Math Methods 1
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Review for Exam 3

1. **Complex integrals with Residue Theorem.** Use the Residue Theorem to evaluate the complex integrals of given function $f(z)$ over the given contour C. Express your answers in $a+bi$ form.

   (a) $f(z) = \frac{1}{z^2(z^2+1)}$, C is the circle of radius 1 centered at $\frac{i}{2}$.

   (b) $f(z) = \frac{e^z}{z^2-4}$, C is the circle of radius 5 centered at the origin.

   (c) $f(z) = \frac{1}{(z-2)^2(4+z)}$, C is the circle of radius 3 centered at the origin.

   (d) $f(z) = \frac{1}{(z-2)^2(4+z)}$, C is the circle of radius 5 centered at the origin.

   (e) $f(z) = \frac{e^z}{(z-1)^5}$, C is the boundary of the right half of the disc with radius 2.

   (f) $f(z) = z\cos\frac{1}{z}$, C is the square with vertices (1,0), (0,1), (-1, 0) and (0,-1).

   (g) $f(z) = \frac{1}{(z^2-2z+2)^2}$, C is the boundary of the upper half of the disc with radius R where $R > 2$.

2. **Fourier Series.**

   (a) The input to an electrical circuit that switches between a high and a low state with time period 2π can be represented by the boxcar function $f(x) = \begin{cases} 1 & 0 \leq x < \pi \\ -1 & -\pi \leq x < 0 \end{cases}$. Find its Fourier series expansion and use it to find the sum of series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

   (b) Find the Fourier cosine series expansion for $f(x) = x^2$ for $0 < x \leq 2$. Then use the expansion and Parseval's Theorem to find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

   (c) Find the Fourier cosine and Fourier sine expansion of $f(x) = \begin{cases} x & 0 < x \leq 1 \\ 2-x & 1 < x < 2 \end{cases}$. Use the Fourier sine expansion to find the sum of series $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

   (d) The output from an electronic oscillator is the sawtooth function $f(t) = t$ for $0 \leq t \leq 1$ that keeps repeating with period 1. Sketch this function and find its complex Fourier series. Using this series and Parseval's Theorem, find the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

3. **Fourier Transform.**

   (a) Find the Fourier and the inverse Fourier transforms of the boxcar function $f(t) = \begin{cases} 1 & -1 < t < 1 \\ 0 & \text{otherwise} \end{cases}$ Express your answer as real functions.

   (b) Find the Fourier and the Fourier cosine transforms of $f(t) = e^{-t}$, $t > 0$, $f(t) = 0$ otherwise.

   (c) Find cosine Fourier transform of $f(t) = 2t - 3$ for $0 < t < 3/2$, $f(x) = 0$ otherwise.

Solutions.

1. Complex integrals with Residue Theorem. More detailed solutions can be found on the class handout.

   (a) Since $z^2+1=(z-i)(z+i)$, f has three singularities $0, i$ and $-i$. 0 is a pole of order 2 and $\pm i$ are poles of the first order. Just 0 and i are inside curve C. The residue at 0 is $\lim_{z\to 0}\frac{d}{dz}\left(z^2 f(z)\right)=\lim_{z\to 0}\frac{d}{dz}\left(\frac{1}{z^2+1}\right)=\lim_{z\to 0}\frac{-2z}{(z^2+1)^2}=0$. The residue at i is $\lim_{z\to i}(z-i)f(z)=\lim_{z\to i}\frac{1}{z^2(z+i)}=\frac{1}{-1(i+i)}=\frac{-1}{2i}=\frac{i}{2}$. Using the Residue Theorem, the integral is equal to $2\pi i(0+\frac{i}{2})=-\pi$.

   (b) $\frac{e^z}{z^2-4}=\frac{e^z}{(z-2)(z+2)}$ has two poles of order 1, 2 and -2 and both are in C. The residue at 2 is $\lim_{z\to 2}(z-2)f(z)=\lim_{z\to 2}\frac{e^z}{(z+2)}=\frac{e^2}{4}$. The residue at -2 is $\lim_{z\to -2}(z+2)f(z)=\lim_{z\to 2}\frac{e^z}{(z-2)}=\frac{-e^{-2}}{4}$. Thus, the integral is $2\pi i(\frac{e^2}{4}-\frac{e^{-2}}{4})=\frac{\pi i}{2}(e^2-e^{-2})$.

   (c) $f(z)=\frac{1}{(z-2)^2(4+z)}$ has two singularities: 2 is a pole of order 2 and -4 is a pole of order 1. 2 is in C and -4 is not.
   The residue at 2 is $\lim_{z\to 2}\frac{d}{dz}\left((z-2)^2 f(z)\right)=\lim_{z\to 2}\frac{d}{dz}\left(\frac{1}{4+z}\right)=\lim_{z\to 2}\frac{-1}{(4+z)^2}=\frac{-1}{36}$ so the integral is $\frac{-\pi i}{18}$.

   (d) Find the residue at -4 for function from part (c). It is $\lim_{z\to -4}(z+4)f(z)=\lim_{z\to -4}\frac{1}{(z-2)^2}=\frac{1}{36}$. So, the integral is the product of $2\pi i(\frac{-1}{36}+\frac{1}{36})=0$.

   (e) $f(z)=\frac{e^z}{(z-1)^5}$. The only singularity is 1 and it is a pole of order 5. The residue at 1 is $\frac{1}{4!}\lim_{z\to 1}\frac{d^4}{dz^4}\left((z-1)^5 f(z)\right)=\frac{1}{24}\lim_{z\to 1}\frac{d^4}{dz^4}\left(e^z\right)=\frac{1}{24}\lim_{z\to 1}e^z=\frac{e^1}{24}=\frac{e}{24}$ so the integral is $2\pi i\frac{e}{24}=\frac{e\pi i}{12}$.

   (f) $f(z)=z\cos\frac{1}{z}=z\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!z^{2n}}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!z^{2n-1}}=z-\frac{1}{2!z}+\frac{1}{4!z^3}-\ldots$. The only singularity is 0 and it is an essential singularity (infinitely many terms with z in denominator). From the power series expansion, the coefficient with $\frac{1}{z}$ is $\frac{-1}{2}$ so that is the residue. Hence, the integral is equal to $2\pi i\frac{-1}{2}=-\pi i$.

   (g) The zeros of the denominator of $f(z)=\frac{1}{(z^2-2z+2)^2}$ are $z=\frac{2\pm\sqrt{4-8}}{2}=\frac{2\pm 2i}{2}=1\pm i$. Thus $f(z)=\frac{1}{(z-(i+1))^2(z-(1-i))^2}$. So, there are two poles the second order $1\pm i$ and just $1+i$ is inside C. The residue at $1+i$ is $\lim_{z\to 1+i}\frac{d}{dz}\left((z-(1+i))^2 f(z)\right)=\lim_{z\to 1+i}\frac{d}{dz}\left(\frac{1}{(z-(1-i))^2}\right)=\lim_{z\to 1+i}\frac{-2}{(z-1+i)^3}=\frac{-2}{(2i)^3}=\frac{-2}{8(-i)}=\frac{1}{4i}=\frac{-i}{4}$. Thus the integral is equal to $2\pi i\frac{-i}{4}=\frac{\pi}{2}$.

2. Fourier Series. More detailed solutions can be found on the class handout.

   (a) Graph the function and note it is odd. Thus, $a_n=0$. Since $L=\pi$ $(T=2\pi)$, the coefficients b_n can be computed as $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx dx=\frac{2}{\pi}\int_0^{\pi}\sin nx dx=\frac{-2}{n\pi}\cos nx|_0^{\pi}=\frac{-2}{n\pi}((-1)^n-1)$. Note that $b_{2k}=0$ and $b_{2k+1}=\frac{-2}{n\pi}(-2)=\frac{4}{(2k+1)\pi}$. Hence, $f(x)=\frac{4}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{1}{n}\sin nx=\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin(2k+1)x}{2k+1}=\frac{4}{\pi}(\sin x+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\ldots)$.

   (b) Note that x^2 is already an even function. So, consider this function on interval [-2, 2] and replicate its graph on this domain outside of this interval too to create a periodic function of period $T=4$ (thus $L=2$). Since the new function is even too, $b_n=0$.
   $a_n=\int_0^2 x^2\cos\frac{n\pi x}{2}dx$. Using integration by parts twice, obtain that $a_n=\frac{2}{n\pi}x^2\sin\frac{n\pi x}{2}|_0^2-\frac{4}{n\pi}\int_0^2 x\sin\frac{n\pi x}{2}dx=\frac{8}{n^2\pi^2}x\cos\frac{n\pi x}{2}|_0^2-\frac{16}{n^3\pi^3}\sin\frac{n\pi x}{2}|_0^2=\frac{16}{n^2\pi^2}\cos n\pi=\frac{16(-1)^n}{n^2\pi^2}$. Note that this

formula works just for $n > 0$ so a_0 has to be computed separately $a_0 = \int_0^2 x^2 dx = \frac{x^3}{3}|_0^2 = \frac{8}{3}$. Thus, the Fourier series is $x^2 = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\frac{n\pi x}{2}$.

By Parseval's Theorem, $\frac{1}{2}\int_0^2 x^4 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{16}{9} + \frac{16^2}{2\pi^4}\sum_{n=1}^{\infty}\left(\frac{1}{n^4} + 0\right)$. Note that integral on the left side is $\frac{1}{2}\int_0^2 x^4 dx = \frac{16}{5}$. Dividing the equation above by 16 produces $\frac{1}{5} = \frac{1}{9} + \frac{8}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$.

(c) **Cosine expansion.** First extend the function symmetrically with respect to y-axis so that it is defined on basic period [-2,2] and that it is *even*. Thus $T = 4$ and $L = 2$. The coefficients b_n are zero in this case and the coefficients a_n can be computed as follows. $a_n = \int_0^2 f(x)\cos\frac{n\pi x}{2}dx = \int_0^1 x\cos\frac{n\pi x}{2}dx + \int_1^2 (2-x)\cos\frac{n\pi x}{2}dx = \left(\frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)|_0^1 + \left(\frac{2(2-x)}{n\pi}\sin\frac{n\pi x}{2} - \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)|_1^2 = \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2}\cos n\pi = \frac{4}{n^2\pi^2}(2\cos\frac{n\pi}{2} - 1 - \cos n\pi)$. If $n = 2k+1$ is odd, $a_n = \frac{4}{(2k+1)^2\pi^2}(0 - 1 + 1) = 0$. If $n = 2k$ is even, $a_n = \frac{4}{(2k)^2\pi^2}(2(-1)^k - 1 - 1)$. Because of the part with $(-1)^k$, we can distinguish two more cases depending on whether k is even or odd. Thus, if $k = 2l$ is even, $a_n = \frac{4}{(4l)^2\pi^2}(2-1-1) = 0$. If $k = 2l+1$ is odd, $a_n = \frac{4}{(2(2l+1))^2\pi^2}(2(-1) - 1 - 1) = \frac{-16}{(4l+2)^2\pi^2} = \frac{-4}{(2l+1)^2\pi^2}$. If $n = 0$, $a_0 = \int_0^2 f(x)dx = \int_0^1 xdx + \int_1^2 (2-x)dx = \frac{1}{2} + \frac{1}{2} = 1$. So, $f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{l=0}^{\infty}\frac{1}{(2l+1)^2}\cos\frac{(4l+2)\pi x}{2} = \frac{1}{2} - \frac{4}{\pi^2}\sum_{l=0}^{\infty}\frac{1}{(2l+1)^2}\cos(2l+1)\pi x$.

Sine expansion. First extend the function symmetrically with respect to the origin so that it is defined on basic period [-2,2] and that it is *odd*. Thus $T = 4$ and $L = 2$. The coefficients a_n are zero in this case and the coefficients b_n can be computed as follows. $b_n = \int_0^2 f(x)\sin\frac{n\pi x}{2}dx = \int_0^1 x\sin\frac{n\pi x}{2}dx + \int_1^2 (2-x)\sin\frac{n\pi x}{2}dx = (\frac{-2x}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2})|_0^1 + \left(\frac{-2(2-x)}{n\pi}\cos\frac{n\pi x}{2} - \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)|_1^2 = \underline{\frac{-2}{n\pi}\cos\frac{n\pi}{2}} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} + \underline{\frac{2}{n\pi}\cos\frac{n\pi}{2}} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} = \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$. This is 0 if n is even. If $n = 2k+1$, this is $\frac{8(-1)^k}{(2k+1)^2\pi^2}$. So, $f(x) = \frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\sin\frac{(2k+1)\pi x}{2}$.

To find the sum of $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$, note that when $x = 1$ the function $f(1)$ is equal to 1 and its Fourier sine expansion is equal to $\frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\frac{(2n+1)\pi}{2} = \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}(-1)^n = \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$. So $\frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = 1 \Rightarrow \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$.

(d) $T = 1$, $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2n\pi i t}$ and $c_n = \int_0^1 te^{-2n\pi i t} = \frac{t}{-2n\pi i}e^{-2n\pi i t}|_0^1 + \frac{1}{4n^2\pi^2}e^{-2n\pi i t}|_0^1 = \frac{1}{-2n\pi i}e^{-2n\pi i} + \frac{1}{4n^2\pi^2}e^{-2n\pi i} - \frac{1}{4n^2\pi^2}$. Note that $e^{-2n\pi i} = \cos(-2n\pi) + i\sin(-2n\pi) = 1$. Thus $c_n = \frac{1}{-2n\pi i} + 0 = \frac{i}{2n\pi}$. Note that $c_{-n} = \frac{-i}{2n\pi} = \overline{c_n}$. $c_0 = \int_0^1 t\,dt = \frac{1}{2}$. This gives us $f(t) = \frac{1}{2} + \sum_{n=-\infty, n\neq 0}^{\infty}\frac{i}{2n\pi}e^{2n\pi i t}$. Note also that $a_0 = 2c_0 = 1$, $a_n = 0$ for $n > 0$ and $b_n = \frac{-1}{n\pi}$. By Parseval's Theorem, $\int_0^1 x^2 dx = \frac{1}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2} \Rightarrow \frac{1}{3} = \frac{1}{4} + \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

3. Fourier Transform.

(a) $F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$. Since $f(t) = 0$ for $t < -1$ and $t > 1$, and $f(t) = 1$ for $-1 \leq t \leq 1$, we have that $F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-i\omega t}dt = \frac{-1}{\sqrt{2\pi}i\omega}e^{-i\omega t}|_{-1}^{1} = \frac{-1}{\sqrt{2\pi}\omega}\frac{e^{-i\omega} - e^{i\omega}}{i} = \frac{1}{\sqrt{2\pi}\omega}\frac{e^{i\omega} - e^{-i\omega}}{i} = \frac{2}{\sqrt{2\pi}\omega}\frac{e^{i\omega} - e^{-i\omega}}{2i} = \frac{2}{\sqrt{2\pi}\omega}\sin\omega = \frac{2}{\sqrt{2\pi}}\text{sinc}\omega$.

The inverse transform of the boxcar function can be obtained as $\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{i\omega t}d\omega = \frac{1}{\sqrt{2\pi}it}(e^{it} - e^{-it}) = \frac{2}{\sqrt{2\pi}t}\frac{e^{it} - e^{-it}}{2i} = \frac{2}{\sqrt{2\pi}t}\sin t = \frac{2}{\sqrt{2\pi}}\frac{\sin t}{t} = \frac{2}{\sqrt{2\pi}}\text{sinc}t$.

(b) Fourier transform: $F(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t} e^{-i\omega t} dt = \frac{-1}{\sqrt{2\pi}(1+i\omega)} e^{-(1+i\omega)t}|_0^\infty = \frac{1}{\sqrt{2\pi}(1+i\omega)}$.

Fourier Cosine transform: $F(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-t} \cos\omega t \; dt$. Using two integration by parts with $u = e^{-t}$, we have that $F(\omega) = \sqrt{\frac{2}{\pi}} (\frac{1}{\omega} e^{-t} \sin\omega t|_0^\infty + \frac{1}{\omega} \int_0^\infty e^{-t} \sin\omega t \; dt) = \sqrt{\frac{2}{\pi}} (0 - \frac{1}{\omega^2} e^{-t} \cos\omega t|_0^\infty - \frac{1}{\omega^2} \int_0^\infty e^{-t} \cos\omega t \; dt) = \sqrt{\frac{2}{\pi}} (\frac{1}{\omega^2} - \frac{1}{\omega^2} \sqrt{\frac{\pi}{2}} F(\omega)$. Solving for $F(\omega)$ gives your $F(\omega)(1 + \frac{1}{\omega^2}) = \sqrt{\frac{2}{\pi}} \frac{1}{\omega^2}$. Multiply by ω^2 to get $F(\omega)(\omega^2 + 1) = \sqrt{\frac{2}{\pi}} \Rightarrow F(\omega) = \sqrt{\frac{2}{\pi}} \frac{1}{\omega^2+1}$.

(c) $F(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{3/2} (2t - 3) \cos\omega t \; dt = \sqrt{\frac{2}{\pi}} (\frac{2t-3}{\omega} \sin\omega t|_0^{3/2} - \frac{2}{\omega} \int_0^{3/2} \sin\omega t \; dt) = \sqrt{\frac{2}{\pi}} (0 + \frac{2}{\omega^2} \cos\omega t|_0^{3/2}) = \sqrt{\frac{2}{\pi}} \frac{2}{\omega^2} (\cos\frac{3\omega}{2} - 1) = \frac{2\sqrt{2}}{\omega^2\sqrt{\pi}} (\cos\frac{3\omega}{2} - 1)$.