Math Methods 1
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Review for Exam 3

1. **Fourier Series.**

   (a) The input to an electrical circuit that switches between a high and a low state with time period 2π can be represented by the boxcar function $f(x) = \begin{cases} 1 & 0 \leq x < \pi \\ -1 & -\pi \leq x < 0 \end{cases}$. Find its Fourier series expansion and use it to find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

   (b) Find the Fourier cosine expansion of $f(x) = \begin{cases} x & 0 < x \leq 1 \\ 2-x & 1 < x < 2 \end{cases}$ and use it find the sum of the series $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

   (c) Consider the function $f(x) = x^2$ for $0 < x \leq 2$.

      (i) Sketch the graphs of the following (1) the periodic extension of $f(x)$, (2) the even periodic extension of $f(x)$, (3) the odd periodic extension of $f(x)$ and write the integrals computing the coefficients of the corresponding Fourier series in all three cases.

      (ii) Find the Fourier cosine expansion for $f(x)$.

      (iii) Using part (ii), show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

      (iv) Using part (ii) and Parseval's Theorem, find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

   (d) The output from an electronic oscillator is the sawtooth function $f(t) = t$ for $0 \leq t \leq 1$ that keeps repeating with period 1. Sketch this function and find its complex Fourier series. Using this series and Parseval's Theorem, find the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

2. **Fourier Transform.**

   (a) Find the Fourier and the inverse Fourier transforms of the boxcar function $f(t) = \begin{cases} 1 & -1 < t < 1 \\ 0 & \text{otherwise} \end{cases}$ Express your answer as real functions.

   (b) Find the Fourier transform of $f(t) = e^{-t}$, $t > 0$, $f(t) = 0$ otherwise.

   (c) Find cosine Fourier transform of $f(t) = 2t - 3$ for $0 < t < 3/2$, $f(x) = 0$ otherwise.

Solutions.

<u>More detailed solutions can be found on the class handout.</u>

1. (a) Graph the function and note it is odd. Thus, $a_n = 0$. Since $L = \pi$ $(T = 2\pi)$, the coefficients b_n can be computed as $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_0^{\pi} \sin nx dx = \frac{-2}{n\pi}\cos nx|_0^{\pi} = \frac{-2}{n\pi}((-1)^n - 1)$. Note that $b_{2k} = 0$ and $b_{2k+1} = \frac{-2}{n\pi}(-2) = \frac{4}{(2k+1)\pi}$. Hence, $f(x) = \frac{4}{\pi}\sum_{n=\text{odd}}^{\infty} \frac{1}{n}\sin nx = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1}{2k+1}\sin(2k+1)x = \frac{4}{\pi}(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots)$.
   To find the sum of the given series, choose $x = \frac{\pi}{2}$ and plug it in both sides of the above equation. Since $f(\frac{\pi}{2}) = 1$ and sin we have that $1 = \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\sin(2n+1)\frac{\pi}{2}}{2n+1} = \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$.

(b) Extend the function symmetrically with respect to y-axis. Repeating the pattern on $(-1,1)$ produces the entire periodic extension so you can take $T=2$ and $L=1$. The coefficients b_n are zero and $a_n = \frac{2}{1}\int_0^1 f(x)\cos\frac{n\pi x}{1}dx = 2\int_0^1 x\cos n\pi x\, dx$. Use integration by parts with $u=x$, $v=\frac{1}{n\pi}\sin n\pi x$. Obtain $a_n = \frac{2x}{n\pi}\sin n\pi x\Big|_0^1 + \frac{2}{n^2\pi^2}\cos n\pi x\Big|_0^1 = 0 + \frac{2}{n^2\pi^2}(\cos n\pi - 1) = \frac{2}{n^2\pi^2}((-1)^n - 1)$. If $n=2k$ is even, $a_n = 0$. If $n = 2k+1$ is odd, $a_n = \frac{2}{(2k+1)^2\pi^2}(-1-1) = \frac{-4}{(2k+1)^2\pi^2}$. Since we are dividing by n already when finding v, the above formula does not apply to $n=0$ so a_0 must be computed separately as $a_0 = \frac{2}{1}\int_0^1 x\cos 0 dx = 2\int_0^1 x dx = 2\,\frac{x^2}{2}\Big|_0^1 = 1$. So, $f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos(2k+1)\pi x$.

To find the sum of $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$, note that when $x=0$ the function $f(x)$ is equal to $f(0)=0$ and its Fourier cosine expansion is equal to $\frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos(2k+1)\pi(0) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$. Rename the index k to be n if you want and obtain the equation. $0 = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$. Solve for the required sum to get $\frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{1}{2} \Rightarrow \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$.

(c) (i) The *periodic* extension of $f(x)$ can be obtained by replicating $f(x)$ on intervals $\ldots[-4,-2]$, $[-2,0]$, $[0,2]$, $[2,4]$, $[4,6]\ldots$ of length $T=2$. Then, $a_n = \int_0^2 x^2\cos n\pi x\, dx$, $b_n = \int_0^2 x^2\sin n\pi x\, dx$, and $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos n\pi x + b_n\sin n\pi x$.

The *even* extension of $f(x)$ is obtained by extending $f(x)$ from $[0,\ 2]$ to $[-2,2]$ by defining $f(x) = (-x)^2 = x^2$ on $[-2,0]$. Thus, the result is the function x^2 defined on interval $[-2,2]$. Then, replicate this function on intervals $\ldots[-6,-2],[-2,2],[2,6],[6,10],\ldots$ of length $T=4$. Thus, $T=4$ and $L=2$. The coefficients of the cosine Fourier series can be calculated by $a_n = \int_0^2 x^2\cos\frac{n\pi x}{2}dx$, $b_n=0$, and $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{2}$.

The *odd* extension of $f(x)$ is obtained by extending $f(x)$ from $[0,\ 2]$ to $[-2,2]$ by defining $f(x) = -(x)^2 = -x^2$ on $[-2,0]$. Replicate this function on intervals $\ldots[-6,-2],[-2,2],[2,6],[6,10],\ldots$ of length $T=4$. Thus, $T=4$ and $L=2$. The coefficients of the corresponding since Fourier series can be calculated by $a_n=0$, $b_n = \int_0^2 x^2\sin\frac{n\pi x}{2}\, dx$, and $f(x) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{2}$.

(ii) $a_n = \int_0^2 x^2\cos\frac{n\pi x}{2}dx$. Using integration by parts twice, obtain that $a_n = \frac{2}{n\pi}x^2\sin\frac{n\pi x}{2}|_0^2 - \frac{4}{n\pi}\int_0^2 x\sin\frac{n\pi x}{2}dx = \frac{8}{n^2\pi^2}x\cos\frac{n\pi x}{2}|_0^2 - \frac{16}{n^3\pi^3}\sin\frac{n\pi x}{2}|_0^2 = \frac{16}{n^2\pi^2}\cos n\pi = \frac{16(-1)^n}{n^2\pi^2}$. Note that this formula works just for $n>0$ so a_0 has to be computed separately $a_0 = \int_0^2 x^2 dx = \frac{x^3}{3}|_0^2 = \frac{8}{3}$. Thus, the Fourier series is $x^2 = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{2}$.

(iii) For both sums, we need to find convenient values of x which reduce the right side of the formula $x^2 = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{2}$ to the series we need to sum. For the first one, it seems that the value of cosine should be 1 so take $x=0$. Plugging $x=0$ in the formula above produces

$$0 = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} \quad\Rightarrow\quad -\frac{4}{3} = \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} \quad\Rightarrow\quad -\frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

Multiply both sides with -1 to have $\frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)(-1)^n}{n^2} \quad\Rightarrow\quad \frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$.

For the second sum, note that the value of cosine should be $(-1)^n$ so that, multiplied by another $(-1)^n$, the numerator becomes $(-1)^{2n}=1$. The value $x=2$ would produce $\cos\frac{n\pi(2)}{2} = \cos n\pi =$

$(-1)^n$. Plugging $x = 2$ in the expansion in part (b) produces

$$2^2 = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(-1)^n \quad\Rightarrow\quad 4 = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2} \quad\Rightarrow\quad \frac{8}{3} = \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} \quad\Rightarrow\quad \frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

(iv) By Parseval's Theorem, $\frac{1}{2}\int_0^2 x^4 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{16}{9} + \frac{16^2}{2\pi^4}\sum_{n=1}^{\infty}\left(\frac{1}{n^4} + 0\right)$. Note that integral on the left side is $\frac{1}{2}\int_0^2 x^4 dx = \frac{16}{5}$. Dividing the equation above by 16 produces $\frac{1}{5} = \frac{1}{9} + \frac{8}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$.

(d) $T = 1$, $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2n\pi i t}$ and $c_n = \int_0^1 t e^{-2n\pi i t} = \frac{t}{-2n\pi i}e^{-2n\pi i t}|_0^1 + \frac{1}{4n^2\pi^2}e^{-2n\pi i t}|_0^1 = \frac{1}{-2n\pi i}e^{-2n\pi i} + \frac{1}{4n^2\pi^2}e^{-2n\pi i} - \frac{1}{4n^2\pi^2}$. Note that $e^{-2n\pi i} = \cos(-2n\pi) + i\sin(-2n\pi) = 1$. Thus $c_n = \frac{1}{-2n\pi i} + 0 = \frac{i}{2n\pi}$. This formula applies just to $n \neq 0$ so compute c_0 separately as $c_0 = \int_0^1 t\ dt = \frac{1}{2}$. Hence, $f(t) = \frac{1}{2} + \sum_{n=-\infty, n\neq 0}^{\infty}\frac{i}{2n\pi}e^{2n\pi i t}$.

This sum can be expressed using formulas with a_n and b_n also. Since $c_{-n} = \frac{-i}{2n\pi} = \overline{c_n}$, we have that $a_n = 0$ for $n > 0$, $a_0 = 2c_0 = 1$, and $b_n = \frac{-1}{n\pi}$.

By Parseval's Theorem, $\int_0^1 x^2 dx = \frac{1}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2} \Rightarrow \frac{1}{3} = \frac{1}{4} + \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

2. (a) $F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$. Since $f(t) = 0$ for $t < -1$ and $t > 1$, and $f(t) = 1$ for $-1 \leq t \leq 1$, we have that $F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-i\omega t}dt = \frac{-1}{\sqrt{2\pi}i\omega}e^{-i\omega t}|_{-1}^{1} = \frac{-1}{\sqrt{2\pi}i\omega}(e^{-i\omega} - e^{i\omega}) = \frac{1}{\sqrt{2\pi}i\omega}(\cos\omega + i\sin\omega - \cos(-\omega) - i\sin(-\omega)) = \frac{1}{\sqrt{2\pi}i\omega}(\cos\omega + i\sin\omega - \cos\omega + i\sin\omega) = \frac{1}{\sqrt{2\pi}i\omega}2i\sin\omega = \frac{2}{\sqrt{2\pi}}\frac{\sin\omega}{\omega} = \frac{2}{\sqrt{2\pi}}\operatorname{sinc}\omega$. The inverse transform of $f(\omega)$ can be obtained as $\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{i\omega t}d\omega = \frac{1}{\sqrt{2\pi}it}(e^{it} - e^{-it})$ Similarly as above, this simplifies to $\frac{1}{\sqrt{2\pi}it}2i\sin t = \frac{2}{\sqrt{2\pi}}\frac{\sin t}{t} = \frac{2}{\sqrt{2\pi}}\operatorname{sinc} t$.

(b) $F(\omega) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-t}e^{-i\omega t}dt = \frac{-1}{\sqrt{2\pi}(1+i\omega)}e^{-(1+i\omega)t}\Big|_0^{\infty} = \frac{1}{\sqrt{2\pi}(1+i\omega)}$.

(c) $F(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{3/2}(2t-3)\cos\omega t\ \ dt = \sqrt{\frac{2}{\pi}}\left(\frac{2t-3}{\omega}\sin\omega t|_0^{3/2} - \frac{2}{\omega}\int_0^{3/2}\sin\omega t\ \ dt\right) = \sqrt{\frac{2}{\pi}}\left(0 + \frac{2}{\omega^2}\cos\omega t|_0^{3/2}\right) = \sqrt{\frac{2}{\pi}}\frac{2}{\omega^2}(\cos\frac{3\omega}{2} - 1) = \frac{2\sqrt{2}}{\omega^2\sqrt{\pi}}(\cos\frac{3\omega}{2} - 1)$.