 = 9$ between the planes $z = 0$ and $z = 4$ (i) without using the Divergence Theorem and (ii) using the Divergence Theorem.

   (c) Compute the flux of the vector field $\mathbf{f} = r^2\hat{\mathbf{r}}_r + 2r^2\sin\phi\hat{\mathbf{r}}_\phi + r^4\sin\phi\hat{\mathbf{r}}_\theta$ over the boundary of the region inside the cone $z = \sqrt{3(x^2 + y^2)}$ and the sphere of radius a. The problem is significantly shorter if you use the Divergence Theorem.

   (d) Find the flux done by the vector field $\mathbf{f} = r\hat{\mathbf{r}}_r + 2rz\hat{\mathbf{r}}_\theta + r\sin\theta\hat{\mathbf{r}}_z$ over the boundary of the region between the paraboloids $z = r^2$ and $z = a^2 - 3r^2$. The problem is significantly shorter if you use the Divergence Theorem.

2. **Line integral in cylindrical or spherical coordinates.**

   (a) Find the work done by the vector field $\mathbf{f} = \cos\theta\hat{\mathbf{r}}_r - r\sin\theta\hat{\mathbf{r}}_\theta - 5r\hat{\mathbf{r}}_z$ acting along the positive oriented intersection of the horizontal plane $z = b$, the cylinder $x^2 + y^2 = a^2$, the xz and yz planes (i) directly and (ii) using Stokes' Theorem.

   (b) Find the work done by the vector field $\mathbf{f} = r\sin\phi\hat{\mathbf{r}}_r + r\hat{\mathbf{r}}_\theta + 4\hat{\mathbf{r}}_z$ acting along the positive oriented intersection of the sphere of radius R centered at the origin and the horizontal plane $z = \frac{1}{2}R$ (i) directly and (ii) using Stokes' Theorem.

   (c) Find the work done by the vector field $\mathbf{f} = r^2\hat{\mathbf{r}}_r + r^2\sin\phi\hat{\mathbf{r}}_\phi + 2\hat{\mathbf{r}}_\theta$ acting along the curve from the previous problem (i) directly and (ii) using Stokes' Theorem.

3. **Analytic Functions. Complex integrals. Cauchy's Integral Formula.**

   (a) Check if the following functions are analytic.
   (i) $f(z) = ze^{5iz^2}$, (ii) $f(x + iy) = x^2y + ixy^2$, (iii) $f(x + iy) = x^2 - y^2 + 2xyi$.

   (b) Check if analytic functions with real part equal to the given functions u exist. If so, find all analytic functions that have real parts equal to u.
   (i) $u = xe^{3y}$ (ii) $u = x^2 + 3x - y^2 + 5y$.

   (c) Evaluate $\int$ Re$(z)\ dz$ over the line segment $x = 1$ from (1,0) to (1,1) and the line segment $y = 1$ from (1,1) to (0,1).

   (d) Evaluate $\int z^4 dz$ (i) over the upper-half of the unit circle traversed counterclockwise, (ii) over the unit circle traversed counterclockwise.

   (e) Let $f(z) = z^3 - 2z + e^{z-2}$ and let C be the circle of radius 3 in xy-plane. Evaluate
   (i) $\int_C f(z)dz$ and (ii) $\int_C \frac{f(z)}{z-2}dz$.

4. **Laurent Series**. Find the power series expansions of the functions $f(z)$ centered at indicated point a. Then, determine all the singularities of $f(z)$, classify their types and find the residues.

   (a) $f(z) = \frac{z}{1-z^2}, \quad z = 0$

   (b) $f(z) = \frac{e^{z-1}}{(z-1)^2}, \quad z = 1$

   (c) $f(z) = \frac{1-\cos z}{z^2}, \quad z = 0$

   (d) $f(z) = z\cos\frac{1}{z}, \quad z = 0$

Solutions. More detailed solutions can be found on the class handout.

1. Flux integral.

   (a) (i) $\mathbf{f}\cdot d\mathbf{S} = \sin^2\phi d\phi d\theta$. $F = \int\int_S \mathbf{f}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi}\sin^2\phi d\phi d\theta = 2\pi\int_0^{\pi}\sin^2\phi d\phi = 2\pi\ \frac{\pi}{2} = \pi^2 \approx 9.87$. (ii) $\mathrm{div}\mathbf{f} = 3\sin\phi$. $F = \int\int\int_V 3\sin\phi\ r^2\sin\phi\ dr\,d\theta\,d\phi = 2\pi\ \frac{\pi}{2} = \pi^2 \approx 9.87$.

   (b) (i) $F = F_1 + F_2 + F_3 = 0 + 216\pi + 0 = 216\pi$. (ii) $\mathrm{div}\mathbf{f} = 3r$. $F = \int\int\int_V 3r\ r\ dr\,d\theta\,dz = \int_0^{2\pi}d\theta\ \int_0^3 3r^2\ dr \int_0^4\ dz = 2\pi\ 3^3\ 4 = 216\pi$.

   (c) $\mathrm{div}\mathbf{f} = 4r(1+\cos\phi)$. $F = \int\int\int_V 4r(1+\cos\phi)\ r^2\sin\phi\ \ dr\,d\theta\,d\phi = \int_0^{2\pi}\int_0^{\pi/6}\int_0^a 4r^3(1+\cos\phi)\sin\phi\ dr\,d\theta\,d\phi = 2a^4\pi\ \left(-\cos\phi - \frac{\cos^2\phi}{2}\right)\Big|_0^{\pi/6} = 2a^4\pi\left(\frac{9}{8} - \frac{\sqrt{3}}{2}\right) \approx 1.63a^4$

   (d) $\mathrm{div}\mathbf{f} = 2$. $F = \int\int\int_V \mathrm{div}\mathbf{f}\ dV = \int_0^{2\pi}\int_0^{a/2}\int_{r^2}^{a^2-3r^2} 2\ r\ dr\,d\theta\,dz = 2\pi\int_0^{a/2}(a^4 - 6a^2r^2 + 9r^4 - r^4)dr = 2\pi\left(\frac{a^5}{2} - \frac{a^5}{4} + \frac{a^5}{20}\right) = \frac{3a^5\pi}{5}$.

2. Line integral.

   (a) (i) Work done $= -a^2+0+a = a-a^2$. (ii) $\mathrm{curl}\mathbf{f} = 5\hat{\mathbf{r}}_\theta - 2\sin\theta\hat{\mathbf{r}}_z + \frac{1}{r}\sin\theta\hat{\mathbf{r}}_z$. $d\mathbf{S} = rdrd\theta\hat{\mathbf{r}}_z \Rightarrow \mathrm{curl}\mathbf{f}\cdot d\mathbf{S} = (-2r\sin\theta + \sin\theta)\,dr\,d\theta$. Work done $= \int_0^{\pi/2}\int_0^a(-2r\sin\theta + \sin\theta)\,dr\,d\theta = \int_0^{\pi/2}\int_0^a(-a^2\sin\theta + a\sin\theta)\,d\theta = -a^2 + a$.

   (b) (i) $W = \oint_C \mathbf{f}\cdot d\mathbf{r} = \int_C \frac{\sqrt{3}R}{2}\frac{\sqrt{3}R}{2}d\theta = \frac{3R^2}{4}\int_0^{2\pi}d\theta = \frac{3R^2\pi}{2}$. (ii) $d\mathbf{S} = rdrd\theta\hat{\mathbf{r}}_z$, $\mathrm{curl}\mathbf{f} = \frac{1}{r}(2r - r\cos\theta)\hat{\mathbf{r}}_z = (2-\cos\theta)\hat{\mathbf{r}}_z \Rightarrow \mathrm{curl}\mathbf{f}\cdot d\mathbf{S} = (2-\cos\theta)rdrd\theta$. $W = \int\int \mathrm{curl}\mathbf{f}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\frac{\sqrt{3}R}{2}}(2-\cos\theta)rdrd\theta = \int_0^{\frac{\sqrt{3}R}{2}}(4\pi - 0)rdr = 2\pi\frac{3R^2}{4} = \frac{3R^2\pi}{2}$.

   (c) (i) $d\mathbf{r} = \frac{\sqrt{3}R}{2}d\theta\hat{\mathbf{r}}_\theta \Rightarrow \mathbf{f}\cdot d\mathbf{r} = \sqrt{3}Rd\theta$. $W = \oint_C \mathbf{f}\cdot d\mathbf{r} = \sqrt{3}R\int_0^{2\pi}d\theta = 2\sqrt{3}R\pi$. (ii) $d\mathbf{S} = r^2\sin\phi d\theta d\phi\hat{\mathbf{r}}_r$, $\mathrm{curl}\mathbf{f} = \frac{1}{r^2\sin\phi}(2r\cos\phi\hat{\mathbf{r}}_r + 2r\sin\phi\hat{\mathbf{r}}_\phi + 3r^3\sin^2\phi\hat{\mathbf{r}}_\theta) \Rightarrow \mathrm{curl}\mathbf{f}\cdot d\mathbf{S} = 2R\cos\phi d\theta d\phi$. $W = \int\int \mathrm{curl}\mathbf{f}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\frac{\pi}{3}} 2R\cos\phi d\theta d\phi = 4R\pi\int_0^{\pi/3}\cos\phi d\phi = 2\sqrt{3}R\pi$.

3. Analytic Functions. Complex integrals. Cauchy's Integral Formula.

   (a) (i) $f(z)$ has derivative $f'(z) = e^{5iz^2} + 10iz^2e^{5iz^2}$ which is continuous. Thus, f is analytic.
   (ii) For $f(x+iy) = x^2y + ixy^2$, $u = x^2y$ and $v = xy^2$. Check the Cauchy-Riemann equations. $\frac{\partial u}{\partial x} = 2xy = \frac{\partial v}{\partial y}$ but $\frac{\partial u}{\partial y} = x^2$ and $\frac{\partial v}{\partial x} = y^2$ so the second equation fails. Thus, f is not analytic.
   (iii) For $f(x+iy) = x^2 - y^2 + 2xyi$, $u = x^2 + y^2$ and $v = 2xy$. Check if the Cauchy-Riemann equations hold. $\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -2y = \frac{\partial v}{\partial x}$ so both equations hold. Thus, f is analytic.

(b) (i) Check if u satisfies Laplace equation. $\frac{\partial u}{\partial x} = e^{3y} \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0$ and $\frac{\partial u}{\partial y} = 3xe^{3y} \Rightarrow \frac{\partial^2 u}{\partial y^2} = 9xe^{3y}$. Since $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 - 9xe^{3y} \neq 0$, no such analytic function exists. (ii) Check the Laplace equation: $\frac{\partial u}{\partial x} = 2x+3 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 2$ and $\frac{\partial u}{\partial y} = -2y+5 \Rightarrow \frac{\partial^2 u}{\partial y^2} = -2$. Since $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2-2 = 0$.

Since $2x+3 = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ you can find v as $v = \int (2x+3)dy = 2xy + 3y + g(x)$ and you can find g from the second Cauchy-Riemann equation $2y - 5 = \frac{-\partial u}{\partial y} = \frac{\partial v}{\partial x} = 2y + g'(x) \Rightarrow g'(x) = -5 \Rightarrow g(x) = -5x + c$. Thus, $v = 2xy + 3y - 5x + c$. In this case $f(z) = x^2 + 3x - y^2 + 5y + i(2xy + 3y - 5x + c) = x^2 + 2ixy + y^2 + 3(x+iy) + 5(-ix+y) = (x+iy)^2 + 3(x+iy) - 5i(x+iy) = z^2 + 3z - 5iz$.

(c) On the first line segment $x = 1$, $y = y$ and $0 \leq y \leq 1$. $z = 1 + iy$ so Rez=1 and $dz = 0dx + idy = idy$. $\int$ Re $(z)dz = \int_0^1 idy = iy|_0^1 = i$. On the second line segment $x = x$, $y = 1$ and the x-values are decreasing from 1 to 0. $z = x + i$ and Re $z = x$, $dz = dx + 0i = dx$. $\int$ Re $(z)dz = \int_1^0 xdx = \frac{x^2}{2}|_1^0 = -\frac{1}{2}$. So, the total integral is $i - \frac{1}{2}$.

(d) (i) On the unit circle $x = \cos t$, and $y = \sin t$, so $z = \cos t + i \sin t = e^{it}$. Thus, $z^4 = e^{4it}$ and $dz = e^{it} idt$. The bounds for t are $0 \leq t \leq \pi$ so $\int z^4 dz = \int_0^\pi e^{4it} e^{it} idt = i \int_0^\pi e^{5it} dt = \frac{1}{5}(e^{5\pi i} - e^{0i}) = \frac{1}{5}(\cos 5\pi + i \sin 5\pi - 1) = \frac{-2}{5}$. (ii) $f(z) = z^4$ is analytic ($f'(z) = 4z^3$ is continuous). Thus, the integral is zero by Cauchy's Theorem.

(e) $f(z) = z^3 - 2z + e^{z-2}$ is analytic (derivative $3z^2 - 2 + e^{z-2}$ is continuous) so the integral in part (i) is zero by Cauchy's Theorem. (ii) You can take $g(z) = \frac{f(z)}{z-2}$ and note that g has one singularity, 2, it is a pole of the first order, and it is in the interior of C. The residue of g at 2 is $\lim_{z\to 2}(z-2)\frac{f(z)}{z-2} = \lim_{z\to 2} f(z) = f(2) = 5$. Hence $\int_C g(z)dz = 2\pi i(5) = 10\pi i$. Alternatively, you can use the Cauchy's integral formula for $f(z)$ and obtain that $\int_C \frac{f(z)}{z-2}dz = 2\pi i f(2) = 2\pi i(8 - 4 + 1) = 10\pi i$.

4. Laurent Series

(a) $\frac{1}{1-z} = \sum_{n=0}^\infty z^n \Rightarrow \frac{1}{1-z^2} = \sum_{n=0}^\infty z^{2n} \Rightarrow f(z) = \frac{z}{1-z^2} = \sum_{n=0}^\infty z^{2n+1}$. The only singularities of $f(z)$ are ± 1 and they are poles of order 1. The residues are $\lim_{z\to 1}(z-1)\frac{z}{(1-z)(1+z)} = \lim_{z\to 1} \frac{z}{-(1+z)} = -\frac{1}{2}$ and $\lim_{z\to -1}(z+1)\frac{z}{(1-z)(1+z)} = \lim_{z\to -1} \frac{z}{1-z} = \frac{-1}{2}$.

(b) $e^z = \sum_{n=0}^\infty \frac{z^n}{n!} \Rightarrow e^{z-1} = \sum_{n=0}^\infty \frac{(z-1)^n}{n!} \Rightarrow \frac{e^{z-1}}{(z-1)^2} = \frac{1}{(z-1)^2}\sum_{n=0}^\infty \frac{(z-1)^n}{n!} = \sum_{n=0}^\infty \frac{(z-1)^{n-2}}{n!} = \frac{1}{(z-1)^2} + \frac{1}{z-1} + \frac{1}{2!} + \frac{z-1}{3!} + \ldots$. The only singularity is $z = 1$ and it is a pole of the order 2. The coefficient with the term with $\frac{1}{z-1}$ is 1 so the residue is 1.

(c) $f(z) = \frac{1-\cos z}{z^2} = \frac{1}{z^2} - \frac{1}{z^2}\cos z = \frac{1}{z^2} - \frac{1}{z^2}\sum_{n=0}^\infty \frac{(-1)^n z^{2n}}{(2n)!} = \frac{1}{z^2} - \sum_{n=0}^\infty \frac{(-1)^n z^{2n-2}}{(2n)!} = \frac{1}{z^2} - \frac{1}{z^2} + \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \ldots = \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \ldots$ Thus, there are no terms with negative exponents. So, the only singularity $z = 0$ is a removable singularity.

(d) $f(z) = z\cos\frac{1}{z} = z\sum_{n=0}^\infty \frac{(-1)^n}{(2n)!z^{2n}} = \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!z^{2n-1}} = z - \frac{1}{2!z} + \frac{1}{4!z^3} - \ldots$. The only singularity is 0 and it is an essential singularity (infinitely many terms with z in denominator). From the power series expansion, the coefficient with $\frac{1}{z}$ is $\frac{-1}{2}$ so that is the residue.